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A FORMULA FOR n - TIMES INTEGRATED C_0 GROUP OF OPERATORS ($n \in \mathbb{N}$)

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ABSTRACT. In this paper we analyze the connection between the group of rotation in the complex plane and n - times integrated group of rotation ($n \in \mathbb{N}$). Later we give and prove a general formula for arbitrary n - times integrated C_0 group of operators in a Banach space ($n \in \mathbb{N}$).

1. Introduction and some preliminaries

Many mathematicians have been developing the semigroup theory and the group theory of operators in a Banach space, and later, also, the theory of integrated semigroups (and groups) of operators in a Banach space (for example, see [1]-[14]). Here we give some important preliminaries.

Definition 1.1. *Let X be a Banach space. A one parameter family $T(t)$, $t \in \mathbb{R}$, of linear and bounded operators from X into X is C_0 group, or strongly continuous group of operators if*

i) $T(0) = I$, *where I is the identity operator on X ,*
ii) $T(t+s) = T(t)T(s)$ *for every $t \in \mathbb{R}$ and $s \in \mathbb{R}$,*
iii) $\lim\limits_{t\to 0^+} T(t)x = x$ *for every $x \in X$.*

The condition iii) is a condition of strongly continuity of the function $T(t)$ in point $t = 0$. The linear operator $A : X \to X$ defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\} \text{ and } Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ for all } x \in D(A),$$

is the infinitesimal generator of the C_0 group $T(t)$, defined on its domain $D(A)$.

It is known that infinitesimal generator of a strongly continuous group is a closed operator. Integrated groups of operators in Banach spaces have been introduced to study abstract Cauchy problems. Also, some differential operators in Euclidean spaces are examples of integrated group (see [3]). In [7] it is proved that α - times integrated groups

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define algebra homomorphism and smooth distribution groups of fractional order are equivalent to α - times integrated groups ($\alpha \geq 0$).

Definition 1.2. *Given $\alpha \geq 0$, a family of strongly continuous, linear and bounded operators $S(t)$, $t \geq 0$, on a Banach space X is said to be an α-times integrated semigroup if*

i) $S(0) = 0$,

ii) $S(t)S(s)x = \frac{1}{\Gamma(\alpha)}\left[\int\limits_s^{t+s} (t+s-r)^{\alpha-1} S(r)x\,dr - \int\limits_0^t (t+s-r)^{\alpha-1} S(r)x\,dr\right]$

for all $x \in X$, $t \geq 0$, $s \geq 0$. Here Γ is the Gamma function.
The generator A of $S(t)$, $t \geq 0$, is defined as it follows:
$D(A)$ is the set of all $x \in X$ such that there exists $y \in X$ satisfying

$$S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x = \int\limits_0^t S(s)y ds \quad (t \geq 0) \text{ and then } Ax := y.$$

The generator of integrated semigroup is a closed operator.

Definition 1.3. *An α-times integrated group $S(t)$, $t \in \mathbb{R}$, is a strongly continuous family of linear and bounded operators on a Banach space X such that $S_+(t) := S(t)$ $(t \geq 0)$ and $S_-(t) := -S(-t)$ $(t \geq 0)$ are α-times integrated semigroups, and if $t < 0 < r$, then*

$$S(t)S(r) = \frac{1}{\Gamma(\alpha)}\left[\int\limits_{t+r}^{r} (s-t-r)^{\alpha-1} S(s)\,ds + \int\limits_t^0 (t+r-s)^{\alpha-1} S(s)\,ds\right]$$

holds when $t + r \geq 0$, and

$$S(t)S(r) = \frac{1}{\Gamma(\alpha)}\left[\int\limits_{t}^{t+r} (t+r-s)^{\alpha-1} S(s)\,ds + \int\limits_0^r (s-t-r)^{\alpha-1} S(s)\,ds\right]$$

holds when $t + r \leq 0$.

If A is the generator of $S_+(t)$, $t \geq 0$, then operator $(-A)$ is the generator of $S_-(t)$, $t \geq 0$. The generator of $S(t)$, $t \in \mathbb{R}$, is defined as the generator of $S_+(t)$, $t \geq 0$.

Specially, if $T(t)$, $t \in \mathbb{R}$, is a C_0 group with infinitesimal generator A and if $S(t)$, $t \in \mathbb{R}$, is obtained by n-times successive integration of $T(t)$ ($n \in \mathbb{N}$), then $S(t)$, $t \in \mathbb{R}$, is n-times integrated group of operators with the same generator A and it holds

$$S(t)x = \frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-1} T(s)x\,ds \quad (x \in X).$$

We want to investigate in more details the connection between this n-times integrated group $S(t)$ and C_0 group $T(t)$ ($t \in \mathbb{R}$).

2. Results

In [14] it is considered the geometric meaning of once integrated group of rotation in the complex plane. Here we continue our investigation and obtain a formula for n-times integrated group of rotation. Later we obtain a general formula for arbitrary C_0 group $T(t)$. It is known that the field of all complex numbers $\mathbb{C}$ is itself a Banach space with norm equals to the absolute value. The group of rotation $T(t)$ $(t \in \mathbb{R})$ about the origin in the complex plane is given with $T(t)z = e^{it} \cdot z\,(z \in \mathbb{C})$, where $t \in \mathbb{R}$ is the angle of rotation and i is the imaginary unit. This family of operators is the group of operators because it is obviously $T(0) = I$, I is the identity operator on $\mathbb{C}$ and $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}$. Also, it holds

$$\lim_{t \to 0^+} T(t)z = \lim_{t \to 0^+} e^{it} \cdot z = z \qquad \text{for every } z \in \mathbb{C}.$$

By Definition 1.1, the group $T(t)$ $(t \in \mathbb{R})$ is a C_0 group of operators on $\mathbb{C}$. The infinitesimal generator of the group $T(t)$ $(t \in \mathbb{R})$ is the operator $A = i \cdot I$ because

$$Az := \lim_{t \to 0^+} \frac{T(t)z - z}{t} = \lim_{t \to 0^+} \frac{(e^{it} - 1)z}{t} = i \cdot z.$$

Let $S_n(t)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$, be defined as

$$S_n(t)z = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} T(s)z\,ds \quad (z \in \mathbb{C}).$$

Then, $S_n(t)$, $t \in \mathbb{R}$, is n-times integrated group of rotation with the same generator $A = i \cdot I$. For $n = 1$ and for every $t \in \mathbb{R}$ and all $z \in \mathbb{C}$, we have

$$\begin{aligned} S_1(t)z &= \int_0^t T(s)z\,ds = z \int_0^t e^{is} ds = \frac{e^{it} - 1}{i} \cdot z = [\sin t + i(1 - \cos t)]\,z \\ &= iz - i(\cos t + i \sin t)z = i \cdot [z - T(t)z] \\ &= e^{i\frac{\pi}{2}} \cdot [z - T(t)z] = T\left(\frac{\pi}{2}\right)[z - T(t)z] = T\left(\frac{\pi}{2}\right) z - T\left(\frac{\pi}{2} + t\right) z, \end{aligned}$$

i.e.

$$S_1(t) = T\left(\frac{\pi}{2}\right) - T\left(\frac{\pi}{2} + t\right) \quad (t \in \mathbb{R}).$$

This shows the connection between the group of rotation and once integrated group of rotation in the complex plane. It means that the point $S_1(t)z \in \mathbb{C}$ we obtain by rotating the point $z - T(t)z$ about the origin by the angle of $\frac{\pi}{2}$. If the points in the complex plane we identify with their radius vectors, then the vector $S_1(t)z$ we obtain as a difference of vectors $T\left(\frac{\pi}{2}\right) z$ and $T\left(\frac{\pi}{2} + t\right) z$, and $S_1(t)z$ is perpendicular to $z - T(t)z$. Because of

$$|S_1(t)z| = |[\sin t + i(1 - \cos t)]z| = (2 - 2\cos t) \cdot |z| = |z| \Leftrightarrow \cos t = \frac{1}{2},$$

we conclude that the operator $S_1(t)$ is an isometry on $\mathbb{C}$ only for $t = \pm\frac{\pi}{3} + 2k\pi \quad (k \in \mathbb{Z})$. Since

$$S_2(t)z = \int_0^t S_1(s)z\,ds = i \cdot \int_0^t (z - T(s)z)\,ds = i \cdot (tz - S_1(t)z),$$

it follows that vector $S_2(t)$ is perpendicular to the vector $tz - S_1(t)z$. Also, the vector

$$S_3(t)z = \int_0^t S_2(s)z\,ds = i \cdot \int_0^t (sz - S_1(s)z)\,ds = i \cdot \left(\frac{t^2}{2}z - S_2(t)z\right)$$

is perpendicular to the vector $\frac{t^2}{2}z - S_2(t)z$.
Analogously, by using the principle of mathematical induction we come up to the conclusion that for every $n \in \mathbb{N}$ it holds

$$S_n(t)z = \int_0^t S_{n-1}(s)z\,ds = i \cdot \left(\frac{t^{n-1}}{(n-1)!}z - S_{n-1}(t)z\right),$$

so the vector $S_n(t)z$ is perpendicular to the vector $\frac{t^{n-1}}{(n-1)!}z - S_{n-1}(t)z$.
Hence, for every $n \in \mathbb{N}$ it holds

$$S_n(t)z = (-i) \cdot \left(S_{n-1}(t)z - \frac{t^{n-1}}{(n-1)!}z\right).$$

Here is $T(t) = S_0(t)$, because the C_0 group $T(t)$, itself, by definition, is the same one as 0-times integrated group.

Using the successive application of this relation by the principle of mathematical induction, we may prove the following formula

$$S_n(t)z = (-i)^n \cdot \left(T(t)z - \sum_{k=0}^{n-1} \frac{(i \cdot t)^k}{k!}z\right). \tag{2.1}$$

For $n = 1$ the relation (2.1) holds, since we have shown that $S_1(t)z = i \cdot [z - T(t)z]$. From the assumption that (2.1) holds for some $n \in \mathbb{N}$ we get

$$\begin{aligned} S_{n+1}(t)z &= (-i) \cdot \left(S_n(t)z - \frac{t^n}{n!}z\right) = (-i)^{n+1} \cdot \left(T(t)z - \sum_{k=0}^{n-1} \frac{(i \cdot t)^k}{k!}z\right) + i \cdot \frac{t^n}{n!}z \\ &= (-i)^{n+1} \cdot \left(T(t)z - \sum_{k=0}^{n} \frac{(i \cdot t)^k}{k!}z\right). \end{aligned}$$

Therefore, we have just proved the following theorem:

Theorem 2.1. *Let $T(t)\,(t \in \mathbb{R})$ be a group of rotations in the complex plane $\mathbb{C}$ and let $S_n(t)\,(t \in \mathbb{R})\,(n \in \mathbb{N})$ be n-times integrated group of rotation in $\mathbb{C}$. Then, for every $n \in \mathbb{N}$ the vector $S_n(t)z$ is perpendicular to the vector $\frac{t^{n-1}}{(n-1)!}z - S_{n-1}(t)z$. Also, the formula (2.1) holds for every $n \in \mathbb{N}$.*

Notice that formula (2.1) presents one relation and connection between the group of rotation and n-times integrated group of rotation in a Banach space $\mathbb{C}$. We perceive that in the case of the group of rotation, specially, an infinitesimal generator of this group is $A = i \cdot I$, so now the formula (2.1) gives us an idea and motivation for the next general result.

Theorem 2.2. *Let $T(t)\,(t \in \mathbb{R})$ be a C_0 group of operators in a Banach space X and let $S_n(t)\,(t \in \mathbb{R})\,(n \in \mathbb{N})$ be n-times integrated group of operators in X, got from the group $T(t)\,(t \in \mathbb{R})$. Then, for every $x \in D(A^n)$ it holds*

$$A^n S_n(t)x = T(t)x - \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x \,. \tag{2.2}$$

Proof. It is known that for every $x \in D(A)$ we have

$$T(t)x - x = \int_0^t T(s)Ax\, ds = A\int_0^t T(s)x\, ds = AS_1(t)x,$$

so the relation (2.2) holds for $n = 1$. Assume now that for some $n \in \mathbb{N}$ and every $x \in D(A^n)$, the relation (2.2) holds. Namely, it is known that $A^n S_n(t)x = S_n(t)A^n x$. Now from the fact that, because of closedness, an operator A may go under the integral sign and using the assumption that (2.2) holds for $n \in \mathbb{N}$, we get that for every $x \in D(A^{n+1})$ we have

$$\begin{aligned} A^{n+1}S_{n+1}(t)x &= A\left(A^n \int_0^t S_n(s)x\, ds\right) = A\left(\int_0^t A^n S_n(s)x\, ds\right) \\ &= A\left(\int_0^t \left(T(s)x - \sum_{k=0}^{n-1} \frac{s^k}{k!}A^k x\right) ds\right) \\ &= A\int_0^t T(s)x\, ds - \sum_{k=0}^{n-1} \frac{t^{k+1}}{(k+1)!}A^{k+1}x \\ &= T(t)x - x - \sum_{k=1}^{n} \frac{t^k}{k!}A^k x = T(t)x - \sum_{k=0}^{n} \frac{t^k}{k!}A^k x\,. \end{aligned}$$

Therefore, we have proved that for every $n \in \mathbb{N}$, the assertion of this theorem holds. □

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