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ON SOME SUBSEQUENCES OF FEJÉR MEANS FOR INTEGRABLE FUNCTIONS ON UNBOUNDED VILENKIN GROUPS

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Abstract. Following the methods in [2] and [4] we prove the almost everywhere convergence of some subsequences of the form $(\sigma_{a_N M_N + a_{N+1} M_{N+1} \dots + a_{N+l} M_{N+l}} f)_N$ to f, for every integrable function f on unbounded Vilenkin groups, under the boudedness condition of $a_N, \dots, a_{N+l}$ and l. This result enables us to include every element of $\sigma_n f$ in some convergent subsequence.

1. Introduction

On bounded groups, mean convergence holds almost everywhere for integrable functions [5]. However, by means of some different methods on unbounded groups, G. Gát [3] proved this result for L^p functions when $p > 1$, and obtained in [2] that $\sigma_{M_n} f \to f$, a.e. for every integrable function f. The same author [1] established the mean convergence almost everywhere of the full sequence for integrable functions on rarely unbounded groups. In [4] almost everywhere convergence of subsequences of the form $(\sigma_{a_N M_N} f)_N$ was established, where the numbers a_N are bounded, to the integrable function f. In the present work we provide a more general result concerning this fact. Namely, almost everywhere convergence is proved for subsequences of the form $(\sigma_{a_N M_N + a_{N+1} M_{N+1} \dots + a_{N+l} M_{N+l}} f)_N$, where l and $a_N, \dots, a_{N+l}$ are bounded. In this way every element of the sequence $\sigma_n f$ belongs to some convergent subsequence.

Let $(m_0, m_1, \dots, m_n, \dots)$ be an unbounded sequence of integers not less than 2. We denote by P the set of positive integers and let $\mathbb{N} = P \cup \{0\}$. Let $G := \prod_{n=0}^{\infty} \mathbb{Z}_{m_n}$, where $\mathbb{Z}_{m_n}$ denotes the discrete group of order m_n, with addition *mod* m_n. Each element from G can be represented as a sequence $(x_n)_n$, where $x_n \in \{0, 1, \dots, m_n - 1\}$, for every integer $n \geq 0$. Addition in G is obtained coordinatewise.

The topology on G is generated by the subgroups

$$I_n := \{x = (x_i)_i \in G,\ x_i = 0, \text{ for } i < n\},$$

and their translations

$$I_n(y) := \{x = (x_i)_i \in G,\ x_i = y_i, \text{ for } i < n\}.$$

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Sometimes we write $I_n(y)$ in the form $I_n(y) = I_n(y_0, \dots, y_{n-1})$. Also we write $I_n(y, j) = I_{n+1}(y_0, \dots, y_{n-1}, j)$, where $j \in \{0, 1, \dots, m_n - 1\}$.

Define the sequence $(M_n)_n$ as follows: $M_0 = 1$ and $M_{n+1} = m_n M_n$.

If $\mu(I_n)$ denotes the normalized product measure of I_n then it can be easily seen that $\mu(I_n) = M_n^{-1}$.

The generalized Rademacher functions are defined by

$$r_n(x) := e^{\frac{2\pi i x_n}{m_n}},\ n \in \mathbb{N},\ x \in G,$$

For every nonnegative integer n, there exists a unique sequence $(n_i)_i$ so that $n = \sum_{i=0}^{\infty} n_i M_i$. Put as in [3], $n^{(j)} = \sum_{i=j}^{\infty} n_i M_i$.

The system of Vilenkin functions is given by

$$\psi_n(x) := \prod_{i=0}^{\infty} r_i^{n_i}(x),\ n \in \mathbb{N},\ x \in G.$$

The Fourier coefficients, the partial sums of the Fourier series, the mean values, the Dirichlet kernels, the Fejér means and the Fejér kernels with respect to the Vilenkin system are respectively defined as follows

$$\hat{f}(n) = \int f(x)\bar{\psi}_n(x)dx, \qquad S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\psi_k, \qquad E_n(f) = S_{M_n} f,$$

$$D_n = \sum_{k=0}^{n-1} \psi_k, \qquad \sigma_n f = \frac{1}{n}\sum_{k=1}^{n} S_k f, \qquad K_n = \frac{1}{n}\sum_{k=1}^{n} D_k,$$

for every $f \in L^1(G)$.

It can be easily seen that

$$S_n f(y) = \int D_n(y - x) f(x)dx, \qquad D_{M_n}(x) = M_n 1_{I_n}(x),$$

and

$$E_n f(y) = M_n \int_{I_n(y)} f(x)dx.$$

Let $l, s_0, \dots, s_{l-1}$ be fixed nonnegative integers and $s_l > 0$. Then define the operator

$$\tilde{H}_{j,A}^{(s_0,s_1,\dots,s_l)} f(y) = M_{A-j} \left| \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},y_{A-j+1},\dots,y_{A-1})} f(x) \cdot \prod_{i=0}^{l} r_{A+i}^{s_i}(y - x) \frac{1}{1 - r_{A-j}(y - x)} dx \right|,$$

for every $f \in L^1(G)$, $j \in P$ and $A \geq j$ satisfying $s_i < m_{A+i}$, for every $i \in \{0, 1, \dots, l\}$.

We also define

$$\tilde{H}_j^{(s_0,s_1,\dots,s_l)} f(y) = \sup_{A: s_i < m_{A+i}, i \in \{0,1,\dots,l\}} \tilde{H}_{j,A}^{(s_0,s_1,\dots,s_l)} f(y).$$

2. Main results

Lemma 1. *For every fixed $l, s_0, \dots, s_l$, the operator $\tilde{H}_j^{(s_0,s_1,\dots,s_l)}$ is bounded on L^2.*

Proof. Let $f \in L^2$. Using the proof of [2, Lemma 2.3.] we can write

$$\tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)} f = H_{1,A}(f \prod_{i=0}^{l} \bar{r}_{A+i}^{s_i}),$$

from which we get

$$\|\tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)} f\|_2^2 = \|H_{1,A}(f \prod_{i=0}^{l} \bar{r}_{A+i}^{s_i})\|_2^2 \le C\|f \prod_{i=0}^{l} \bar{r}_{A+i}^{s_i}\|_2^2 \le C\|f\|_2^2.$$

Since

$$\tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)} f = \tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)}(E_{A+l+1}f),$$

and

$$\tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)}(E_{A+l}f) = 0,$$

it follows

$$\begin{aligned} &\| \sup_{A: s_i < m_{A+i}, i \in \{0,1,\dots,l\}} \tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)} f\|_2^2 \\ &= \| \sup_{A: s_i < m_{A+i}, i \in \{0,1,\dots,l\}} \tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)}(E_{A+l+1}f - E_{A+l}f)\|_2^2 \\ &\le \sum_{A: s_i < m_{A+i}, i \in \{0,1,\dots,l\}} \|\tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)}(E_{A+l+1}f - E_{A+l}f)\|_2^2 \\ &\le C \sum_{A} \|(E_{A+l+1}f - E_{A+l}f)\|_2^2 \le C\|f\|_2^2. \end{aligned}$$

□

Lemma 2. *For every fixed $l, s_0, \dots, s_l$ the operator $\tilde{H}_1^{(s_0,s_1,\dots,s_l)}$ is of weak type (L^1, L^1).*

Proof. We use the same decomposition as in [4] and [6]. Namely, for an arbitrary function $f \in L^1$, if $\lambda > 0$ is such that $\|f\|_1 \le \lambda$ and $(\alpha_k)_k$ is a sequence of integers defined by $\alpha_k = -s_l$ if $s_l < m_k$ and $\alpha_k = 0$ otherwise, there exist mutually disjoint intervals $J_j = \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(z^j, l)$, $j \in P$, and integrable functions b and g such that

1. $f = b + g$,
2. $\|g\|_\infty \le C\lambda$,
3. $\|g\|_1 \le C\|f\|_1$,
4. $supp(b) \subset \bigcup_{j=1}^{\infty} J_j$,
5. $\int_{J_j} b d\mu = \int_{J_j} b r_{k_j}^{\alpha_{k_j}} d\mu = 0$, for every $j \in P$,
6. $\int_{J_j} |b| d\mu \le C \int_{J_j} |f| d\mu$, for every $j \in P$,
7. $\sum_{j=1}^{\infty} \mu(J_j) \le \frac{\|f\|_1}{\lambda}$.

Let the sets $(6J_j)$ and $(6F)$ be defined as in [2, Lemma 2.4.].

In [6, Lemma 2] it was proved that for every $j \in P$ there exist constants a_{k_j} and b_{k_j} such that

$$b(x) = f(x) - a_{k_j} - b_{k_j} \bar{r}_{k_j}^{\alpha_{k_j}}(x), \forall x \in J_j.$$

We introduce the functions

$$h_j(x) = [b(x) - (\mu(J_j))^{-1} \sum_{i=1}^{l} \varepsilon_i^j \left(\int_{J_j} f \prod_{t=0}^{i} \bar{r}_{k_j+t}^{s_{l-i+t}} d\mu \right) \prod_{t=0}^{i} r_{k_j+t}^{s_{l-i+t}} - \varepsilon^j \mu(J_j))^{-1} \left(\int_{J_j} f \bar{r}_{k_j+1}^{s_0} \bar{r}_{k_j+2}^{s_1} \dots \bar{r}_{k_j+1+l}^{s_l} \right) r_{k_j+1}^{s_0} r_{k_j+2}^{s_1} \dots r_{k_j+1+l}^{s_l}] 1_{J_j}(x), j \in P,$$

where $\varepsilon_i^j = 1$ if $s_{l-i+t} < m_{k_j+t}$ for every $t \in \{0, 1, \dots, i\}$, and $\varepsilon_i^j = 0$ otherwise, similarly $\varepsilon^j = 1$ if $s_t < m_{k_j+1+t}$ for every $t \in \{0, 1, \dots, l\}$, and $\varepsilon^j = 0$ otherwise.

Then, it is easily seen that

$$\int_{J_j} h_j d\mu = \int_{J_j} h_j \bar{r}_{k_j}^{s_l} d\mu = \varepsilon_i^j \int_{J_j} h_j \prod_{t=0}^{i} \bar{r}_{k_j+t}^{s_{l-i+t}} d\mu = \varepsilon^j \int_{J_j} h_j \prod_{t=0}^{l} \bar{r}_{k_j+1+t}^{s_t} d\mu = 0,$$

for every $i \in \{1, \dots, l\}$.

Suppose that $y \in G \setminus I_{k_j}(z^j)$. Then $y \in I_{k_j-l+i-1}(z^j) \setminus I_{k_j-l+i}(z^j)$, for some $i \in \{l - k_j + 1, \dots, l\}$.

Hence,

$$\tilde{H}_1^{(s_0,s_1,\dots,s_l)} h_j(y) = H_{1,k_j-l+i} \left(h_j \prod_{t=0}^{l} \bar{r}_{k_j-l+i+t}^{s_t} \right)(y) = H_{1,k_j-l+i} \left(h_j \prod_{t=-l+i}^{i} \bar{r}_{k_j+t}^{s_{t+l-i}} \right)(y)$$
$$= H_{1,k_j-l+i} \left(h_j \prod_{t=0}^{i} \bar{r}_{k_j+t}^{s_{t+l-i}} \right)(y) = 0,$$

if $i > 0$, because

$$\varepsilon_i^j \int_{J_j} h_j \prod_{t=0}^{i} \bar{r}_{k_j+t}^{s_{l-i+t}} d\mu = \varepsilon_i^j \int_{I_{k_j-l+i}(z^j)} h_j \prod_{t=0}^{i} \bar{r}_{k_j+t}^{s_{l-i+t}} d\mu = 0.$$

Now if $i = 0$, we get

$$\tilde{H}_1^{(s_0,s_1,\dots,s_l)} h_j(y) = H_{1,k_j-l+i}(h_j \bar{r}_{k_j}^{s_l})(y) = 0,$$

because

$$\int_{J_j} h_j \bar{r}_{k_j}^{s_l} d\mu = \int_{I_{k_j-l+i}(z^j)} h_j \bar{r}_{k_j}^{s_l} d\mu = 0.$$

For $i < 0$, we obtain

$$\tilde{H}_1^{(s_0,s_1,\dots,s_l)} h_j(y) = H_{1,k_j-l+i} \left(h_j \prod_{t=-l+i}^{i} \bar{r}_{k_j+t}^{s_{t+l-i}} \right)(y) = H_{1,k_j-l+i} h_j(y) = 0,$$

because $\int_{J_j} h_j d\mu = \int_{I_{k_j-l+i}(z^j)} h_j d\mu = 0$.

If $y \in I_{k_j}(z^j) \setminus (6J_j)$, then

$$\tilde{H}_1^{(s_0,s_1,\dots,s_l)} h_j(y) = H_{1,k_j+1} \left(h_j \prod_{t=0}^{l} \bar{r}_{k_j+1+t}^{s_t} \right)(y).$$

Since

$$\varepsilon^j \int_{J_j} h_j \prod_{t=0}^{l} \bar{r}_{k_j+1+t}^{s_t} d\mu = 0,$$

using the method of [2, Lemma 2.4.], we get

$$\int_{G\setminus(6J_j)} \tilde{H}_1^s h_j(y)dy \le C\|h_j\|_1 \le C\int_{J_j} |f|d\mu, \quad \forall j \in P.$$

Repeating the steps of the proof of [4, Lemma 2.2.], the result immediately follows. □

Lemma 3. *There exists an absolute constant $C > 0$ such that for all $j \in P$, $f \in L^1$ and $\lambda > 0$, we have*

$$\mu\left(\tilde{H}_j^{(s_0,s_1,\ldots,s_l)} f > \lambda\right) \le \frac{j^2 C}{2^j \lambda}\|f\|_1.$$

Proof. Since

$$\tilde{H}_j^{(s_0,s_1,\ldots,s_l)} f = \sup_{j\le A, s_i<m_{A+i}} \tilde{H}_{j,A}^{(s_0,s_1,\ldots,s_l)} f \le \sum_{k=0}^{j-1} \sup_{\substack{j\le A, s_i<m_{A+i}\\ A\equiv k (mod\ j)}} \tilde{H}_{j,A}^{(s_0,s_1,\ldots,s_l)} f$$
$$\le \sum_{k=0}^{j-1}\sum_{z=0}^{l+1} \sup_{\substack{j\le A, s_i<m_{A+i}\\ A\equiv k+zj\ (mod\ (l+2)j)}} \tilde{H}_{j,A}^{(s_0,s_1,\ldots,s_l)} f,$$

following the steps of [4, Lemma 2.3.], it suffices to prove that for every $k \in \{0, 1, \ldots, j-1\}$, $z \in \{0, 1, \ldots, l+1\}$ the operators

$$2^j \sup_{\substack{j\le A\le Nj+k\\ A\equiv k+zj\ (mod\ (l+2)j)}} \tilde{H}_{j,A}^{(s_0,s_1,\ldots,s_l)} f$$

are of weak type (L^1, L^1) uniformly on $N \in \mathbb{N}$. We consider the permutations

- $\alpha_z(n) = n$, if $n \ge Nj + k$ or $n \ne k + zj + t(l+2)j, k + (z+1)j - 1 + t(l+2)j$, for any $t \in \mathbb{N}$,
- $\alpha_z(k + zj + t(l+2)j) = k + (z+1)j - 1 + t(l+2)j$, if $z + t(l+2) < N$,
- $\alpha_z(k + (z+1)j - 1 + t(l+2)j) = k + zj + t(l+2)j$, if $z + t(l+2) < N$.

Let G_z be the Vilenkin group generated by the sequence $(m_{\alpha_z(i)})_i$.

Then, for $A \le Nj + k$, $A \equiv k + (z+1)j\ (mod\ (l+2)j)$, we have $\alpha_z(A - j) = A - 1$, $\alpha_z(A - 1) = A - j$, but $\alpha_z(A + i) = A + i$, for every $i \in \{0, 1, \ldots, l\}$. Besides if $A = k + t(l+2)j$ for some $t \ge 1$, then

$$\alpha_z(A-j) = \alpha_z(k+(t(l+2)-1)j) = \alpha_z(k+(l+1)j+(t-1)(l+2)j) = k+t(l+2)j-1 = A-1$$

and

$$\tilde{H}_{j,A}^{(s_0,s_1,\dots,s_l)}f(y) = M_{A-j}\left|\int_{\bigcup_{x_{A-j}\neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},y_{A-j+1},\dots,y_{A-1})} f(x)\prod_{i=0}^{l}\bar{r}_{A+i}^{s_i}(x)\frac{1}{1-r_{A-j}(y-x)}dx\right|$$

$$= M_{A-j}\left|\int_{\bigcup_{x'_{A-1}\neq y'_{A-1}} I_A(y'_0,\dots,y'_{A-j-1},y'_{A-j},y'_{A-j+1},\dots,x'_{A-1})} f'(x')\left(\prod_{i=0}^{l}\bar{r}'_{A+i}(x)\right)^{s_i}\frac{1}{1-r'_{A-1}(y'-x')}dx'\right|$$

$$= \frac{M_{A-j}}{M_{A-1}}\tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)}f'(y') \leq 2^{1-j}\tilde{H}_{1,A}^{(s_0,s_1,\dots,s_l)}f'(y'),$$

where $(x'_i)_i = (x_{\alpha_z(i)})_i \in G_z$, for every $x \in G$, $(r'_n)_n$ is the convenient set of Rademacher functions for G_z and f' is defined on G_z by $f'(x') = f(x)$.

Following the steps of [2, Lemma 2.5.] we get that

$$2^j \sup_{\substack{j\leq A\leq Nj+k \\ A\equiv k+zj\ (mod\ (l+2)j)}} \tilde{H}_{j,A}^{(s_0,s_1,\dots,s_l)}f$$

is of weak type (L^1, L^1) uniformly on N. □

Theorem 1. *Let $f \in L^1$, $L, S \in P$ fixed. Then*

$$S_{a_N M_N + a_{N+1}M_{N+1}\dots + a_{N+l}M_{N+l}}f \to f$$

almost everywhere uniformly on

$$1 \leq l \leq L,\quad a_{N+t} \in \{0, 1, \dots, \min(S, m_{N+t} - 1)\},$$

for $t \in \{1, 2, \dots, l-1\}$,

$$a_N \in \{1, 2, \dots, \min(S, m_N - 1)\}$$

and

$$a_{N+l} \in \{1, 2, \dots, \min(S, m_{N+l} - 1)\}.$$

Proof. By induction, since $S_{a_N M_N}f \to f$ almost everywhere uniformly on

$$a_N \in \{1, 2, \dots, \min(S, m_N - 1)\},$$

(see [4, Theorem 2.5.]), it suffices to prove that

$$S_{a_{N+1}M_{N+1} + a_{N+2}M_{N+2}\dots + a_{N+l}M_{N+l}}f \to f$$

implies that

$$S_{a_N M_N + a_{N+1}M_{N+1}\dots + a_{N+l}M_{N+l}}f \to f.$$

Let $n = a_N M_N + a_{N+1}M_{N+1}\dots + a_{N+l}M_{N+l}$, we write

$$D_n = D_{n^{(N+1)}} + \psi_{n^{(N+1)}} D_{a_N M_N} = D_{n^{(N+1)}} + \psi_{n^{(N+1)}} D_{M_N} + \sum_{s=1}^{a_N - 1} \psi_{n^{(N+1)}} r_N^s D_{M_N}.$$

Clearly, it suffices to prove that

$$\left(\psi_{n^{(N+1)}} D_{M_N} + \sum_{s=1}^{a_N - 1} \psi_{n^{(N+1)}} r_N^s D_{M_N} \right) * f \to 0,$$

almost everywhere. Since

$$\left(\psi_{n^{(N+1)}} D_{M_N} + \sum_{s=1}^{a_N - 1} \psi_{n^{(N+1)}} r_N^s D_{M_N} \right) * g \to 0,$$

almost everywhere, whenever g is a polynomial, then we only need to prove that for every fixed $l, s_0, \ldots, s_{l-1}$ being nonnegative integers and $s_l > 0$, we have

$$\sup_{\substack{N: s_i < m_{N+i} \\ i=0,\ldots,l}} \left| \prod_{i=0}^{l} r_{N+i}^{s_i} D_{M_N} * g \right|$$

is of weak type (L^1, L^1) uniformly on N.

Using the decomposition of Lemma 2, we have $f = \sum_{j=1}^{\infty} h_j + G$.

It can be easily seen that if $y \in G \setminus (6F)$, we have that

$$\left(\prod_{i=0}^{l} r_{N+i}^{s_i} D_{M_N} * h_j \right)(y) = 0,$$

for every $N, j \in P$.

Following the steps of [4, Lemma 2.4.], it can be easily seen that

$$\sup_{\substack{N: s_i < m_{N+i} \\ i=0,\ldots,l}} \left| \prod_{i=0}^{l} r_{N+i}^{s_i} D_{M_N} * g \right|$$

is bounded on L^2. Then it can be concluded as in [4, Lemma 2.4.] that

$$\sup_{\substack{N: s_i < m_{N+i} \\ i=0,\ldots,l}} \left| \prod_{i=0}^{l} r_{N+i}^{s_i} D_{M_N} * g \right|$$

is of weak type (L^1, L^1) uniformly on N and $l \le L$,

$$a_{N+t} \in \{0, 1, \ldots, \min(S, m_{N+t} - 1)\},$$

for $t \in \{1, 2, \ldots, l-1\}$,

$$a_N \in \{1, 2, \ldots, \min(S, m_N - 1)\}$$

and

$$a_{N+l} \in \{1, 2, \ldots, \min(S, m_{N+l} - 1)\}.$$

□

Theorem 2. *Let $f \in L^1$, $L, S \in P$ fixed. Then*

$$\sigma_{a_N M_N + a_{N+1} M_{N+1} \ldots + a_{N+l} M_{N+l}} f \to f,$$

almost everywhere uniformly on $l \leq L$,

$$a_{N+t} \in \{0, 1, \ldots, \min(S, m_{N+t} - 1)\},$$

for $t \in \{1, 2, \ldots, l-1\}$,

$$a_N \in \{1, 2, \ldots, \min(S, m_N - 1)\}$$

and

$$a_{N+l} \in \{1, 2, \ldots, \min(S, m_{N+l} - 1)\}.$$

Proof. By induction, since $\sigma_{a_N M_N} f \to f$ almost everywhere uniformly on

$$a_N \in \{1, 2, \ldots, \min(S, m_N - 1)\},$$

it suffices to prove that

$$\sigma_{a_{N+1} M_{N+1} + a_{N+2} M_{N+2} \ldots + a_{N+l} M_{N+l}} f \to f$$

implies that

$$\sigma_{a_N M_N + a_{N+1} M_{N+1} \ldots + a_{N+l} M_{N+l}} f \to f.$$

Let $n = a_N M_N + a_{N+1} M_{N+1} \ldots + a_{N+l} M_{N+l}$, we write

$$\begin{aligned} K_n &= \frac{1}{n} \sum_{i=1}^{n^{(N+1)}} D_i + \frac{1}{n} \sum_{i=n^{(N+1)}+1}^{n} D_i \\ &= \frac{n^{(N+1)}}{n} K_{n^{(N+1)}} + \frac{a_N M_N}{n} D_{n^{(N+1)}} + \frac{a_N M_N}{n} \psi_{n^{(N+1)}} K_{a_N M_N}. \end{aligned}$$

From Lemma 1, we only need to prove that

$$\psi_{n^{(N+1)}} K_{a_N M_N} * f \to 0,$$

almost everywhere.

In other words, we need to prove that

$$\sup_{\substack{N: s_i < m_{N+i} \\ i=0,\ldots,l}} \left| \prod_{i=1}^{l} r_{N+i}^{s_i} K_{a_N M_N} * g \right|$$

is of weak type (L^1, L^1) uniformly on N and $s_i \leq S$.

We use the decomposition of $K_{a_N M_N}$ obtained in [4, Theorem 2.6.], we have

$$\begin{aligned} \prod_{i=1}^{l} r_{N+i}^{s_i} K_{a_N M_N} &= \prod_{i=1}^{l} r_{N+i}^{s_i} \left(\frac{1}{a_N} \sum_{j=1}^{a_N - 1} D_{j M_N} + \frac{1}{a_N} K_{M_N} + \frac{1}{a_N} \sum_{j=1}^{a_N - 1} r_N^j K_{M_N} \right) \\ &= \frac{1}{a_N} \prod_{i=1}^{l} r_{N+i}^{s_i} \sum_{j=1}^{a_N - 1} \left(D_{M_N} + \sum_{t=1}^{j-1} r_N^t D_{M_N} \right) + \prod_{i=1}^{l} r_{N+i}^{s_i} \frac{1}{a_N} K_{M_N} + \prod_{i=1}^{l} r_{N+i}^{s_i} \frac{1}{a_N} \sum_{j=1}^{a_N - 1} r_N^j K_{M_N}. \end{aligned}$$

In Theorem 1, it was proved that

$$\sup_{\substack{N: s_i < m_{N+i} \\ i=0,\ldots,l}} \left| \prod_{i=0}^{l} r_{N+i}^{s_i} D_{M_N} * g \right|$$

is of weak type (L^1, L^1) uniformly on $s_i \leq S$ and N.

Therefore, it suffices to prove that

$$\sup_{\substack{N: s_i < m_{N+i} \\ i=0,\dots,l}} \left| \prod_{i=0}^{l} r_{N+i}^{s_i} K_{M_N} * g \right|$$

is of weak type (L^1, L^1) uniformly on $s_i \le S$ and N. We write

$$\begin{aligned}
&\left| \left(\prod_{i=0}^{l} r_{N+i}^{s_i} K_{M_N} * f \right)(y) \right| = \left| \int K_{M_N}(y-x) \prod_{i=0}^{l} \bar{r}_{N+i}^{s_i}(x) f(x) dx \right| \\
&\le \left| \int_{I_N(y)} K_{M_N}(y-x) \prod_{i=0}^{l} \bar{r}_{N+i}^{s_i}(x) f(x) dx \right| \\
&+ \sum_{t=0}^{N-1} \left| \int_{I_t(y) \setminus I_{t+1}(y)} K_{M_N}(y-x) \prod_{i=0}^{l} \bar{r}_{N+i}^{s_i}(x) f(x) dx \right| \\
&\le S_{M_N}|f|(y) + \sum_{t=0}^{N-1} M_t \left| \int_{\bigcup_{x_t \ne y_t} I_N(y_0,\dots,y_{t-1},x_t,y_{t+1},\dots,y_{N-1})} \right. \\
&\left. f(x) \prod_{i=0}^{l} \bar{r}_{N+i}^{s_i}(x) \frac{1}{1 - r_t(y-x)} dx \right| \\
&\le S_{M_N}|f|(y) + \sum_{t=0}^{N-1} \tilde{H}_{N-t,N}^{(s_0,s_1,\dots,s_l)} f(y) = S_{M_N}|f|(y) + \sum_{j=1}^{N} \tilde{H}_{j,N}^{(s_0,s_1,\dots,s_l)} f(y).
\end{aligned}$$

Hence,

$$\sup_{\substack{N: s_i < m_{N+i} \\ i=0,\dots,l}} \left| \prod_{i=0}^{l} r_{N+i}^{s_i} K_{M_N} * f \right|(y) \le \sup_{\substack{N: s_i < m_{N+i} \\ i=0,\dots,l}} S_{M_N}|f|(y) + \sum_{j=1}^{\infty} \tilde{H}_j^{(s_0,s_1,\dots,s_l)} f(y).$$

Following the steps in the proof of [2, Theorem 2.1.] and replacing H_j by $\tilde{H}_j^{(s_0,s_1,\dots,s_l)}$, the result follows by applying Lemma 3. □

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