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INTEGRATED SEMIGROUPS AND ONCE INTEGRATED GROUP OF ROTATION IN THE COMPLEX PLANE

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ABSTRACT. We prove two theorems for α–times integrated semigroups $\left(\alpha \in \mathbb{R}^+\right)$, and analyze the geometric meaning of once integrated group of rotation in the complex plane.

1. INTRODUCTION

The semigroup theory of operators and integrated semigroups of operators in Banach space is very well developed and many mathematicians have been developing the theory. Hille and Phillips in [5] have proved several formulas, so called exponentially formulas, for strongly continuous semigroups. Also, Pazy in [10], Butzer and Berens in [3], and some other authors have proved those or similar formulas in different ways. Arendt in [1] introduces the notion of $1-$times (or once) integrated semigroup. Later, Kellermann in [6], Arendt in [2] and Neubrander in [9] have developed the theory of $n-$times integrated semigroups ($n \in \mathbb{N}$). The motivation for definition of $n-$times integrated semigroup was $n-$times successive integration of strongly continuous semigroup of operators. Hieber in [4] introduces $\alpha-$times integrated semigroups ($\alpha \in \mathbb{R}^+$). In [8] authors have obtained generalizations of some earlier results and some applications of $\alpha-$times integrated semigroups ($\alpha \in \mathbb{R}^+$). Generalizations of some exponential formulas for strongly continuous semigroups to integrated semigroups have been proven in [12], [13] and [15]. Maslov and Fedoryuk in [7] have obtained a result for strongly continuosly group of linear and bounded operators, using the method of stationary phase. Motivated by that, in [14] authors have obtained the certain result for once integrated group of linear and exponentially bounded operators in a Banach space. In this paper we give some other results for $\alpha-$times integrated semigroups ($\alpha \in \mathbb{R}^+$). Also, we analyze the geometric meaning of once integrated group of rotation in the complex plane.

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2. Results

Some important preliminaries and results from the theory of semigroups and integrated semigroups of operators in a Banach space one can find for example in [3, 5, 8, 9, 10, 11, 12]. Here we give and prove two theorems for integrated semigroups and later analyze once integrated group of rotation in the complex plane, and explain geometric meaning of integrated group of rotation.

Definition 2.1. *The integral of order α of the function $f(t)$ $(t \geq 0)$ $(\alpha \in \mathbb{R}^+)$, or $\alpha-$times integrated function $f(t)$, is the function $g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$, where Γ is the gamma function.*

Theorem 2.1. *The integral of order $1-\alpha$ of $\alpha-$times integrated function $f(t)$ $(t \geq 0)$ for $0 < \alpha < 1$ is once integrated function.*

Proof. The function $g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$ is $\alpha-$times integrated function $f(t)$. The integral of order $1-\alpha$ of $g(t)$ is equal

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{(1-\alpha)-1} g(s)ds = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} ds \int_0^s (s-u)^{\alpha-1} f(u)du.$$

Therefore, we need to prove that for $0 < \alpha < 1$ holds

$$\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} ds \int_0^s (s-u)^{\alpha-1} f(u)du = \int_0^t f(u)du \tag{2.1}$$

With that in mind we consider the integral

$$I = \int_0^t (t-s)^{-\alpha} ds \int_0^s (s-u)^{\alpha-1} f(u)du.$$

If we interchange the order of integration , then

$$I = \int_0^t f(u)du \int_u^t (t-s)^{-\alpha}(s-u)^{\alpha-1} ds = \int_0^t f(u) I_1 du,$$

where

$$I_1 = \int_u^t (t-s)^{-\alpha}(s-u)^{\alpha-1} ds.$$

The substitution $s-u=z$ gives

$$I_1 = \int_0^{t-u} (t-u-z)^{-\alpha} z^{\alpha-1} dz.$$

The substitution $\frac{z}{t-u} = w$ gives further

$$I_1 = \int_0^1 (1-w)^{-\alpha} w^{\alpha-1} dw = B(\alpha,\ 1-\alpha) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)},$$

where B is the beta function. Since $\Gamma(1) = 1$, then we have $I_1 = \Gamma(\alpha)\Gamma(1-\alpha)$ and $I = \Gamma(\alpha)\Gamma(1-\alpha) \int_0^t f(u)du$. Hence, (2.1) holds. □

Specially, if in Theorem 2.1 we put $f(t) = T(t)$, where $T(t)$ $(t \geq 0)$ is a strongly continuous semigroup of operators in a Banach space, then we obtain the next result.

Corollary 2.1. *The integral of order $1-\alpha$ of $\alpha-$times integrated strongly continuous semigroup $T(t)$ $(t \geq 0)$ for $0 < \alpha < 1$ is once integrated semigroup $S(t) = \int_0^t T(u)du$ $(t \geq 0)$.*

The motivation for the following investigation is a Hille-Phillips result for semigroups of operators of class (0, A) in a Banach space (Theorem 6.3.2. and Theorem 11.6.2. in [5]) and Theorem 3.3. in [12]. Hille and Phillips in [5] have proved the next theorem.

Theorem 2.2. [5; *Theorem* 11.6.2.] *Let $T(t)$ $(t \geq 0)$ be a semigroup of class (0, A) on a Banach space X such that $\|T(t)\| \leq Me^{\omega_0 t}$ for all $t \geq 0$ and for suitable constants $M \geq 1$ and $\omega_0 \geq 0$. If A is the infinitesimal generator of $T(t)$ $(t \geq 0)$, then*

$$T(t)x = (C,1) - \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} R(\lambda, A)x d\lambda,$$

for every $x \in X$, $t \geq 0$, $\gamma > \omega_0$. Here (C,1)-lim means Cesaro-1 limit.

Definition 2.2. [5, 12] *Let $f(\omega)$ be a function on $[0,\infty)$ with values in a complex Banach space X, such that for every $\lambda > 0$, $e^{-\lambda\omega} f(\omega) \in L([0,\infty), X)$ (the space of linear bounded functions from $[0,\infty)$ into X). Then, for $\beta > 0$, the Cesaro-β limit of the function $f(\omega)$ as $\omega \to \infty$ is defined*

$$(C,\beta) - \lim_{\omega\to\infty} f(\omega) := \lim_{T\to\infty} \frac{\beta}{T^\beta} \int_0^T (T-\omega)^{\beta-1} f(\omega) d\omega.$$

In [12] it is proved the next result for integrated semigroups.

Theorem 2.3. [12; *Theorem* 3.3.] *Let $S(t)$ $(t \geq 0)$ be an $\alpha-$times integrated exponentially bounded semigroup defined on a Banach space X $(\alpha \in \mathbb{R}^+)$, with generator A. Let $M \geq 0$ and $\omega_0 \in \mathbb{R}$ satisfy $\|S(t)\| \leq Me^{\omega_0 t}$ for all $t \geq 0$. Let $0 < \beta < 1$. If $\gamma > \max(\omega_0, 0)$, $x \in X$ and $t \geq 0$, then we have*

$$S(t)x = (C,\beta) - \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha} d\lambda,$$

and the limit is uniform in t on any bounded interval $[a,b] \subset [0,\infty)$.

Remark 2.1. *From Corollary 3.1. in* [12] *it follows that the assertion of Theorem 2.3 holds for every $\beta > 0$.*

In this paper we want to prove the assertion specially for $\beta = 1$ in a different way.

Theorem 2.4. *Let $S(t)$ $(t \geq 0)$ be an $\alpha-$times integrated exponentially bounded semigroup defined on a Banach space X $(\alpha \in \mathbb{R}^+)$, with generator A, and let $R(\lambda, A)$ be the resolvent of operator A. Let the constants $M \geq 1$ and $\omega_0 \geq 0$ satisfy $\|S(t)\| \leq Me^{\omega_0 t}$ for all $t \geq 0$. If $\gamma > \omega_0$, $x \in X$ and $t \geq 0$, then we have*

$$S(t)x = (C,1) - \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha} d\lambda.$$

Proof. First of all we want to prove that for every $x \in X$ it holds
(2.2)

$$(C,1) - \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha} d\lambda = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha} d\tau$$

Using Definition 2.2 for $\beta = 1$ and interchanging the order of integration we obtain

$$(C,1) - \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha} d\lambda = \lim_{T\to\infty} \frac{1}{T} \int_0^T d\omega \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha} d\lambda$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \int_0^T d\omega \int_{-\omega}^{\omega} e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha} d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \left[\int_{-T}^{0} e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha} d\tau \int_{-\tau}^{T} d\omega + \int_0^T e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha} d\tau \int_{\tau}^{T} d\omega \right]$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \left[\int_{-T}^{0} (T+\tau)\, e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha} d\tau + \int_0^T (T-\tau)\, e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha} d\tau \right]$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha} d\tau$$

Hence, (2.2) holds. It is known that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ it holds

$$\frac{R(\lambda, A)}{\lambda^\alpha} = \int_0^\infty e^{-\lambda t} S(t) dt \tag{2.3}$$

For that λ it is $\left\|\frac{R(\lambda,A)}{\lambda^\alpha}\right\| \leq \frac{M}{\operatorname{Re}\lambda - \omega_0}$. Then, for $\lambda = \gamma + i\tau$ it is $\left\|e^{\lambda t}\frac{R(\lambda,A)}{\lambda^\alpha}\right\| \leq e^{\gamma t}\frac{M}{\gamma-\omega_0}$. Hence, for every $\omega \in \mathbb{R}$ and for every $x \in X$, the integral $\int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\frac{R(\lambda,A)x}{\lambda^\alpha} d\lambda$ absolute converges. Let $\lambda = \gamma + i\tau$ and $\gamma > \omega_0$. Then from (2.3) we have

$$\frac{R(\gamma+i\tau,A)}{(\gamma+i\tau)^{\alpha}} = \int_0^{\infty} e^{-(\gamma+i\tau)t} S(t)dt \tag{2.4}$$

For any fixed $x \in X$ and $t \geq 0$ consider the function

$$S_1(t,T)x = \frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)e^{(\gamma+i\tau)t}\frac{R(\gamma+i\tau,A)x}{(\gamma+i\tau)^{\alpha}}d\tau \tag{2.5}$$

Now we use (2.4) and later interchange the order of integration, and obtain

$$S_1(t,T)x = \frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)e^{(\gamma+i\tau)t}d\tau\int_0^{\infty}e^{-(\gamma+i\tau)u}S(u)xdu \tag{2.6}$$

$$= \frac{1}{2\pi}\int_0^{\infty}e^{\gamma(t-u)}S(u)xdu\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)e^{i\tau(t-u)}d\tau$$

The interior integral equals

$$\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)e^{i\tau(t-u)}d\tau = \int_{-T}^{T}e^{i\tau(t-u)}d\tau - \int_0^{T}\frac{\tau}{T}e^{i\tau(t-u)}d\tau - \int_{-T}^{0}\frac{-\tau}{T}e^{i\tau(t-u)}d\tau$$

$$= \frac{e^{i\tau(t-u)}}{i(t-u)}\Big|_{-T}^{T} - \int_0^{T}\frac{2\tau}{T}\cos\tau(t-u)d\tau = \frac{2}{T}\frac{1-\cos T(t-u)}{(t-u)^2}$$

Therefore, from (2.6) we have

$$S_1(t,T)x = \frac{1}{2\pi}\int_0^{\infty}e^{\gamma(t-u)}\frac{2}{T}\frac{1-\cos T(t-u)}{(t-u)^2}S(u)xdu$$

$$= e^{\gamma t}\int_0^{\infty}\frac{2}{\pi T}\frac{\sin^2\frac{T(t-u)}{2}}{(t-u)^2}e^{-\gamma u}S(u)xdu.$$

Put $h(u) = e^{-\gamma u}S(u)x$ for $u \geq 0$ and $h(u) = 0$ for $u < 0$. Then, for $\gamma > \omega_0$, $h(u) \in L\left((-\infty,\infty),X\right)$, and for Fejer's kernel $F(\beta,T) = \frac{2}{\pi T}\frac{\sin^2\frac{\beta T}{2}}{\beta^2}$ it holds

$$S_1(t,T)x = e^{\gamma t}\int_0^{\infty}F(t-u,T)\,h(u)du \tag{2.7}$$

From theorem 6.3.2. in [5] we see that the Fejer's kernel satisfies the necessary conditions such that

$$\lim_{T\to\infty}\int_0^{\infty}F(t-u,T)\,h(u)du = h(t) \tag{2.8}$$

Using (2.7), (2.8) and $h(t)=e^{-\gamma t}S(t)x$ for $t\geq 0$, we obtain

$$\lim_{T\to\infty} S_1(t,T)x = S(t)x \quad (t\geq 0) \tag{2.9}$$

On the other hand, using (2.2) and (2.5) it follows

$$\lim_{T\to\infty} S_1(t,T)x = (C,1)-\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\frac{R(\lambda,A)x}{\lambda^{\alpha}}d\lambda \qquad (t\geq 0) \tag{2.10}$$

From (2.9) and (2.10) we conclude that for every $x\in X$ and all $t\geq 0$ it holds

$$S(t)x = (C,1)-\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\frac{R(\lambda,A)x}{\lambda^{\alpha}}d\lambda.$$

□

Now we want to consider the geometric meaning of once integrated group of rotation in the complex plane. It is known that the field of all complex number $\mathbb{C}$ is itself a Banach space with norm equals to the absolute value (modulus). The strongly continuous group of rotation $T(t)$ $(t\in\mathbb{R})$ about the origin in the complex plane is given with $T(t)z=e^{it}z$ $(z\in\mathbb{C})$, where $t\in\mathbb{R}$ is the angle of rotation and i is the imaginary unit. This family of operators is the group of operators because it is obviously $T(0)=I$, I is the identity operator on $\mathbb{C}$, and $T(t+s)=T(t)T(s)$ for all t, $s\in\mathbb{R}$. The infinitesimal generator of the group $T(t)$ $(t\in\mathbb{R})$ is the operator $A=i\cdot I$ because

$$Az := \lim_{t\to 0^+}\frac{T(t)z-z}{t} = \lim_{t\to 0^+}\frac{\left(e^{it}-1\right)z}{t} = i\cdot z.$$

If we define the family of operators $S(t)$ $(t\in\mathbb{R})$ with

$$S(t)z=\int_0^t T(s)z\,ds = z\int_0^t e^{is}ds \quad (t\in\mathbb{R},\ z\in\mathbb{C}\ ),$$

then this family of operators is once integrated group of linear and bounded operators on $\mathbb{C}$, and hold the properties

$$S(0)=0 \text{ and } S(r)S(t)=\int_0^{r+t} S(u)du-\int_0^{r} S(u)du-\int_0^{t} S(u)du \quad (r,\ t\in\mathbb{R}).$$

We have for every $t\in\mathbb{R}$ and all $z\in\mathbb{C}$,

$$\begin{aligned} S(t)z &= \frac{e^{it}-1}{i}z = [\sin t + i(1-\cos t)]\,z = iz - i(\cos t + i\sin t)\,z = i\cdot[z-T(t)z] \\ &= e^{i\frac{\pi}{2}}\cdot[z-T(t)z] = T\left(\frac{\pi}{2}\right)[z-T(t)z] = T\left(\frac{\pi}{2}\right)z - T\left(\frac{\pi}{2}+t\right)z, \end{aligned}$$

i.e.

$$S(t)=T\left(\frac{\pi}{2}\right)-T\left(\frac{\pi}{2}+t\right) \quad (t\in\mathbb{R}) \tag{2.11}$$

The relation (2.11) shows the connection between the group of rotation and once integrated group of rotation in the complex plane. It means that the point $S(t)z\in\mathbb{C}$ we obtain by rotating the point $z-T(t)z$ about the origin by the angle of $\frac{\pi}{2}$. If the points in the complex plane we identify with their radius vectors, then the vector $S(t)z$ we obtain as a difference of vectors $T\left(\frac{\pi}{2}\right)z$ and $T\left(\frac{\pi}{2}+t\right)z$, and the vector $S(t)z$ is perpendicular to $z-T(t)z$ for every $t\neq 0$ and every $z\neq 0$. Because of

$$|S(t)z| = |[\sin t + i\,(1-\cos t)]\,z| = (2-2\cos t)\cdot |z| = |z| \Longleftrightarrow \cos t = \frac{1}{2},$$

we conclude that the operator $S(t)$ is an isometry on $\mathbb{C}$ only for $t=\pm\frac{\pi}{3}+2k\pi\ (k\in\mathbb{Z})$.

The semigroup theory and the theory of integrated semigroups and integrated groups of linear bouded operators in a Banach space has many respectable applications in functional analysis, algebra and geometry. Particularly important applications of this theory are in solving of some types of ordinary and partial differential equations. Every obtained result in this paper, and in general, has a special interpretation in the theory of the functions of real or complex variable. We think that the further investigation of integrated groups of rotations in the complex plane can give some other interesting relations and meanings in geometry.

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