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$Q-$CESÀRO SEQUENCE SPACES DERIVED BY $Q-$ANALOGUE

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ABSTRACT. In the present paper, we mainly focus on $q-$analogs of the Cesàro sequence spaces. The $q-$Cesàro sequence spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ which are the $BK-$spaces including the spaces c_0 and c have been introduced and proved that the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ are linearly isomorphic to the spaces c_0 and c, respectively. Additionally, the $\alpha-, \beta-$ and $\gamma-$duals of the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ have been computed and their basis have been constructed. Finally, the necessary and sufficient conditions on an infinite matrix belonging to the classes $([\tilde{c}] : \ell_p)$ and $([\tilde{c}] : c)$ have been determined, where $1 \leq p \leq \infty$.

1. INTRODUCTION

By ω, we shall denote the space of all real valued sequences. Any vector subspace of ω is called as a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p ; we denote the spaces of all bounded, convergent, absolutely and $p-$ absolutely convergent series, respectively; where $1 < p < \infty$.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, ...\}$. A K-space λ is called an *FK-space* provided λ is a complete linear metric space. An FK-space whose topology is normable is called a *BK-space* (see [2, p.272-273]).

Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus,

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$A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \ldots + \alpha_n b_n)\| = 0,$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

For a sequence space λ, the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{1.2}$$

which is a sequence space. The new sequence space λ_A generated by the limitation matrix A from the space λ either includes the space λ or is included by the space λ, in general, i.e., the space λ_A is the expansion or the contraction of the original space λ. The sequence space $c_A := \{x = (x_k) \in \omega : Ax \in c\}$ is called the convergence domain of A. The matrix A is said to be conservative if the convergence of the sequence x implies the convergence of $A(x)$, (or equivalently $c \subset c_A$). In addition, if $A(x)$ converges to the limit x, for each convergent sequence x, then it is called regular. The following theorem states the well known characterization of conservative matrices and can be found in any standard summability book [1].

Theorem 1.1. *An infinite matrix* $A = (a_{nk})$ $n, k = 0, 1, 2, \ldots$ *is conservative if and only if*

(i) $\lim_{n\to\infty} a_{nk} = \lambda_k$ *for each* $k = 0, 1, \ldots$

(ii) $\lim_{n\to\infty} \sum_k a_{nk} = \lambda$*, and*

(iii) $\sup_n \sum_k |a_{nk}| \leq M < \infty$ *for some* $M > 0$.

Here, of course, the limits λ_k and λ are finite. If $\lambda_k = 0$ for all k and $\lambda = 1$ then the above theorem reduces to the well known theorem of Silverman and Toeplitz which provides necessary and sufficient conditions for regularity of the infinite matrix $A = (a_{nk})$ $n, k = 0, 1, 2, \ldots$.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [7], Ng and Lee [5], Malkowsky [4] and Altay and Başar [3, 8]. They introduced the sequence spaces $(\ell_p)_{N_q}$ in [7], $(\ell_p)_{C_1} = X_p$ in [5], $(\ell_\infty)_{R^t} = r^t_\infty$, $c_{R^t} = r^t_c$ and $(c_0)_{R^t} = r^t_0$ in [4] and $(\ell_p)_{E^r} = e^r_p$ in [3]; where N_q, C_1, R^t and E^r denote the Nörlund, arithmetic, Riesz and Euler means, respectively and $1 \leq p \leq \infty$. Şengönül and Başar [9] have studied the sequence spaces $\tilde{c}_0 = (c_0)_{C_1}$ and $\tilde{c} = c_{C_1}$; where C_1 denotes the matrix $C_1 = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \dfrac{1}{n+1}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$, and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. We will also use the convention that any term with negative subscript is equal to naught.

In the present paper, we introduce the sequence spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ and derive some results related to those sequence spaces. Furthermore, we construct a basis and compute the $\alpha-, \beta-$ and $\gamma-$duals of the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$. Finally, we characterize the matrix classes $([\tilde{c}] : \ell_p)$ and $([\tilde{c}] : c)$, where $1 \leq p \leq \infty$.

2. $q-$Cesàro Methods

In this section, we will first briefly mention about the Cesaro methods. Later, we will give definition of $q-$Cesaro method.

Definition 2.1. *Let α be a real number with $-\alpha \notin \mathbb{N}$ then the regular matrices $C_\alpha = (c_{nk}^\alpha)$ defined by*

$$c_{nk}^\alpha = \begin{cases} \dfrac{\dbinom{n-k+\alpha-1}{n-k}}{\dbinom{n+\alpha}{n}}, & (0 \leq k \leq n) \\ \\ 0, & (k > n) \end{cases}$$

and the associated matrix summability methods, are called the Cesàro matrix and Cesàro summability method of order α, respectively.

In particular if we choose $\alpha = 1$, we get the first order Cesàro matrix C_1 with the following explicit form,

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

Corresponding summability method is called the first order Cesàro summability method and denoted by $(C, 1)$. The Cesàro methods have played a central role in connection with the applications of summability theory to different branches of mathematics. The following theorem is the direct result of the theorem of Silverman and Toeplitz which provides necessary and sufficient conditions for regular matrices.

Theorem 2.1. *(i) If $\alpha \geq 0$, then C_α is regular.*
(ii) If $\alpha < 0$, then C_α is not conservative or regular.

We now give a brief introduction to the symbols of $q-$mathematics and $q-$Cesàro matrices. The subject of $q-$mathematics has many applications in mathematics and the beginnings of $q-$mathematics date back to time of Euler.

Definition 2.2. *The value* $[r]_q$ *denotes the* $q-$*integer of* r*, which is given by*

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q}, & q \in \mathbb{R}^+ - \{1\} \\ r, & q = 1. \end{cases}$$

For a given $q > 0$ let us define

$$\mathbb{N}_q = \{[r] : r \in \mathbb{N}\}.$$

We see from the definition of $[r]_q$ that

$$\mathbb{N}_q = \{0, 1, 1+q, 1+q+q^2, 1+q+q^2+q^3, ...\}. \tag{2.1}$$

Obviously, if we put $q = 1$ in (2.1), the set of all $q-$integers $\mathbb{N}_q$ reduces to the set of all natural numbers, the set of nonnegative integers $\mathbb{N}$.

Definition 2.3. *Given a value* $q > 0$, $q-$*shifted factorial is defined as*

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1})$$

for all $n \geq 1$ *and*

$$(a;q)_0 = 1.$$

The infinite version of this product is defined by

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Then one can define the $q-$analogue of the factorial, the $q-$factorial, as

$$[n]_q! = \begin{cases} \dfrac{q-1}{q-1} \cdot \dfrac{q^2-1}{q-1} \cdot ... \cdot \dfrac{q^n-1}{q-1}, & n = 1, 2, ... \\ 1, & n = 0. \end{cases}$$

Definition 2.4. *For any integer* n *and* k*,* $q-$*binomial coefficient is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q,q)_{n-k}} \tag{2.2}$$

for any $n \geq k \geq 0$.

Another way to write (2.2) is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}$$

which satisfies the following two pascal rules:

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ j \end{bmatrix} = q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

where $1 \leq j \leq n-1$.

For the last thirty years, studies involving $q-$integers and their applications (for example, $q-$analogs of positive linear operators and their approximation properties) have become active research areas. During the same period a large number of research papers on $q-$analogs of existing theories, involving interesting results, have been published (see [11, 12, 13]). The motivation of the present paper is the following question "What kind of results can be achieved by considering $q-$analogs of Cesàro matrices in the existing sequence spaces theory?"

There are many ways to define $q-$analogs of Cesàro matrices. In the following theorem, we suggest a suitable $q-$analog of the Cesàro matrix of order one.

Theorem 2.2. [10, Theorem 6] *$C_1(q^k) = (c^1_{nk}(q^k))$ with*

$$c^1_{nk}(q^k) = \begin{cases} \dfrac{q^k}{[n+1]_q}, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{2.3}$$

for all $n, k \in \mathbb{N}$.

The matrix method $C_1(q^k)$ and the corresponding summability method are called $q-$Cesàro matrix and $q-$Cesàro summability method of order one, respectively.

In the rest of this paper we shall focus on the matrix $C_1(q^k)$ which has the following explicit form;

$$C_1(q^k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{[2]_q} & \frac{q}{[2]_q} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{[3]_q} & \frac{q}{[3]_q} & \frac{q^2}{[3]_q} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{[n+1]_q} & \frac{q}{[n+1]_q} & \frac{q^2}{[n+1]_q} & \cdots & \frac{q^n}{[n+1]_q} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

Theorem 1.1 give us the following characterization for $C_1(q^k)$:

Theorem 2.3. [10, Lemma 7] *(i) $C_1(q^k)$ is conservative for each $q \in \mathbb{R}^+$,*
(ii) $C_1(q^k)$ is regular for each $q \ge 1$.

3. The $q-$Cesàro sequence spaces $[\tilde{c}_0]$ and $[\tilde{c}]$

We introduce the sequence spaces $[\tilde{c}_0]$ and $[\tilde{c}]$, as the set of all sequences such that $C_1(q^k)-$transforms of them are in the spaces c_0 and c, that is

$$[\tilde{c}_0] = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k x_k = 0 \right\}$$

and

$$[\tilde{c}] = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k x_k \quad \text{exists} \right\},$$

where $C_1(q^k)$ denotes the method of $q-$Cesàro matrix of order one defined by (2.3). With the notation of (1.2), we can redefine the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ by

$$[\tilde{c}_0] = \{c_0\}_{C_1(q^k)} \quad \text{and} \quad [\tilde{c}] = \{c\}_{C_1(q^k)}. \tag{3.1}$$

If λ is any normed sequence space then we call the matrix domain $\{\lambda\}_{C_1(q^k)}$ as the *$q-$Cesàro sequence space.* Define the sequence $y = (y_k^q)$, which will be frequently used, as the $C_1(q^k)-$transform of a sequence $x = (x_k)$, i.e.,

$$y_k^q = \frac{1}{[k+1]_q} \sum_{j=0}^{k} q^j x_j; \qquad (k \in \mathbb{N}). \tag{3.2}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 3.1. *The sets $[\tilde{c}_0]$ and $[\tilde{c}]$ are linear spaces with coordinatewise addition and scalar multiplication that are $BK-$spaces with norm $\|x\|_{[\tilde{c}_0]} = \|x\|_{[\tilde{c}]} = \|x\|_{\ell_\infty}$.*

Proof. The proof of the first part of the theorem is a routine verification, and so we omit it. Furthermore, since (3.1) holds, c_0 and c are $BK-$spaces with respect to their natural norm (see [14, pp.217-218]), and the matrix $C_1(q^k)$ is normal, i.e., $c^1_{nn}(q^k) \neq 0$ and $c^1_{nk}(q^k) = 0$, $k > n$, for all $k, n \in \mathbb{N}$, Theorem 4.3.2 of Wilansky [15, pp.61] implies that the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ are $BK-$spaces. □

Theorem 3.2. *The $q-$Cesàro sequence spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ are linearly isomorphic to the spaces c_0 and c, respectively, i.e., $[\tilde{c}_0] \cong c_0$ and $[\tilde{c}] \cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $[\tilde{c}_0]$ and c_0. Consider the transformation T defined, with the notation of (3.2), from $[\tilde{c}_0]$ to c_0 by $x \mapsto y = Tx$. The linearity of T is clear. Further, it is trivial that $x = \theta = (0, 0, 0, ...)$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in c_0$ and define the sequence $x = (x_k^q)$ by

$$x_k^q = \frac{[k+1]_q}{q^k} y_k - \frac{[k]_q}{q^k} y_{k-1}; \quad (k \in \mathbb{N}).$$

Then, we have

$$\lim_{n\to\infty} \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k x_k^q = \lim_{n\to\infty} \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k \left\{ \frac{[k+1]_q}{q^k} y_k - \frac{[k]_q}{q^k} y_{k-1} \right\} = \lim_{n\to\infty} y_n = 0$$

which says us that $x \in [\tilde{c}_0]$. Additionally, we observe that

$$\begin{aligned} \|x\|_{[\tilde{c}_0]} &= \sup_{n\in\mathbb{N}} \left| \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k \left\{ \frac{[k+1]_q}{q^k} y_k - \frac{[k]_q}{q^k} y_{k-1} \right\} \right| \\ &= \sup_{n\in\mathbb{N}} |y_n| = \|y\|_{c_0} < \infty. \end{aligned}$$

Consequently, we see from here that T is surjective and is norm preserving. Hence, T is a linear bijection which therefore shows us that the spaces $[\tilde{c}_0]$ and c_0 are linearly isomorphic, as was desired. □

We can now give theorems on inclusion relations concerning the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$.

Theorem 3.3. *The inclusion $c \subset [\tilde{c}]$ strictly holds for each $q \in \mathbb{R}^+$.*

Proof. To prove the validity of the inclusion $c \subset [\tilde{c}]$, let us take any $y \in c$. Since, the method $C_1(q^k)$ is conservative for each $q \in \mathbb{R}^+$ we immediately observe that $C_1(q^k)y \in c$ which means that $y \in [\tilde{c}]$. Hence, the inclusion $c \subset [\tilde{c}]$ holds. Furthermore, let us consider the sequence $x = \{x_k(q)\}$ defined by

$$x_k(q) = \begin{cases} \frac{1}{q}, & k = 0, 2, ... \\ \frac{-1}{q^2}, & k = 1, 3, ... \end{cases}$$

for each $q \in \mathbb{R}^+$. Then, since

$$\{C_1(q^k)x\}_n = \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k x_k(q) = \begin{cases} 0, & n = 1, 3, ... \\ \frac{q^{n-1}}{[n+1]_q}, & n = 0, 2, ... \end{cases}$$

we obtain

$$\lim_{n\to\infty} \{C_1(q^k)x\}_n = \begin{cases} 0, & q \leq 1 \\ \frac{q-1}{q^2}, & q > 1 \end{cases} .$$

This shows that x is in $[\tilde{c}]$ but not in c. Hence, the inclusion $c \subset [\tilde{c}]$ is strict. This completes the proof. □

Theorem 3.4. *The inclusion $c_0 \subset [\tilde{c}_0]$ strictly holds for $q \geq 1$.*

Proof. To prove the validity of the inclusion $c_0 \subset [\tilde{c}_0]$, let us take any $y \in c_0$. Then, bearing in mind the regularity of the method $C_1(q^k)$ for $q \geq 1$ we immediately observe that $C_1(q^k)y \in c_0$ which means that $y \in [\tilde{c}_0]$. Hence, the inclusion $c_0 \subset [\tilde{c}_0]$ holds. Now, let us consider the sequence $u = \{u_k(q)\}$ defined by

$$u_k(q) = \frac{(-1)^k}{q}$$

for each $q \geq 1$. Then, we obtain that

$$C_1(q^k)u = \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k (-1)^k \frac{1}{q} = \frac{1-(-1)^{n+1}}{2[n+1]_q}$$

which shows that $C_1(q^k)u \to 0$ as $n \to \infty$. That is to say that $u \in [\tilde{c}_0] \backslash c_0$. □

Theorem 3.5. *The inclusion $[\tilde{c}_0] \subset [\tilde{c}]$ strictly holds.*

Proof. It is clear that the inclusion $[\tilde{c}_0] \subset [\tilde{c}]$ holds. Further, to show that this inclusion is strict, consider the sequence $x = (x_k) = (1)$ for all $k \in \mathbb{N}$. Then, we obtain by (3.2) for all $k \in \mathbb{N}$ that

$$C_1(q^k)x = \frac{1}{[n+1]_q} \sum_{k=0}^{n} q^k x_k = 1$$

which shows that $C_1(q^k)x \to 1$ as $n \to \infty$. That is to say that $C_1(q^k)x \in c \backslash c_0$. Thus, the sequence x is in $[\tilde{c}]$ but not in $[\tilde{c}_0]$. Hence, the inclusion $[\tilde{c}_0] \subset [\tilde{c}]$ is strict.

□

Theorem 3.6. *The space ℓ_∞ does not include the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ for $q \le 1$.*

Proof. For any fixed $q \le 1$, choosing an index sequence as $r_j = j^{j+1}$ $(j \in \mathbb{N})$ and $r_0 = 0$, unbounded sequence $v = \{v_k(q)\}$ with

$$v_k(q) = \begin{cases} \displaystyle\sum_{i=0}^{j} \frac{1}{q(i+1)}, & k = 0, 2, \ldots \text{ and } r_j \le k < r_{j+1} \\ \displaystyle\sum_{i=0}^{j} \frac{-1}{q^2(i+1)}, & k = 1, 3, \ldots \text{ and } r_j \le k < r_{j+1} \end{cases},$$

is $C_1(q^k)-$summable to 0. Hence, the sequence $v = \{v_k(q)\}$ is in the space $[\tilde{c}_0]$ but is not in the space ℓ_∞. This shows that the space ℓ_∞ does not include both the space $[\tilde{c}_0]$ and the space $[\tilde{c}]$, as desired. □

Because of the isomorphism T, defined in Theorem 3.2, is onto the inverse image of the basis of those spaces c_0 and c are the basis of the new spaces $[\tilde{c}_0]$ and $[\tilde{c}]$, respectively. Therefore, we have the following:

Theorem 3.7. *Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}_{n\in\mathbb{N}}$ of the elements of the space $[\tilde{c}_0]$ by*

$$b_n^{(k)}(q) = \begin{cases} (-1)^{k-n} \dfrac{[k+1]_q}{q^k}, & k \le n \le k+1 \\ 0, & 0 \le n < k \text{ or } n > k+1 \end{cases} \tag{3.3}$$

for every fixed $k \in \mathbb{N}$. Then:

(i) The sequence $\{b^{(k)}(q)\}_{k\in\mathbb{N}}$ is a basis for the space $[\tilde{c}_0]$ and any $x \in [\tilde{c}_0]$ has a unique representation of the form

$$x = \sum_k \lambda_k(q) b^{(k)}(q),$$

where $\lambda_k(q) = \{C_1(q^k)x\}_k$ for all $k \in \mathbb{N}$.

(ii) The set $\{z, b^{(k)}(q)\}$ is a basis for the space $[\tilde{c}]$ and any $x \in [\tilde{c}]$ has a unique representation of the form

$$x = lz + \sum_k [\lambda_k(q) - l] b^{(k)}(q),$$

where $z = (1/q^k)$ and $l = \lim_{k\to\infty}\{C_1(q^k)x\}_k$.

4. The $\alpha-,\beta-$ and $\gamma-$duals of the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$

In this section, we state and prove the theorems determining the $\alpha-,\beta-$ and $\gamma-$duals of the sequence spaces $[\tilde{c}_0]$ and $[\tilde{c}]$.

For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k)\in\omega : xz=(x_kz_k)\in\mu \text{ for all } x\in\lambda\}. \tag{4.1}$$

With the notation of (4.1), the $\alpha-,\beta-$ and $\gamma-$duals of a sequence space λ, which are respectively denoted by $\lambda^\alpha,\lambda^\beta$ and λ^γ, are defined by

$$\lambda^\alpha=S(\lambda,\ell_1),\quad \lambda^\beta=S(\lambda,cs)\quad \text{and } \lambda^\gamma=S(\lambda,bs).$$

We shall begin with to quote the lemmas, due to Stieglitz and Tietz [6], which are needed in proving Theorems 4.3-4.5, below.

Lemma 4.1. *$A\in(c_0:\ell_1)=(c:\ell_1)$ if and only if*

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}a_{nk}\right|<\infty.$$

Lemma 4.2. *$A\in(c:c)$ if and only if*

$$\lim_{n\to\infty}a_{nk}=\alpha_k,\quad (k\in\mathbb{N}), \tag{4.2}$$

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|<\infty, \tag{4.3}$$

$$\lim_{n\to\infty}\sum_k a_{nk}\ \textit{exists.} \tag{4.4}$$

Theorem 4.1. *The $\alpha-$dual of the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ is the set*

$$c_1(q)=\left\{a=(a_k)\in\omega : \sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(-1)^{n-k}\frac{[k+1]_q}{q^n}a_n\right|<\infty\right\}.$$

Proof. Let $a=(a_n)\in\omega$ and define the matrix $B=(b_{nk})$ via the sequence $a=(a_n)$ by

$$b_{nk}=\begin{cases}(-1)^{n-k}\dfrac{[k+1]_q}{q^n}a_n, & n-1\le k\le n\\ 0, & 0\le k<n-1 \text{ or } k>n\end{cases}\quad ;\quad (n,k\in\mathbb{N}).$$

Bearing in mind the relation (3.2) we immediately derive that

$$a_nx_n=\sum_{k=n-1}^{n}(-1)^{n-k}\frac{[k+1]_q}{q^n}a_ny_k=(By)_n,\quad (n\in\mathbb{N}). \tag{4.5}$$

We therefore observe by (4.5) that $ax=(a_nx_n)\in\ell_1$ whenever $x\in[\tilde{c}_0]$ or $[\tilde{c}]$ if and only if $By\in\ell_1$ whenever $y\in c_0$ or c. Then, we derive by Lemma 4.1 that

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(-1)^{n-k}\frac{[k+1]_q}{q^n}a_n\right|<\infty$$

which yields the result that $\{[\tilde{c}_0]\}^\alpha=\{[\tilde{c}]\}^\alpha=c_1(q)$. □

Theorem 4.2. *Define the sets* $c_2(q)$ *and* $c_3(q)$ *by*

$$c_2(q) = \left\{ a = (a_k) \in \omega : \sum_k [k+1]_q \left| \Delta\left(\frac{a_k}{q^k}\right) \right| < \infty \right\}$$

and

$$c_3(q) = \{ a = (a_k) \in \omega : ([k+1]_q a_k) \in \ell_\infty \},$$

where

$$\Delta\left(\frac{a_k}{q^k}\right) = \frac{a_k}{q^k} - \frac{a_{k+1}}{q^{k+1}}$$

for all $k \in \mathbb{N}$. *Then,* $\{[\tilde{c}]\}^\beta = c_2(q) \cap cs$ *and* $\{[\tilde{c}_0]\}^\beta = c_2(q) \cap c_3(q)$.

Proof. Because of the proof may also be obtained for the space $[\tilde{c}_0]$ in the similar way, we omit it and give the proof only for the space $[\tilde{c}]$. Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \frac{a_k}{q^k} \{[k+1]_q y_k - [k]_q y_{k-1}\} \\ &= \sum_{k=0}^{n-1} [k+1]_q \Delta\left(\frac{a_k}{q^k}\right) y_k + [n+1]_q a_n y_n \\ &= (Ty)_n, \quad (n \in \mathbb{N}), \end{aligned} \tag{4.6}$$

where $T = (t_{nk})$ is defined by

$$t_{nk} = \begin{cases} [k+1]_q \Delta\left(\frac{a_k}{q^k}\right), & 0 \le k \le n-1 \\ [n+1]_q a_n, & k = n \\ 0, & k > n \end{cases} ; \quad (n, k \in \mathbb{N}). \tag{4.7}$$

Thus, we deduce from Lemma 4.2 with (4.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in [\tilde{c}]$ if and only if $Ty \in c$ whenever $y = (y_k) \in c$. It is obvious that the columns of that matrix T, defined by (4.7), are in the space c. Therefore, we derive the consequences from (4.2), (4.3) and (4.4) that

$$\sum_k [k+1]_q \left| \Delta\left(\frac{a_k}{q^k}\right) \right| < \infty$$

and

$$a = (a_k) \in cs,$$

respectively. This shows that $\{[\tilde{c}]\}^\beta = c_2(q) \cap cs$. □

Theorem 4.3. *The* $\gamma-$*dual of the spaces* $[\tilde{c}_0]$ *and* $[\tilde{c}]$ *is the set* $c_2(q) \cap c_3(q)$.

Proof. The proof of this Theorem is similar to the proof the Theorem 4.2 and so we leave the detail to the reader. □

5. Some Matrix Mappings Related to $q-$Cesàro Sequence Spaces

In this section, we characterize the matrix mappings from $[\breve{c}]$ into some of the known sequence spaces.

We shall write throughout for brevity that

$$\tilde{a}_{nk} = [k+1]_q \Delta\left(\frac{a_{nk}}{q^k}\right) = [k+1]_q \left(\frac{a_{nk}}{q^k} - \frac{a_{n,k+1}}{q^{k+1}}\right)$$

for all $n, k \in \mathbb{N}$. We will also use the similar notation with other letters and use the convention that any term with negative subscript is equal to naught. We shall begin with two lemmas due to Wilansky [15, p.57 and p.128] which are needed in the proof of our theorems.

Lemma 5.1. *The matrix mappings between the $BK-$spaces are continuous.*

Lemma 5.2. *$A \in (c : \ell_p)$ if and only if*

$$\sup_{F\in\mathcal{F}} \sum_n \left|\sum_{k\in F} a_{nk}\right|^p < \infty, \quad (1 \le p < \infty). \tag{5.1}$$

Theorem 5.1. *$A \in ([\breve{c}] : \ell_p)$ if and only if*

(i) For $1 \le p < \infty$,

$$\sup_{F\in\mathcal{F}} \sum_n \left|\sum_{k\in F} \tilde{a}_{nk}\right|^p < \infty, \tag{5.2}$$

$$\sum_k |\tilde{a}_{nk}| < \infty \text{ for all } n \in \mathbb{N}, \tag{5.3}$$

$$\{[k+1]_q a_{nk}\}_{k\in\mathbb{N}} \in cs \text{ for all } n \in \mathbb{N}. \tag{5.4}$$

(ii) For $p = \infty$, (5.4) holds, and

$$\sup_{n\in\mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty. \tag{5.5}$$

Proof. Suppose the conditions (5.2)-(5.4) hold and take any $x \in [\breve{c}]$. Then, $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[\breve{c}]\}^\beta$ for all $n \in \mathbb{N}$ and this implies that Ax exists. Let us define the matrix $B = (b_{nk})$ with $b_{nk} = \tilde{a}_{nk}$ for all $n, k \in \mathbb{N}$. Then, since (5.1) is satisfied for that matrix B we have $B \in (c : \ell_p)$. Let us now consider the following equality obtained from the m^{th} partial sum of the series $\sum_k a_{nk}x_k$:

$$\begin{aligned} \sum_{k=0}^{m} a_{nk}x_k &= \sum_{k=0}^{m-1} [k+1]_q \Delta\left(\frac{a_{nk}}{q^k}\right) y_k + [m+1]_q a_{nm} y_m \\ &= \sum_{k=0}^{m-1} \tilde{a}_{nk} y_k + [m+1]_q a_{nm} y_m; \quad (n, m \in \mathbb{N}). \end{aligned} \tag{5.6}$$

Following the way that used in the proof of Theorem 4.2, one can derive by combining the conditions (5.3) and (5.4) that $\{[m+1]_q a_{nm}\}_{m\in\mathbb{N}} \in c_0$ for each $n \in \mathbb{N}$. Thus, bearing in mind this fact if we pass to limit in (5.6) as $m \to \infty$ then the second term on the right hand term tends to zero and we derive that

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}y_k, \quad (n \in \mathbb{N}) \tag{5.7}$$

which yields by taking ℓ_p-norm that

$$\|Ax\|_{\ell_p} = \|By\|_{\ell_p} < \infty.$$

This means that $A \in ([\tilde{c}] : \ell_p)$.

Conversely, suppose that $A \in ([\tilde{c}] : \ell_p)$. Then, since $[\tilde{c}]$ and ℓ_p are the $BK-$spaces we have from Lemma 5.1 that there exists some real constant $K > 0$ such that

$$\|Ax\|_{\ell_p} \leq K \cdot \|x\|_{[\tilde{c}]} \tag{5.8}$$

for all $x \in [\tilde{c}]$. Since the inequality (5.8) is also satisfied for the sequence $x = (x_k) = \sum_{k\in F} b^{(k)}(q)$ belonging to the space $[\tilde{c}]$, where $b^{(k)}(q) = \{b_n^{(k)}(q)\}_{n\in\mathbb{N}}$ is defined by (3.3), we thus have for any $F \in \mathcal{F}$ that

$$\|Ax\|_{\ell_p} = \left(\sum_n \left|\sum_{k\in F} \tilde{a}_{nk}\right|^p\right)^{1/p} \leq K \cdot \|x\|_{[\tilde{c}]}$$

which shows the necessity of (5.2).

Since A is applicable to the space $[\tilde{c}]$ by the hypothesis, the necessities of (5.3) and (5.4) are trivial. This completes the proof of the part (i) of Theorem.

Since the part (ii) may also be proved in the similar way that of the part (i), we leave the detailed proof to the reader. □

Theorem 5.2. $A \in ([\tilde{c}] : c)$ *if and only if (5.4) and (5.5) hold, and*

$$\lim_{n\to\infty} \tilde{a}_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{5.9}$$

$$\lim_{n\to\infty} \sum_k \tilde{a}_{nk} = \alpha. \tag{5.10}$$

Proof. Suppose that A satisfies the conditions (5.4), (5.5), (5.9) and (5.10). Let us take any $x = (x_k)$ in $[\tilde{c}]$. Then, Ax exists and it is trivial that the sequence $y = (y_k)$ connected with the sequence $x = (x_k)$ by the relation (3.2) is in c such that $y_k \to l$ as $k \to \infty$. At this stage, we observe from (5.9) and (5.5) that

$$\sum_{j=0}^{k} |\alpha_j| \leq \sup_{n\in\mathbb{N}} \sum_j |\tilde{a}_{nj}| < \infty$$

holds for every $k \in \mathbb{N}$. This leads us to the consequence that $(\alpha_k) \in \ell_1$. Considering (5.7), let us write

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}(y_k - l) + l\sum_k \tilde{a}_{nk}. \tag{5.11}$$

In this situation, by letting $n \to \infty$ in (5.11) we see that the first term on the right tends to $\sum_k \alpha_k(y_k - l)$ by (5.5) and (5.9), and the second term tends to $l\alpha$ by (5.10) and we thus have that

$$(Ax)_n \to \sum_k \alpha_k(y_k - l) + l\alpha$$

which shows that $A \in ([\tilde{c}] : c)$.

Conversely, suppose that $A \in ([\tilde{c}] : c)$. Then, since the inclusion $c \subset \ell_\infty$ holds, the necessities of (5.4) and (5.5) are immediately obtained from Theorem 5.1. To prove the necessity of (5.9), consider the sequence $x = x^{(k)} = \{x_n^{(k)}(q)\}_{n\in\mathbb{N}} \in [\tilde{c}]$ defined by

$$x_n^{(k)}(q) = \begin{cases} (-1)^{k-n}\dfrac{[k+1]_q}{q^k}, & k \le n \le k+1 \\ 0, & 0 \le n \le k-1 \text{ or } n > k+1 \end{cases}$$

for each $k \in \mathbb{N}$. Since Ax exists and is in c for every $x \in [\tilde{c}]$, one can easily see that $Ax^{(k)} = \{\tilde{a}_{nk}\}_{n\in\mathbb{N}} \in c$ for each $k \in \mathbb{N}$ which shows the necessity of (5.9).

Similarly by putting $x = e$ in (5.7), we also obtain that $Ax = \{\sum_k \tilde{a}_{nk}\}_{n\in\mathbb{N}}$ which belongs to the space c and this shows the necessity of (5.10). This step concludes the proof. □

6. Conclusion

For the last thirty years, studies involving $q-$integers and their applications (for example, $q-$analogs of positive linear operators and their approximation properties) have become active research areas. During the same period a large number of research papers on $q-$analogs of existing theories, involving interesting results, have been published (see [11, 12, 13, 16, 17]). The motivation of the present paper is the following question "What kind of results can be achieved by considering $q-$analogs of Cesàro matrices in the existing sequence spaces theory?" In the present paper, we introduce the sequence spaces $[\tilde{c}_0]$ and $[\tilde{c}]$ and derive some results related to those sequence spaces. Furthermore, we construct a basis and compute the $\alpha-, \beta-$ and $\gamma-$duals of the spaces $[\tilde{c}_0]$ and $[\tilde{c}]$. Finally, we characterize the matrix classes $([\tilde{c}] : \ell_p)$ and $([\tilde{c}] : c)$, where $1 \le p \le \infty$.

Finally, we should note from now on that the investigation of the domain of some particular $q-$limitation matrices, namely $q-$Cesàro means of order α, $q-$Euler means of order r, $q-$Riesz means, etc., in the spaces c_0, c, ℓ_∞ and ℓ_p will lead us to new results which are not comparable with the present results.

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