ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **5** (2016), no.2, 111–116
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

SCHWARZIAN DERIVATIVE AS A CONDITION FOR UNIVALENCE

EDMOND ALIAGA[1] AND NIKOLA TUNESKI

ABSTRACT. This paper a method introduced by D. Aharonov and U. Elias is used for obtaining new condition for univalence that involves the Shwartzian derivative and Nehari's functions. An example that illustrates the condition is provided.

1. INTRODUCTION

Let $\mathcal{S}$ denote the class of function $f(z)$ which are univalent (analytic and one-to-one) in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$.

Using the Schwarzian derivative

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2,$$

Nehari in [3] proved the following conditions for univalence:

$$|Sf(z)| \le 2(1 - |z|^2)^{-2} \quad (z \in \mathbb{D}) \quad \Rightarrow \quad f \in \mathcal{S} \tag{1.1}$$

and

$$|Sf(z)| \le \frac{\pi^2}{2} \ (z \in \mathbb{D}) \ \Rightarrow \ f \in \mathcal{S}. \tag{1.2}$$

Implication (1.1) was shown to be sharp by Hille in [2]. Sharpness was considered as in the following definition.

Definition 1.1. *A criteria for univalence of form*

$$|Sg(z)| \le 2p(|z|) \qquad (z \in \mathbb{D})$$

is sharp, if for an analytic function $g(z)$, conditions

$$Sg(x) \ge 2p(x) \qquad (-1 < x < 1)$$

and

$$Sg(z) \ne 2p(z) \qquad (z \in \mathbb{D})$$

imply that $g(z)$ is not univalent in $\mathbb{D}$.

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic function, univalence, sufficient condition, parametric function.

Furthermore, Pokornyi in [5] claimed (without proof) and Nehari in [4] proved that

$$|Sf(z)| \le 4(1-|z|^2)^{-1} \quad (z \in \mathbb{D}) \quad \Rightarrow \quad f \in \mathcal{S}. \tag{1.3}$$

These results opened new area of research in the geometric function theory. One of the most important results is the following.

Theorem 1.1. (*Nehari* [3]) *Suppose that:*

(*i*) *$p(x)$ is a positive and continuous even function for $-1 < x < 1$,*
(*ii*) *$p(x)(1-x^2)^2$ is nonincreasing for $0 < x < 1$,*
(*iii*) *the differential equation*

$$y''(x) + p(x)y(x) = 0$$

has a solution which does not vanish in $-1 < x < 1$.

Then any analytic function $f(z)$ in $\mathbb{D}$ satisfying

$$|Sf(z)| \le 2p(|z|) \quad (z \in \mathbb{D})$$

is univalent in the unit disk $\mathbb{D}$.

Definition 1.2. *A function $p(x)$ satisfying conditions (i), (ii) and (iii) from Theorem 1.1 is called a Nehari's function.*

Using this definition and Theorem 1.1, univalence criteria (1.1), (1.2) and (1.3) can be proven by showing that functions $p(x) = -y''(x)/y(x)$ obtained for

$$y(x) = (1-x^2)^{1/2}, \quad y(x) = cos(\pi x/2) \quad \text{and} \quad y(x) = 1 - x^2,$$

respectively, are Nehari's functions.

Aharonov and Elias in [1] used Theorem 1.1 to introduce a new method for finding sufficient conditions for univalence. Namely, starting with an analytic function $u(z, \lambda_1, \lambda_2, \ldots, \lambda_n)$ with n parameters $(\lambda_1, \lambda_2, \ldots \lambda_n)$ they received the function

$$p(z, \lambda_1, \lambda_2, \ldots, \lambda_n) = -\frac{u''(z, \lambda_1, \lambda_2, \ldots, \lambda_n)}{u(z, \lambda_1, \lambda_2, \ldots, \lambda_n)}.$$

Conditions over the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ when p is a Nehari's function, according to Theorem 1.1, make inequality

$$|Sf(z)| \le 2p(|z|) \quad (z \in \mathbb{D})$$

an univalence criteria. They studied cases

$$u(x, \lambda) = (1-x^2)(1-\lambda x^2)$$

and

$$u(x, \lambda, \mu) = (1-x^2)^{\lambda} cos^{\mu}(\pi x/2),$$

and received the following two results, respectively.

Theorem 1.2. ([1], case $u = (1-x^2)(1-\lambda x^2)$) *Let*

$$p(x, \lambda) = \frac{2(1+\lambda) - 12\lambda x^2}{(1-x^2)(1-\lambda x^2)}.$$

If $f(z)$ is an analytic function in $\mathbb{D}$ satisfying

$$|Sf(z)| \leq 2p(|z|), \quad (z \in \mathbb{D})$$

with $0 \leq \lambda \leq 1/5$, *then* $f(z)$ *is univalent in* $\mathbb{D}$. *Moreover the theorem is sharp, according to Definition 1.1.*

Theorem 1.3. ([1], case $u = (1-x^2)^\lambda cos^\mu(\pi x/2))$) *Let*

$$p(x) = 4\lambda(1-\lambda)x^2\left(1-x^2\right)^{-2} + 2\left(1-x^2\right)^{-1} + \mu\pi^2/4 + \mu(1-\mu)\pi^2\tan^2(\pi x/2)/4 - 2\mu\lambda\pi x\tan(\pi x/2)\left(1-x^2\right)^{-1}$$

and let λ *and* μ *satisfy*

$$\lambda \geq 0, \quad \mu \geq 0, \quad 1/2 \leq \lambda + \mu \leq 1, \quad 2\lambda + \mu \geq 1.$$

If $f(z)$ *is an analytic function in* $\mathbb{D}$ *satisfying*

$$|Sf(z)| \leq 2p(|z|) \quad (z \in \mathbb{D}),$$

then $f(z)$ *is univalent in* $\mathbb{D}$. *The theorem is sharp, according to Definition 1.*

The aim of this paper is using the same method as Aharonov and Elias in [1] to find other conditions for univalence that involve the Schwarzian derivative. Such conditions are of particular importance because the Schwarzian derivative provides information on how "different" is function f from the Möbius (bilinear) transformations

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

since the only functions with zero Schwarzian derivative are the bilinear transformations.

2. Condition for univalence depending on two parameters

Our result is formulated in the following theorem.

Theorem 2.1. *Let* $u(x) = \frac{1-x^2}{a-x^2} \cdot exp\left(\lambda x^2\right)$ $(a > 1,\ -1/4 \leq \lambda < 0)$,

$$q(t) = \frac{8(a-1)t(1-t)}{(a-t)^2} + \frac{2(a-1)(1-t)}{a-t}(1+4\lambda t) - 2\lambda(2\lambda t+1)(1-t)^2.$$

and

$$p(x,a,\lambda) = \frac{8(a-1)x^2}{(1-x^2)(a-x^2)^2} + \frac{2(a-1)(1+4\lambda x^2)}{(1-x^2)(a-x^2)} - 2\lambda.$$

Then, there exist real numbers a *and* λ *such that* $q(t)$ *is non-increasing on the interval* $(0,1)$ *and* $p(x,a,\lambda$ *is a Nehari's function. Even more, any analytic function* $f(z)$ *in* $\mathbb{D}$ *satisfying*

$$|Sf(z)| \leq 2p(|z|) \quad (z \in \mathbb{D}),$$

is univalent in the unit disk $\mathbb{D}$.

Proof. Let define a function $G(x) = exp\left(\lambda x^2\right)$. Then

$$u(x) = \frac{1-x^2}{a-x^2} \cdot G(x) = \left(1 + \frac{1-a}{a-x^2}\right) \cdot G(x) \tag{2.1}$$

does not vanish for $-1 < x < 1$,

$$u'(x) = \frac{2(1-a)x}{(a-x^2)^2} \cdot G(x) + \frac{1-x^2}{a-x^2} \cdot G'(x)$$

and

$$u''(x) = \frac{2\,(1-a)\left(a+3x^2\right)}{\left(a-x^2\right)^3} \cdot G\,(x) + \frac{4x\,(1-a)}{\left(a-x^2\right)^2} \cdot G'\,(x) + \frac{1-x^2}{a-x^2} \cdot G''\,(x)\,.$$

Bearing in mind Theorem 1.1, in order to complete the proof, it is enough to show that

$$p(x) = -\frac{u''(x)}{u(x)} = \frac{8\,(a-1)\,x^2}{(1-x^2)\,(a-x^2)^2} + \frac{2\,(a-1)\left(1+4\lambda x^2\right)}{(1-x^2)\,(a-x^2)} - 2\lambda(2\lambda x^2+1)$$

is a Nehari's function for some real numbers a and λ.

Indeed, $p(x)$ is even and continuous function for $-1 < x < 1$.

Also, the first expression in $p(x)$ is positive, and for $-1/4 \leq \lambda < 0$, also the second expression is positive, as well as the expression $-4\lambda^2x^2 - 2\lambda$, i.e., function $p(x)$ is positive for $-1 < x < 1$.

The function $q\,(t)$ defined by

$$p(x)(1-x^2)^2 = q(x^2)$$

is continuous on $(0,1)$ and the functions $h_1(t)$ and $h_2(t)$ defined by

$$h_1(t) \equiv \lim_{a\to 1^+} q\,(t) = -2\lambda\,(2\lambda t + 1\,)\,(1-t)^2$$

and

$$h_2(t) \equiv \lim_{a\to +\infty} q\,(t) = 2\,(1-t)\,(1+4\lambda t) - 2\lambda\,(2\lambda t + 1\,)\,(1-t)^2$$

are non-increasing on the interval $(0,1)$. So, there exist values of a and λ such that $p(x)(1-x^2)^2$ is non-increasing for $0 < x < 1$. $\square$

Remark 2.1. *In the view of Theorem 2.1 there are two open probvlems. One is to determine specific values (intervals) for a and λ such that $p(x)$ is Nehari's function and the other is to verify sharpness of Theorem 4.*

Example 2.1. *The function*

$$f(z) = \int_0^z \frac{(2-t^2)^2}{(1-t^2)^2 \cdot e^{-t^2/2}}\,dt$$

is an odd univalent function in U.

Proof. If we put $a = 2$ and $\lambda = -1/4$ in expression (2.1), we receive

$$u(x) = \frac{1-x^2}{2-x^2} \cdot \exp\left(-x^2/4\right)$$

and

$$p(z) = -u''(z)/u(z) = \frac{8z^2}{(1-z^2)(2-z^2)^2} + \frac{2}{2-z^2} - \frac{z^2}{4} + \frac{1}{2}$$
$$= \frac{8}{1-z^2} - \frac{6}{2-z^2} - \frac{16}{(2-z^2)^2} - \frac{z^2}{4} + \frac{1}{2},$$

such that $Sf(z) = 2p(z)$. Further, using

$$\frac{1}{1-z^2} = \sum_{k=0}^{+\infty} z^{2k}, \quad \frac{1}{2-z^2} = \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \cdot z^{2k}$$

and

$$\frac{1}{(2-z^2)^2} = \sum_{k=0}^{+\infty} \frac{k+1}{2^{k+2}} \cdot z^{2k},$$

we obtain $p(z) = \sum_{k=0}^{+\infty} a_k z^{2k}$ with $a_0 = 3/2$, $a_1 = 9/4$ and $a_k = 8 - (4k+7)/2^k$ for $k = 2, 3, \dots$. So, $a_k \geq 0$ for all k, which implies $|p(z)| \leq p(|z|)$.

Thus,

$$|Sf(z)| = 2|p(z)| \leq 2p(|z|).$$

Now, in order to conclude univalence of f, bearing in mind Theorem 2.1, it is enough to show that

$$\varphi(x) \equiv p(x)(1-x^2)^2 = \frac{8x^2(1-x^2)}{(2-x^2)^2} + \frac{2(1-x^2)^2}{2-x^2} + \frac{1}{4}(2-x^2)(1-x^2)^2$$

is non-increasing for $0 < x < 1$, i.e., that

$$\psi(t) = \frac{8t(1-t)}{(2-t)^2} + \frac{2(1-t)^2}{2-t} + \frac{1}{4}(2-t)(1-t)^2$$

is non-increasing for $0 < t < 1$. The analysis continues with $\psi'(t) = \frac{1}{4(2-t)^3} \cdot g(t)$, where

$$\begin{aligned} g(t) &= 3t^5 - 26t^4 + 97t^3 - 198t^2 + 116t - 24 \\ &< -23t^4 + 97t^3 - 198t^2 + 116t - 24 \\ &= 23 \cdot t^2 \cdot t(1-t) + 74t^3 - 198t^2 + 116t - 24 \\ &\leq \frac{23}{4}t^2 + 74t^3 - 198t^2 + 116t - 24 \\ &\leq 74t^3 - 192t^2 + 116t - 24 \\ &< 4(19t^3 - 48t^2 + 29t - 6). \end{aligned}$$

The above inequalities hold since $3t^5 < 3t^4$, $t(1-t) \leq 1/4$, $23/4t^2 < 6t^2$ and $74t^3 < 76t^3$, respectively. Finally,

$$h(t) = 19t^3 - 48t^2 + 29t - 6 < 0$$

for all $t \in (0,1)$ since it can be checked that on the interval $(0,1)$ it attains its maximal value for $t = \frac{16}{19} - \frac{\sqrt{651}}{57} = 0.394\dots$ and that value is $-0.863\dots$. Therefore, $g(t) < 0$ and $\psi(t) < 0$ for all $t \in (0,1)$, i.e., $\varphi(x)$ is non-increasing for $0 < x < 1$. $\square$

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Faculty of Mathematical and Natural Sciences
University of Prishtina
M. Tereza N.N.10000 Prishtina, Kosovo
E-mail address: edmond.aliaga@uni-pr.edu

Faculty of Mechanical Engineering
SS. Cyril and Methodius University in Skopje
Karpoš 1000 Skopje, Republic of Macedonia
E-mail address: nikola.tuneski@mf.edu.mk