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A SURVEY AND STRENGTHENING OF ERDOS GYARFAS CONJECTURE

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ABSTRACT. Erdos Gyarfas conjecture States that every graph with minimum degree 3 contains a simple cycle whose length is a power of two i.e. in the power of 2^n. In this paper we shall prove that under what conditions cubic graphs do not possess a cycle of length 2^n.

1. INTRODUCTION

For the purpose of this paper, we assume that the reader is familiar with standard graph theoretic terminology. Unsolved problem in [7] states what has become known as Erdos-Gyarfas conjecture, made in 1995 by the prolific mathematician Paul Erdos and his collaborator Andras Gyarfas, states that every graph with minimum degree 3 contains a simple cycle whose length is a power of two it is one of many conjectures of Erdos. If the conjecture is false, a counter example would take the form of a graph with minimum degree three having no power-of-two cycles. It is known through computer searches of Gordon Royle and Klas Markstrom that any counterexample must have at least 17 vertices, and any cubic counterexample must have at least 30 vertices. Markstrom's searches found four graphs on 24 vertices in which the only power-of-two cycles have 16 vertices. One of these four graphs is planar; however, the Erdos-Gyarfas conjecture is now known to be true for the special case of 3-connected cubic planar graphs [2] (Heckman and Krakovski 2013.) Weaker results relating the degree of a graph to unavoidable sets of cycle lengths are known: there is a set S of lengths, with $|S| = \bigcirc(n^{0.99})$, such that every graph with average degree ten or more contains a cycle with its length in S [6] (Verstraete 2005), and every graph whose average degree is exponential in the iterated logarithm of n necessarily contains a cycle whose length is a power of two [5] (Sudakov and Verstraete 2008). The conjecture is also known to be true for planar claw-free graphs [1] (Daniel and Shauger 2001) and for graphs that avoid large induced stars and satisfy additional constraints on their degrees [4] (Shauger 1998). Markstrom Klas in (2004), [3] gives "Extremal graphs for some problems on cycles in graphs" .

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A graph is planar if it can be embedded in the plane without crossing edges. A plane graph is an embedded planar graph. A graph G is 3-connected if $|V(G)| \geq 4$ and there is no $S \subset V(G)$ such that $|S| < 3$ and $G§$ is disconnected (denotes deletion). A graph G is cubic if every vertex of G is of degree three. By computer searches, Markstrom [2] verified the conjecture for cubic graphs of order at most 29, and found that the smallest cubic planar graph with no 4- or 8-cycles has 24 Vertices (see Figure 1). Note that this graph contains a 16-cycle. Shauger [3] proved the Conjecture for $K_{1,m}$ -free graphs of minimum degree at least $m+1$ or maximum degree at Least $2m-1$. Every 3-connected cubic planar graph contains a $2m$-cycle, for some $2 \leq m \leq 7$, see [2].

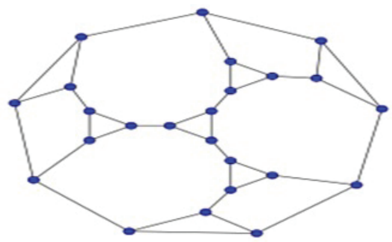


FIGURE 1

Markstrom 24-vertex cubic planar graph with no 4- or 8-cycles, found in a computer search for counter examples to the Erdos-Gyarfas conjecture. It has, however, cycles with 16 vertices.

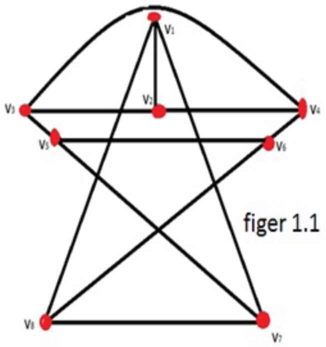
Now it follows our main result.

2. Main result

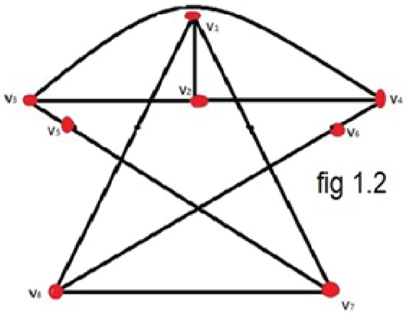
Theorem 2.1. *Let $G(V,E)$ be a cubic graph with vertex set V and edge set E by deleting a certain edge e from G in such a way that remaining graph $G-e$ is $n-2$ cubic without any cycle of 2^n. Let the two vertices v_i and v_j are of degree two if the distance between v_i and v_j is $\frac{n}{2}+1$ provided $\frac{n}{2}+1$ is odd, then we can construct a graph which is cubic but does not contain a cycle of length 2^n.*

Proof. Let G be a cubic graph by removing an edge of graph, the graph $G-e$ is $n-2$ cubic two of its vertices has degree two as every edge contributes two to the degree, let the vertex v_i and v_j be of degree two. And the resultant graph $G-e$ which is $n-2$ cubic has no cycle of length 2^n as shown in figure 1.1 and figure 1.2. Figure 1.1 shows a cubic graph of 8 vertices and contains a cycle of length 4 and 8, both the cycles are power of two i.e. length 2^n . But by removing an edge from vertex v_5 and v_6 graph $G-e$ becomes $n-2$ cubic and without any cycle of length 2^n as shown in figure 1.2 below. But the distance between the vertex v_5 and v_6 is not equal to $\frac{n}{2}+1$

Without loss of generality let G be a cubic graph by removing a certain edge in a graph the graph becomes $n-2$ cubic and without any cycle of length 2^n let the vertex v_i and v_j be two vertices whose degree is two if the distance between these vertices is $\frac{n}{2}+1$



this graph is cubic contains two cycles of lenth 4 and 8 both are power of two



this is n-2 cubic graph without any cycle of length power of two

provided $\dfrac{n}{2}+1$ is odd. Because every cubic graph must has even number of vertices that is n is even but $\dfrac{n}{2}+1$ may be even or odd, so we take only the case when $\dfrac{n}{2}+1$ is odd. In that case we can construct similar graphs and join these graphs by those vertices whose degree is two and every vertex is now degree three and graph is cubic but we have to join them in such a way that resultant graph has minimum length $2^{n-1}+1$ and maximum length 2^n-1 as shown in in figure 2.1 and figure 2.2 below.

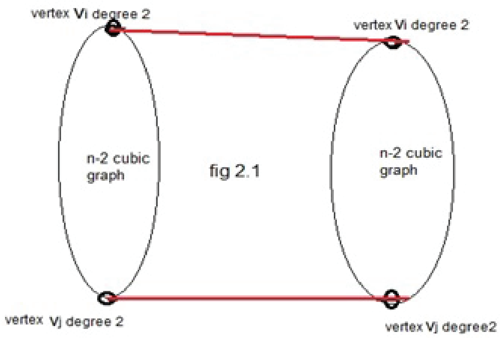
We know that sum of two are more than two equal odds is never power of two, even though sum of two odd numbers is even. Since $(2n+1)+(2n+1)+(2n+1)+\ldots\ n$ times

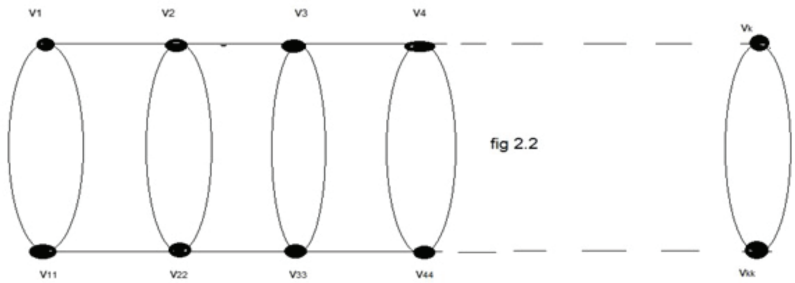
$$\sum_{i=1}^{n}(2n+1)=n(2n+1)\neq 2^n\,,$$

the length of cycle lies between $2^{n-1}+1\leq S_n\leq 2^n-1$, where (S_n) represents the number of cycles in the graph. There does not exist any cycle between

$$2^{n-1}+1\leq S_n\leq 2^n-1\,,$$

whose length is a power of two i.e. length is 2^n. That proves the result. □





Conclusion

From the above result we conclude that if such $n-2$ cubic graph exists we can simply construct a cubic graph without any cycle of length 2^n.

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