ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **5** (2016), no.2, 131–142
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

INCLUSION RELATIONS FOR A CERTAIN SUBCLASS OF STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER BASED ON DZIOK-RAINA OPERATOR

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ABSTRACT. The main object of this paper is to establish a set of inclusion relations for a certain subclass of analytic functions of complex order, by making use of a linear operator. Special cases of these inclusion relations are discussed.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}), \tag{1.1}$$

normalized by $f(0) = 0 = f'(0) - 1$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$. The class of starlike functions $\mathcal{S}^*$ and convex functions $\mathcal{C}$ are well known subclasses of $\mathcal{S}$.

In 1993, Goodman [12, 13] introduced the concept of uniform convexity and uniform starlikeness of functions in $\mathcal{A}$. The classes consisting of uniformly convex and uniformly starlike functions are denoted by $\mathcal{UCV}$ and $\mathcal{UST}$ respectively. Further, Rønning [23] introduced the class $\mathcal{S}_P$, the class of parabolic starlike functions.

Two interesting subclasses of $\mathcal{S}$, denoted by k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ consisting, respectively, of functions which are k - uniformly convex and k-uniformly starlike in $\mathbb{U}$, were studied by Kanas and Wisniowska [15, 16] whose analytic characterizations are as follows:

$$k - \mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, 0 \le k < \infty \quad (z \in \mathbb{U}) \right\} \text{ and}$$

$$k - \mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, 0 \le k < \infty \quad (z \in \mathbb{U}) \right\}.$$

We note that 1 - $\mathcal{UCV}$ = $\mathcal{UCV}$ and 1 - $\mathcal{ST}$ =$\mathcal{S}_P$.

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2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions,Univalent functions, Hadamard product, Wright's generalized hypergeometric functions, k-uniformly convex functions, parabolic starlike functions.

For functions f of the form (1.1), if $f \in k - \mathcal{UCV}$, then the following holds true (cf. [15]):

$$|a_n| \leq \frac{(P_1)_{n-1}}{n!}, \quad n \in \mathbb{N}\backslash\{1\}, \tag{1.2}$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & k=0 \\ a(a+1)(a+2)\ldots(a+k-1) & k=\mathbb{N} \end{cases}$$

and $P_1 = P_1(k)$ is the coefficient of z in the function

$$p_k(z) = 1 + \sum_{n=1}^{\infty} P_n(k) z^n \tag{1.3}$$

which is the extremal function for the class $\mathcal{P}(p_k)$ related to the class k-$\mathcal{UCV}$ by the range of the expression $1 + \frac{zf''(z)}{f'(z)}$ $(z \in \mathbb{U})$.
Similarly, if $f \in \mathcal{A}$ of the form (1.1) belongs to the class k-$\mathcal{ST}$,
then (cf. [16])

$$|a_n| \leq \frac{(P_1)_{n-1}}{(n-1)!}, \quad n \in \mathbb{N}\backslash\{1\} \tag{1.4}$$

where $P_1 = P_1(k)$ is as above, by (1.3).
A function $f \in \mathcal{A}$ is said to be in the class $\Re^{\tau}(A,B)$, $(\tau \in \mathbb{C}\backslash\{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left| \frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]} \right| < 1 \quad (z \in \mathbb{U}).$$

This class $\Re^{\tau}(A,B)$ was introduced earlier by Dixit and Pal [5].
The subclass $\mathcal{T} \subset \mathcal{A}$ consisting of univalent functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$

was studied by Silverman [25].
A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(b)$ if it satisfies the following inequality:

$$\Re\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] > 0, \quad (z \in \mathbb{U}, b \in \mathbb{C}\backslash\{0\}).$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}(b)$ if it satisfies the following inequality:

$$\Re\left[1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right] > 0, \quad (z \in \mathbb{U}, b \in \mathbb{C}\backslash\{0\}).$$

The function classes $\mathcal{S}^*(b)$ and $\mathcal{C}(b)$ were considered earlier by Nasr and Aouf ([18], [19], [20]) and Wiatrowski [32] repectively (see also [6], [17], [29].)

For $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$ and $k \geq 0$, let $\mathcal{U}(k,\alpha,\beta)$ be a subclass of $\mathcal{A}$ consisting of functions of the form (1.1) that satisfy the condition

$$\Re\left(\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)}\right) \geq k \left| \frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} - 1 \right| + \beta\,.$$

This class was studied by Aqlan et al. [3].

For $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $b \in \mathbb{C}\backslash\{0\}$ and $z \in \mathbb{U}$, let $P(\lambda, b)$ be a subclass of $\mathcal{A}$ consisting of functions of the form (1.1) that satisfy the condition

$$\Re\left(1+\frac{1}{b}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}-1\right)\right)>0.$$

This class was considered by Aouf [2].

For $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $b \in \mathbb{C}\backslash\{0\}$ and $z \in \mathbb{U}$, let $SC(b, \lambda, \beta)$ be a subclass of $\mathcal{A}$ consisting of functions of the form (1.1) that satisfy the condition

$$\Re\left(1+\frac{1}{b}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}-1\right)\right)>\beta.$$

or which satisfy the following inequality:

$$\left|\frac{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}-1}{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}-1+2b(1-\beta)}\right|<1. \tag{1.5}$$

The class $SC(b, \lambda, \beta)$ was considered by Altintas et al. [1].
Clearly, we have the following relationships:

$$SC(b,0,0) \equiv \mathcal{S}^*(b)\ , SC(b,1,0) \equiv \mathcal{C}(b) \text{ and } SC(1,\lambda,\beta) \equiv \mathcal{U}(0,\alpha,\beta).$$

The Dziok-Raina operator $W_q^p[\alpha_1]$ was introduced by Dziok and Raina in [9], motivated by the Wright's generalized hypergeometric function as below.

For $\alpha_i \in \mathbb{C}(\frac{\alpha_i}{A_i} \neq 0, -1, -2, ..., A_i > 0; i = 1, 2, ..., p)$ and $\beta_i \in \mathbb{C}(\frac{\beta_i}{B_i} \neq 0, -1, -2, ..., B_i > 0; i = 1, 2, ..., q)$ such that $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$, Wright's generalized hypergeometric function ${}_p\psi_q(z)$ ([33],[28]) is defined by

$$ {}_p\psi_q[z] = {}_p\psi_q[(\alpha_i, A_i)_{1,p}, (\beta_i, B_i)_{1,q}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+nA_i)}{\prod_{i=1}^{q}\Gamma(\beta_i+nB_i)}\frac{z^n}{n!} \tag{1.6}$$

which is analytic for bounded values of $|z|$. In particular, if $A_i = 1$, $B_i = 1$, ${}_p\psi_q[z]$ reduces to the the generalized hypergeometric function ${}_pF_q[z]$ given by

$$ {}_pF_q[z] = {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\beta_i)_n}\frac{z^n}{n!}, \quad p \leq 1+q.$$

In view of (1.6), the Dziok-Raina operator [9](see also [8], [10], [21], [22], [26])

$$W_q^p[\alpha_1] = W_q^p[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}] : \mathcal{S} \to \mathcal{S}$$

is defined by

$$W_q^p[\alpha_1]f(z) = z\frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\Big({}_p\psi_q[(\alpha_i, A_i)_{1,p}, (\beta_i, B_i)_{1,q}; z]\Big) * f(z),$$

where * denotes convolution (Hadamard product) of two functions.
For $f(z)$ of the form (1.1), we have

$$W_q^p[\alpha_1]f(z) = z + \sum_{n=2}^{\infty} a_n \Omega_n z^n, \ z \in \mathbb{U}, \tag{1.7}$$

where

$$\Omega_n = \frac{\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+(n-1)A_i)}{\Gamma(\alpha_i)}}{\frac{\prod_{i=1}^{q}\Gamma(\beta_i+(n-1)B_i)}{\Gamma(\alpha_i)}} \frac{1}{(n-1)!}, \ n \geq 2. \tag{1.8}$$

Taking $A_i = 1 (i = 1, 2, ..., p)$ and $B_i = 1 (i = 1, 2, ..., q)$ the linear operator $W_q^p[\alpha_1]$ given by (1.5) reduces to the Dziok -Srivastava operator $H_q^p[\alpha_1]$ [7], which inturn contains many other operators as special cases, such as the Hohlov operator [14], the Carlson - Shaffer operator [4], the Ruscheweyh derivative operator [24] denoted by I, L and D respectively as detailed below.

$$I(\alpha_1, \alpha_2; \beta_1)f(z) = H_1^2(\alpha_1, \alpha_2; \beta_1)f(z) \tag{1.9}$$

$$L(\alpha_1, ; \beta_1)f(z) = H_1^2(\alpha_1, 1; \beta_1)f(z) \tag{1.10}$$

$$D^\lambda f(z) = H_1^2(\lambda + 1, 1; 1)f(z)\,. \tag{1.11}$$

Motivated by the works of Srivastava et al. [30], Gangadharan et al. [11], Sivasubramanian et al. [27], Sudharsan et al. [31], in this paper by making use of the linear operator defined by (1.7), we establish a number of relations between the classes k-$\mathcal{UCV}$, k-$\mathcal{ST}$, $\Re^\tau(A, B)$ and $SC(b, \lambda, \beta)$.
In order to prove the main results, we need the following lemmas.

Lemma 1. **(Aouf** [2]**)** *Let the function* $f(z)$ *be defined by* (1.1). *If*

$$\sum_{n=2}^{\infty}(1 + \lambda(n-1))[(n-1) + |2b + n - 1|]|a_n| \leq 2|b|, \quad (\lambda \geq 0; b \in \mathbb{C}\backslash\{0\}),$$

then $f(z) \in P(\lambda, b)$.

Lemma 2. *Let the function* $f(z)$ *be defined by* (1.1). *If*

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[(n-1)+|2b(1-\beta)+n-1|]|a_n| \leq 2|b|(1-\beta), \quad (\lambda \geq 0; b \in \mathbb{C}\backslash\{0\}), \tag{1.12}$$

then $f(z) \in SC(b, \lambda, \beta)$.

Proof. For $\beta = 0$ the above lemma was proved by Aouf [2] and hence the details for this class $SC(b, \lambda, \beta)$ is omitted. □

Lemma 3. **(Dixit and Pal**[5]**)** *If a function* $f \in \Re^\tau(A, B)$ *is of form* (1.1), *then*

$$|a_n| \leq (A - B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{1.13}$$

The result is sharp.

2. Main Results

Theorem 1. *Suppose that* $\alpha_i \in \mathbb{C}\backslash\{0\}(i = 1, ..., p)$, $\Re(\beta_i) > 0(i = 1, ..., q)$ *and that* $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i| + p - q$. *If* $f \in \Re^{\tau}(A, B)$ *of the form* (1.1), *and let the inequality*

$$\begin{aligned}
&(1-\lambda+\lambda|b|(1-\beta))\Big[{}_p\psi_q((|\alpha_i|, A_i)_{1,p}, (\Re(\beta_i), B_i)_{1,q}; 1)\Big] \\
&\qquad + \lambda\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\Big[{}_p\psi_q((|\alpha_i|+1, A_i)_{1,p}, (\Re(\beta_i)+1, B_i)_{1,q}; 1)\Big] \\
&\qquad - \frac{(1-|b|(1-\beta))(1-\lambda)\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[{}_p\psi_q((|\alpha_i|-1, A_i)_{1,p}, (\Re(\beta_i)-1, B_i)_{1,q}; 1)\Big] \\
&\le |b|(1-\beta)\frac{1}{(A-B)|\tau|} + (1-\lambda+\lambda|b|(1-\beta)) \\
&\qquad - (1-|b|(1-\beta))(1-\lambda)\frac{\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[1 + \frac{(|\alpha_1|-1)_{A_1}...(|\alpha_p|-1)_{A_p}}{(\Re(\beta_1)-1)_{B_1}...(\Re(\beta_q)-1)_{B_q}}\Big]
\end{aligned} \tag{2.1}$$

hold. Then $W_q^p[|\alpha_1|](f(z)) \in SC(b, \lambda, \beta)$.

Proof. Let f of the form (1.1) belong to the class $\Re^{\tau}(A, B)$. In view of Lemma 2 and (1.7), it suffices to show that

$$\sum_{n=2}^{\infty}(1+\lambda(n-1)[(n-1)+|2b(1-\beta)+n-1|]\,|\Omega_n a_n| \le 2|b|(1-\beta)\,,$$

where the coefficients $\Omega_n (n \in \mathbb{N}\backslash\{0\})$ are given by the equation (1.8). Using (1.13) and the relation $|(a)_{n-1}| \le (|a|)_{n-1}$, we deduce that

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(n-1)(1-\lambda+\lambda|b|(1-\beta)) + \lambda n(n-1) + |b|(1-\beta)] \\
&\qquad \times \left|\frac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{(\beta_1)_{B_1(n-1)}...(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}a_n\right| \\
&\le (A-B)|\tau|\Big[(1-\lambda+\lambda|b|(1-\beta))\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!} \\
&\qquad + \lambda\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-2)!} \\
&\qquad - (1-|b|(1-\beta))(1-\lambda)\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n)!}\Big]
\end{aligned}$$

$$= (A-B)|\tau|\Big[(1-\lambda+\lambda|b|(1-\beta))[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},(\Re(\beta_i),B_i)_{1,q};1)-1]$$
$$+\lambda\frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)}[{}_p\psi_q((|\alpha_i|+1,A_i)_{1,p},(\Re(\beta_i)+1,B_i)_{1,q};1)]$$
$$-(1-\lambda)(1-|b|(1-\beta))\frac{\Re(\beta_1)...\Re(\beta_q)}{(|\alpha_1|)...(|\alpha_p|)}[{}_p\psi_q((|\alpha_i|-1,A_i)_{1,p},(\Re(\beta_i)-1,B_i)_{1,q};1)$$
$$-1-\frac{(|\alpha_1-1|)_{A_1}...(|\alpha_p|-1)_{A_p}}{\Re(\beta_1-1)_{B_1}...\Re(\beta_q-1)_{B_q}}]\Big]$$
$$\leq |b|(1-\beta).$$

This completes the proof of Theorem 1 by virtue of (2.1). □

With $A_i = 1$, $B_i = 1$ we have,

Corollary 1. *Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i) > 0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i|+p-q$. If $f\in\Re^{\tau}(A,B)$ of the form (1.1) and let the inequality*

$$(1-\lambda+\lambda|b|(1-\beta))\Big[{}_pF_q(|\alpha_1|,...,|\alpha_p|,\Re(\beta_1),...,\Re(\beta_q);1)\Big]$$
$$+\lambda\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\Big[{}_pF_q(|\alpha_1|+1,...,|\alpha_p|+1,\Re(\beta_1)+1,...,\Re(\beta_q)+1;1)\Big]$$
$$-\frac{(1-\lambda)(1-|b|(1-\beta))\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[{}_pF_q(|\alpha_1|-1,...,|\alpha_p|-1,\Re(\beta_1)-1,...,\Re(\beta_q)-1;1)\Big]$$
$$\leq (1-\beta)|b|\frac{1}{(A-B)|\tau|}+(1-\lambda+\lambda|b|(1-\beta))$$
$$-(1-\lambda)(1-|b|(1-\beta))\frac{\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[1+\frac{(|\alpha_1|-1)...(|\alpha_p|-1)}{(\Re(\beta_1)-1)...(\Re(\beta_q)-1}\Big]$$

hold. Then ${}_pF_q(f(z))\in SC(b,\lambda,\beta)$.

Theorem 2. *Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i) > 0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i|+p-q$. If $f\in\mathcal{S}$ of the form (1.1), and let the inequality*

$$\lambda\frac{|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1)+1...\Re(\beta_q)+1}\frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{\Re(\beta_1)+2...\Re(\beta_q)+2}$$
$$\times\Big[{}_p\psi_q((|\alpha_i|+3,A_i)_{1,p},(\Re(\beta_i)+3,B_i)_{1,q};1)\Big]$$
$$+[1+4\lambda+\lambda|b|(1-\beta)]\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1)+1...\Re(\beta_q)+1}$$
$$\times\Big[{}_p\psi_q((|\alpha_i|+2,A_i)_{1,p},(\Re(\beta_i+2),B_i)_{1,q};1)\Big]$$
$$+[2(\lambda+1)+|b|(1-\beta)(2\lambda+1)]\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}$$
$$\times\Big[{}_p\psi_q((|\alpha_i|+1,A_i)_{1,p},(\Re(\beta_i)+1,B_i)_{1,q};1)\Big]$$
$$+|b|(1-\beta)\big[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},(\Re(\beta_i),B_i)_{1,q};1)]\leq 2|b|(1-\beta) \tag{2.2}$$

hold. Then $W_q^p[|\alpha_1|](f(z))\in SC(b,\lambda,\beta)$.

Proof. Let $f \in \mathcal{S}$ be of the form (1.1). By virtue of the de Branges theorem it suffices to show that

$$\begin{aligned} S_1 := \textstyle\sum_{n=2}^{\infty} n[(1+\lambda(n-1)[(n-1)+|2b(1-\beta)+n-1|]] \\ \times \left| \tfrac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{(\beta_1)_{B_1(n-1)}...(\beta_q)_{B_q(n-1)}} \tfrac{1}{(n-1)!} \right| \\ \leq 2|b|(1-\beta). \end{aligned}$$

Using the inequality $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$\begin{aligned} S_1 \leq \sum_{n=2}^{\infty} [n^3\lambda + n^2[1+\lambda|b|(1-\beta)-2\lambda] - n(1-\lambda)(1-|b|(1-\beta)] \\ \times \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-1)!}. \end{aligned}$$

Writing $n^3 = (n-1)(n-2)(n-3) + 6(n-1)(n-2) + 7(n-1) + 1$,
$n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$,
the above inequality can be written as

$$\begin{aligned} S_1 \leq\; & \lambda \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-4)!} \\ & + [1+4\lambda+\lambda|b|(1-\beta)] \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-3)!} \\ & + [2(\lambda+1)+|b|(1-\beta)(2\lambda+1)] \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-2)!} \\ & + |b|(1-\beta) \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-1)!} \\ \leq\; & \lambda \frac{(|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)} \frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{\Re(\beta_1+2)...\Re(\beta_q+2)} \\ & \times \sum_{n=4}^{\infty} \frac{(|\alpha_1|+3)_{A_1(n-4)}...(|\alpha_p|+3)_{A_p(n-4)}}{\Re(\beta_1+3)_{B_1(n-4)}...\Re(\beta_q+3)_{B_q(n-4)}} \frac{1}{(n-4)!} \\ & + [1+4\lambda+\lambda|b|(1-\beta)] \frac{(|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)} \\ & \times \sum_{n=3}^{\infty} \frac{(|\alpha_1|+2)_{A_1(n-3)}...(|\alpha_p|+2)_{A_p(n-3)}}{\Re(\beta_1+2)_{B_1(n-3)}...\Re(\beta_q+2)_{B_q(n-3)}} \frac{1}{(n-3)!} \\ & + [2(\lambda+1)+|b|(1-\beta)(2\lambda+1)] \frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \\ & \times \sum_{n=2}^{\infty} \frac{(|\alpha_1|+1)_{A_1(n-2)}...(|\alpha_p|+1)_{A_p(n-2)}}{\Re(\beta_1+1)_{B_1(n-2)}...\Re(\beta_q+1)_{B_q(n-2)}} \frac{1}{(n-2)!} \\ & + |b|(1-\beta) \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-1)!} \end{aligned}$$

$$
\begin{aligned}
&= \lambda \frac{(|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)} \frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{\Re(\beta_1+2)...\Re(\beta_q+2)} \\
&\qquad \times \left[{}_p\psi_q((|\alpha_i|+3, A_i)_{1,p}, (\Re(\beta_i+3), B_i)_{1,q}; 1) \right] \\
&+ [1+4\lambda+\lambda|b|(1-\beta)] \frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)} \\
&\qquad \times \left[{}_p\psi_q((|\alpha_i|+2, A_i)_{1,p}, (\Re(\beta_i+2), B_i)_{1,q}; 1) \right] \\
&+ [2(\lambda+1)+|b|(1-\beta)(2\lambda+1)] \frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \\
&\qquad \times \left[{}_p\psi_q((|\alpha_i|+1, A_i)_{1,p}, (\Re(\beta_i+1), B_i)_{1,q}; 1) \right] \\
&+ |b|(1-\beta) \left[{}_p\psi_q((|\alpha_i|, A_i)_{1,p}, (\Re(\beta_i), B_i)_{1,q}; 1) - 1 \right] \leq 2|b|(1-\beta),
\end{aligned}
$$

by using the inequality (2.2). □

With $A_i = 1$, $B_i = 1$ we have,

Corollary 2. *Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i = 1, ..., p)$, $\Re(\beta_i) > 0(i = 1, ..., q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p} |\alpha_i| + p - q$. If $f \in \mathcal{S}$ of the form (1.1), and let the inequality*

$$
\begin{aligned}
&\lambda \frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1)+1...\Re(\beta_q)+1} \frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{\Re(\beta_1)+2...\Re(\beta_q)+2} \\
&\quad \times \left[{}_pF_q((|\alpha_1|+3, ..., |\alpha_p|+3, \Re(\beta_1)+3, ..., \Re(\beta_1)+3, ; 1) \right] \\
&+ [1+4\lambda+\lambda|b|(1-\beta)] \frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1)+1...\Re(\beta_q)+1} \\
&\quad \times \left[{}_pF_q((|\alpha_1|+2, ..., |\alpha_p|+2, \Re(\beta_1)+2, ..., \Re(\beta_q)+2; 1) \right] \\
&+ [2(\lambda+1)+|b|(1-\beta)(2\lambda+1)] \frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)} \\
&\quad \times \left[{}_pF_q(|\alpha_1|+1, ..., |\alpha_p|+1, \Re(\beta_1)+1, ..., \Re(\beta_q)+1; 1) \right] \\
&+ |b|(1-\beta) \left[{}_pF_q((|\alpha_1|, ..., |\alpha_p|, \Re(\beta_1), ..., \Re(\beta_q); 1) \right] \leq 2|b|(1-\beta)
\end{aligned}
$$

hold. Then ${}_pF_q(f(z)) \in SC(b, \lambda, \beta)$.

Theorem 3. *Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i = 1, ..., p)$, $\Re(\beta_i) > 0(i = 1, ..., q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p} |\alpha_i| + P_1$, where $P_1 = P_1(k)$ is given by (1.3). If $f \in k\text{-}\mathcal{UCV}$ of the form (1.1), for some k $(0 \leq k < \infty)$ and let the inequality*

$$
\begin{aligned}
&(1-\lambda+\lambda|b|(1-\beta)) \left[{}_p\psi_q((|\alpha_i|, A_i)_{1,p}, P_1; (\Re(\beta_i), B_i)_{1,q}; 1) \right] \\
&+ \lambda \frac{|\alpha_1|...|\alpha_p|(P_1(k))}{\Re(\beta_1)...\Re(\beta_q)} \left[{}_p\psi_q((|\alpha_i|+1, A_i)_{1,p}, P_1; (\Re(\beta_i+1), B_i)_{1,q}, 2; 1) \right] \\
&\qquad - (1-|b|(1-\beta))(1-\lambda) \left[{}_p\psi_q((|\alpha_i|, A_i)_{1,p}, P_1; (\Re(\beta_i), B_i)_{1,q}, 2; 1) \right] \\
&\leq 2|b|(1-\beta)
\end{aligned}
$$

hold. Then $W_q^p[|\alpha_1|](f(z)) \in SC(b, \lambda, \beta)$.

Proof. Let f of the form (1.1) belong k-$\mathcal{UCV}$. By virtue of (1.12) and (1.2), it is enough to prove that

$$\sum_{n=2}^{\infty}[(1+\lambda(n-1)[(n-1)+|2b(1-\beta)+n-1|]] \\ \times\left|\frac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{n!}\right|\leq 1-\beta.$$

Using $|(a)_{n-1}|\leq(|a|)_{n-1}$, the proof is complete as in Theorem 1. □

With $A_i=1$, $B_i=1$ we have,

Corollary 3. *Suppose that $\alpha_i\in\mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i)>0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i))>\sum_{i=1}^{p}|\alpha_i|+P_1$, where $P_1=P_1(k)$ is given by* (1.3)*. If $f\in k$-$\mathcal{UCV}$ of the form* (1.1)*, for some k $(0\leq k<\infty)$ and let the inequality*

$$(1-\lambda+\lambda|b|(1-\beta))[{}_pF_q((|\alpha_1|,...,|\alpha_p|,P_1;\Re(\beta_1),...,\Re(\beta_q);1)] \\ +\lambda\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}[{}_pF_q(|\alpha_1|+1,...,|\alpha_p|+1,P_1;\Re(\beta_1)+1,...,\Re(\beta_q)+1,2;1)] \\ -(1-|b|(1-\beta))(1-\lambda)[{}_pF_q(|\alpha_1|,...,|\alpha_p|,P_1;\Re(\beta_1),...,\Re(\beta_1),2;1)] \\ \leq 2|b|(1-\beta)$$

hold. Then ${}_pF_q(f(z))\in SC(b,\lambda,\beta)$.

Theorem 4. *Suppose that $\alpha_i\in\mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i)>0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i))>\sum_{i=1}^{p}|\alpha_i|+P_1+1$, where $P_1=P_1(k)$ is given by* (1.3)*. If $f\in k$-$\mathcal{ST}$ of the form* (1.1) *for some k $(0\leq k<\infty)$ and let the inequality*

$$\frac{\lambda}{2}\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)(P_1+1)}{\Re(\beta_1)+1...\Re(\beta_q)+1} \\ \times[{}_p\psi_q((|\alpha_i|+2,A_i)_{1,p},P_1+2;(\Re(\beta_i)+2,B_i)_{1,q}3;1)] \\ +[(1+\lambda+\lambda|b|(1-\beta))]\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)} \\ \times[{}_p\psi_q(|\alpha_i|+1,A_i)_{1,p},P_1+1;(\Re(\beta_i)+1,B_i)_{1,q},2;1] \\ +|b|(1-\beta)[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},P_1;(\Re(\beta_i),B_i)_{1,q};1)] \\ \leq 2|b|(1-\beta)$$

hold. Then $W_q^p[|\alpha_1|](f(z))\in SC(b,\lambda,\beta)$.

Proof. Let $f \in k-\mathcal{ST}$ be of the form (1.1). Applying the estimates for the coefficients given by (1.4), and using $|(a)_{n-1}| \leq (|a|)_{n-1}$, we get

$$\sum_{n=2}^{\infty}[(1+\lambda(n-1)[(n-1)+|2b(1-\beta)+n-1|]]$$
$$\times\left|\frac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{(n-1)!}\right|$$
$$\leq\sum_{n=2}^{\infty}[(1+\lambda(n-1)[(n-1)+|2b(1-\beta)+n-1|]]$$
$$\times\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{(n-1)!}$$
$$=\sum_{n=2}^{\infty}(\lambda n^2+n[1-2\lambda+\lambda|b|(1-\beta)]-(1-|b|(1-\beta))(1-\lambda))$$
$$\times\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{(n-1)!}.$$

Rewriting $n^2=(n-1)(n-2)+3(n-1)+1$, $n=(n-1)+1$, and proceeding as in Theorem 2, we get the required result. □

With $A_i = 1$, $B_i = 1$ we have,

Corollary 4. *Suppose that* $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i)>0(i=1,...,q)$ *and that* $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i|+P_1+1$, *where* $P_1=P_1(k)$ *is given by* (1.3). *If* $f\in k\text{-}\mathcal{ST}$ *of the form* (1.1) *for some* k $(0\leq k<\infty)$ *and let the inequality*

$$\frac{\lambda}{2}\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)(P_1+1)}{\Re(\beta_1)+1...\Re(\beta_q)+1}$$
$$\times\left[{}_pF_q((|\alpha_1|+2,...,|\alpha_p|+2,P_1+2;\Re(\beta_1)+2,...,\Re(\beta_1)+2,3;1)\right]$$
$$+[1+\lambda+\lambda|b|(1-\beta)]\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}$$
$$\times\left[{}_pF_q(|\alpha_1|+1,...,|\alpha_p|+1,P_1+1;\Re(\beta_1)+1,...,\Re(\beta_q)+1,2;1)\right]$$
$$+|b|(1-\beta)\left[{}_pF_q(|\alpha_1|,...,|\alpha_p|,P_1;\Re(\beta_1),...,\Re(\beta_q);1)\right]$$
$$\leq 2|b|(1-\beta)$$

hold. Then ${}_pF_q(f(z))\in SC(b,\lambda,\beta)$.

Remark 1 By specializing the parameters A_i, B_i, p, q, α_i and β_i , the main results derived can be easily restated interms of the operators defined by (1.9), (1.10) and (1.11).

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