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PROPERTIES OF A NEW CLASS OF BIHARMONIC UNIVALENT FUNCTION

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ABSTRACT. A new class of biharmonic univalent functions using the modified Salagean differential operator was introduced. Properties of functions in the class were established. The work generalized some earlier results.

1. INTRODUCTION

Let A_λ denote the class of functions of the form:

$$f(z) = z^{\lambda+1} + \sum_{k=2}^{\infty} a_k z^{\lambda+k},\ 0 \leq \lambda < 1\,,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If for convenience, we set $A_0 = A$, we see that A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U}$, normalized by $f(0) = 0$ and $f'(0) = 1$. Moreover, we have $f(z) \in A \Longrightarrow z^\lambda f(z) \in A_\lambda,\ 0 \leq \lambda < 1$.

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain, we can write

$$F = H + \overline{G} \tag{1.1}$$

where H and G are analytic in D. We call H the analytic part and G the coanalytic part of F. A necessary and sufficient condition for F to be locally univalent and sense-preserving in D is that $|H'| > |G'|$ in D. Denote by $BH^0(U)$, the class of functions F of the form (1.1) that are univalent and sense-preserving in the unit disk U. The subclasses of biharmonic univalent functions have been studied by some authors for different purposes and different properties (see [1],[7] for details).

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2. Preliminaries

Let $BH^0_\lambda(\mathbf{U})$, $(0 \leq \lambda < 1)$ denote the set of all biharmonic functions F in $\mathbf{U}$ with the following form

$$\begin{aligned} F(z) &= \sum_{k=1}^{2} |z|^{2(k-1)}(h_k(z) + \overline{g_k(z)}) \\ &= \sum_{k=1}^{2} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{j,k} z^{\lambda+j} + \sum_{j=1}^{\infty} \overline{b_{j,k} z^{\lambda+j}} \right), \\ F(z) &= \sum_{k=1}^{2}\sum_{j=1}^{\infty} |z|^{2(k-1)} a_{j,k} z^{j+\lambda} + \sum_{k=1}^{2}\sum_{j=1}^{\infty} \overline{|z|^{2(k-1)} b_{j,k} z^{j+\lambda}}. \end{aligned}$$

with $a_{1,1} = 1; a_{1,2} = b_{1,1} = b_{1,2} = 0$. Note that g_k and h_k are analytic in the open unit disk $\mathbf{U}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Also, for convenience, we set $BH^0_0(U) = BH^0(U)$; we see that the class $BH^0(U)$ consists of functions of the form

$$F(z) = \sum_{k=1}^{2}\sum_{j=1}^{\infty} |z|^{2(k-1)} a_{j,k} z^{j} + \sum_{k=1}^{2}\sum_{j=1}^{\infty} \overline{|z|^{2(k-1)} b_{j,k} z^{j}}$$

with

$$\left.\begin{aligned} H(z) &= \textstyle\sum_{k=1}^{2}\sum_{j=1}^{\infty} |z|^{2(k-1)} a_{j,k} z^{j+\lambda} \\ G(z) &= \textstyle\sum_{k=1}^{2}\sum_{j=1}^{\infty} |z|^{2(k-1)} b_{j,k} z^{j+\lambda} \end{aligned}\right\}. \tag{2.1}$$

Let

$$F(z)^\beta = H(z)^\beta + \overline{G(z)^\beta}$$

Now, we expand (2.1) by binomial theorem. Thus,

$$\left.\begin{aligned} H(z)^\beta &= z^{\beta(\lambda+1)} + \beta \textstyle\sum_{k=1}^{2}\sum_{j=2}^{\infty} |z|^{2(k-1)} a_{j,k} z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\ G(z)^\beta &= \beta \textstyle\sum_{k=1}^{2}\sum_{j=1}^{\infty} |z|^{2(k-1)} b_{j,k} z^{(\beta-1)(\lambda+1)+(\lambda+j)} \end{aligned}\right\}. \tag{2.2}$$

We define the modified Salagean operator, $D^n F(z)^\beta$ as:

$$D^n F(z)^\beta = D^n H(z)^\beta + (-1)^n \overline{D^n G(z)^\beta}, (n \in \mathbb{N}_0) \tag{2.3}$$

where,

$$\left.\begin{aligned} D^n H(z)^\beta &= z^{\beta(\lambda+1)} + \beta[2(k-1)+j+\lambda]^n \textstyle\sum_{k=1}^{2}\sum_{j=2}^{\infty} |z|^{2(k-1)} a_{j,k} z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\ D^n G(z)^\beta &= \beta[2(k-1)+j+\lambda]^n \textstyle\sum_{k=1}^{2}\sum_{j=1}^{\infty} |z|^{2(k-1)} b_{j,k} z^{(\beta-1)(\lambda+1)+(\lambda+j)} \end{aligned}\right\}.$$

Remark 2.1. *If $\beta = 1$ and $\lambda = 0$, then $D^n f(z)^\beta$ becomes the known modified Salagean differential operator.*

Let $BH^0_\lambda(n, \beta, \alpha)$ be the family of biharmonic function of the form (2.3) such that,

$$Re\left\{ \frac{D^{n+1} F(z)^\beta}{D^n F(z)^\beta} \right\} > \alpha, \qquad \beta > 0, 0 \leq \alpha < 1,\ 0 \leq \lambda < 1\ (n \in \mathbb{N}_0) \tag{2.4}$$

We also let TBH^0 consists of biharmonic functions of the form $F(z)^\beta = H(z)^\beta + \overline{G(z)^\beta}$ in $BH^0_\lambda(n, \beta, \alpha)$ so that H and G are of the form:

$$\left.\begin{aligned} H^\beta &= z^{\beta(\lambda+1)} - \textstyle\sum_{k=1}^{2}\sum_{j=2}^{\infty} |z|^{2(k-1)} |a_{j,k}| z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\ G^\beta &= (-1)^n \textstyle\sum_{k=1}^{2}\sum_{j=2}^{\infty} |z|^{2(k-1)} |b_{j,k}| z^{(\beta-1)(\lambda+1)+(\lambda+j)} \end{aligned}\right\}, \tag{2.5}$$

which is a subclass of $BH^0_\lambda(n,\beta,\alpha)$ and satisfying the conditions (2.4). The class $BH^0_\lambda(n,\beta,\alpha)$ includes a variety of well known subclasses of S_H.

Remark 2.2. (1) *If $k=1$, $n=0$, and $\beta=1$ and $\lambda=0$ then $BH^0_0(0,1,\alpha)\equiv S^*_H(\alpha)$, which is the class of sense preserving biharmonic univalent functions of f which are starlike of order α in U.*

(2) *If $k=1$, $n=1$, and $\beta=1$ and $\lambda=0$ then $BH^0_0(1,1,\alpha)\equiv K^*_H(\alpha)$, which is the class of sense preserving biharmonic univalent functions of f which are convex of order α in U.*

3. Main Results

In this section we prove the main results. We now give results on the family of $BH^0_\lambda(n,\beta,\alpha)$ and its subclass.

Theorem 3.1. *Let $F(z)^\beta=H(z)^\beta+G(z)^\beta$, where $H(z)^\beta$ and $G(z)^\beta$ are given by (2.2). If*

$$\sum_{k=1}^{2}\sum_{j=1}^{\infty}\left[(2(k-1)+j+\lambda-\alpha)|a_{j,k}|+(2(k-1)+j+\lambda+\alpha)|b_{j,k}|\right]\beta(2(k-1)+j+\lambda)^n \leq (1+\beta)(1-\alpha) \tag{3.1}$$

where $a_{1,1}=1, a_{1,2}=b_{1,2}=b_{1,1}=0, n\in\mathbb{N}_0, 0\leq\lambda<1,\ \beta\geq1$ and $0\leq\alpha<0$. Then $F(z)^\beta$ is univalent and sense-preserving in U and $F\in BH^0_\lambda(n,\beta,\alpha)$.

Proof. Suppose $z_1^\beta\neq z_2^\beta$. Then,

$$\begin{aligned}
\left|\frac{F(z_1)^\beta-F(z_2)^\beta}{H(z_1)^\beta-H(z_2)^\beta}\right| &= \left|\frac{H(z_1)^\beta+\overline{G(z_1)^\beta}-H(z_2)^\beta+\overline{G(z_2)^\beta}}{H(z_1)^\beta-H(z_2)^\beta}\right| \\
&= \left|\frac{H(z_1)^\beta-H(z_2)^\beta+\overline{G(z_1)^\beta}-\overline{G(z_2)^\beta}}{H(z_1)^\beta-H(z_2)^\beta}\right| \\
&= \left|1+\frac{G(z_1)^\beta-G(z_2)^\beta}{H(z_1)^\beta-H(z_2)^\beta}\right| \\
&\geq 1-\left|\frac{G(z_1)^\beta-G(z_2)^\beta}{H(z_1)^\beta-H(z_2)^\beta}\right| \\
&> \frac{\sum_{k=1}^{2}\sum_{j=1}^{\infty}[2(k-1)+(\beta-1)(\lambda+1)+(\lambda+j)]b_{j,k}}{1-\sum_{k=1}^{2}\sum_{j=2}^{\infty}[2(k-1)+(\beta-1)(\lambda+1)+(\lambda+j)]a_{j,k}} \\
&\geq \frac{\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^n\frac{2(k-1)+j+\lambda+\alpha}{1-\alpha}|b_{j,k}|}{\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^n\frac{2(k-1)+j-\alpha}{1-\alpha}|a_{j,k}|}>0 \\
&: \left|\frac{F(z_1)^\beta-F(z_2)^\beta}{H(z_1)^\beta-H(z_2)^\beta}\right|>0\,.
\end{aligned}$$

Hence, $F(z_1)^\beta\neq F(z_2)^\beta$, which proves the univalence. To show that F^β is sense-preserving, it is sufficient to show that $|H'(z)^\beta|>|G'(z)^\beta|$.

Now,

$$
\begin{aligned}
|H'(z)^{\beta}| &= \left|\left(z^{\beta}(\lambda+1)+\beta\sum_{k=1}^{2}\sum_{j=2}^{\infty}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}\right)'\right| \\
&= \beta\Big[(\lambda+1)\,|z|^{\beta(\lambda+1)-1} \\
&\quad -\sum_{k=1}^{2}\sum_{j=2}^{\infty}(2(k-1)+j+(\beta-1)(\lambda+1)+(\lambda+j))|z|^{2(k-1)}|a_{j,k}|\,|z|^{(\beta-1)(\lambda+1)+(\lambda+j)-1}\Big] \\
&> \beta\left[(\lambda+1)-\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}\frac{2(k-1)+j+\lambda-\alpha}{1-\alpha}|a_{j,k}|\right] \\
&\geq \beta\left[\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}\frac{2(k-1)+j+\lambda+\alpha}{1-\alpha}|b_{j,k}|\right] \\
&\geq \beta\sum_{k=1}^{2}\sum_{j=1}^{\infty}|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)-1} \\
&\geq |G'(z)^{\beta}|\,.
\end{aligned}
$$

Lastly, we show that $F^{\beta}\in BH^{0}_{\lambda}(n,\beta,\alpha)$. By (2.4),

$$
\operatorname{Re}\left\{\frac{D^{n+1}F(z)^{\beta}}{D^{n}F(z)^{\beta}}\right\}>\alpha=\operatorname{Re}\left\{\frac{D^{n+1}H(z)^{\beta}+(-1)^{n+1}\overline{D^{n+1}G(z)^{\beta}}}{D^{n}H(z)^{\beta}+(-1)^{n}\overline{D^{n}G(z)^{\beta}}}\right\}>\alpha\,.
$$

Using the fact that $\operatorname{Re}\{\omega\}>\alpha$ if and only if $|1-\alpha+\omega|>|1+\alpha-\omega|$, it suffices to show that:

$$
\begin{aligned}
&\left|1-\alpha+\frac{D^{n+1}F(z)^{\beta}}{D^{n}F(z)^{\beta}}\right|-\left|1+\alpha-\frac{D^{n+1}F(z)^{\beta}}{D^{n}F(z)^{\beta}}\right|\geq 0 \\
&\left|\frac{D^{n+1}F(z)^{\beta}+(1-\alpha)D^{n}F(z)^{\beta}}{D^{n}F(z)^{\beta}}\right|-\left|\frac{-D^{n+1}F(z)^{\beta}+(1+\alpha)D^{n}F(z)^{\beta}}{D^{n}F(z)^{\beta}}\right|\geq 0\,.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left|D^{n+1}F(z)^{\beta}+(1-\alpha)D^{n}F(z)^{\beta}\right|-\left|D^{n+1}F(z)^{\beta}+(1+\alpha)D^{n}F(z)^{\beta}\right| \\
=\;&\left|D^{n+1}H(z)^{\beta}+(-1)^{n+1}\overline{D^{n+1}G(z)^{\beta}}+(1-\alpha)[D^{n}H(z)^{\beta}+(-1)^{n}\overline{D^{n}G(z)^{\beta}}]\right| \\
&-\left|D^{n+1}H(z)^{\beta}+(-1)^{n+1}\overline{D^{n+1}G(z)^{\beta}}-(1+\alpha)[D^{n}H(z)^{\beta}+(-1)^{n}\overline{D^{n}G(z)^{\beta}}]\right|
\end{aligned}
$$

$$
\begin{aligned}
= \quad & |z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k=1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& +(-1)^{n+1}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& +(1-\alpha)[z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& +(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k-1)}\overline{b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}]| \\
& -|z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& +(-1)^{n+1}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k-1)}\overline{b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}} \\
& -(1+\alpha)[z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& +(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}]| \\
= \quad & |(2-\alpha)z^{\beta(\lambda+1)} \\
& +\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda+1-\alpha)|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& -(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda-1+\alpha)|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& -|(-\alpha)z^{\beta(\lambda+1)} \\
& +\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda-1-\alpha)|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& -(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda+1+\alpha)|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}| \\
\geq \quad & (2-\alpha)|z|^{\beta} \\
& -\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda+1-\alpha)|z|^{2(k-1)}|a_{j,k}||z|^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& -\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda-1+\alpha)|z|^{2(k-1)}|b_{j,k}||z|^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& (-\alpha)|z|^{\beta(\lambda+1)} \\
& -\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)-j+\lambda-1-\alpha)|z|^{2(k-1)}|a_{j,k}||z|^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& -\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda+1+\alpha)|z|^{2(k-1)}|b_{j,k}||z|^{(\beta-1)(\lambda+1)+(\lambda+j)}|
\end{aligned}
$$

$$
\begin{aligned}
&= 2(1-\alpha) - 2\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^n(2(k-1)+j+\lambda-\alpha)|a_{j,k}| \\
&\quad -2\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^n(2(k-1)+j+\lambda+\alpha)|b_{j,k}| \\
&= 2(1-\alpha)\left[1-\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^n\left(\frac{2(k-1)+j+\lambda-\alpha}{1-\alpha}\right)|a_{j,k}|-\right. \\
&\quad \left.\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^n\left(\frac{(2(k-1)+j+\lambda+\alpha)}{1-\alpha}\right)|b_{j,k}|\right].
\end{aligned}
$$

The last expression is non negative by the hypothesis of the theorem. This completes the proof. ☐

In the following theorem, it is shown that the inequality condition in (2.4) is also necessary for the functions in the subclass $TBH_{\lambda}^{0}(n,\beta,\alpha)$.

Theorem 3.2. *Let $F^{\beta} = H^{\beta} + G^{\beta}$ be given by (2.5). Then $F^{\beta} \in TBH_{\lambda}^{0}(n,\beta,\alpha)$ if and only if*

$$\sum_{k=1}^{2}\sum_{j=1}^{\infty}[(2(k-1)+j+\lambda-\alpha)|a_{j,k}|+(2(k-1)+j+\lambda+\alpha)|b_{j,k}|]\beta(2(k-1)+j+\lambda)^n \leq (1+\beta)(1-\alpha)$$

Proof. Since $TBH_{\lambda}^{0}(n,\beta,\alpha) \subset BH_{\lambda}^{0}(n,\beta,\alpha)$, we prove only the if part of the theorem. Thus,

$$\operatorname{Re}\left\{\frac{D^{n+1}F(z)^{\beta}}{D^{n}F(z)^{\beta}}\right\} \geq \alpha\,,$$

where $\beta \geq 1 \quad, 0 \leq \alpha < 1, \quad n \in \mathbb{N}_0$

$$\Rightarrow \operatorname{Re}\left\{\frac{D^{n+1}F(z)^{\beta}}{D^{n}F(z)^{\beta}} - \alpha\right\} \geq 0\,.$$

We have:

$$\operatorname{Re}\left\{\frac{D^{n+1}F(z)^{\beta} - \alpha D^{n}F(z)^{\beta}}{D^{n}F(z)^{\beta}}\right\} \geq 0\,,$$

$$
\begin{aligned}
D^{n+1}F(z)^{\beta} &= z^{\beta(\lambda+1)} - \sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
&\quad +(-1)^{n+1}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n+1}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
D^{n}F(z)^{\beta} &= z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
&\quad +(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}
\end{aligned}
$$

$$
= \mathrm{Re}\left\{ \frac{\frac{z^{\beta(\lambda+1)}-\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}.(2(k-1)+j+\lambda)|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}{-\alpha z^{\beta(\lambda+1)}+\sum_{k=1}^{2}\sum_{j=2}^{\infty}\alpha\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}}{z^{\beta(\lambda+1)}-\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}} \right.
$$

$$
\left. \frac{\frac{-(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}.(2(k-1)+j+\lambda)|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}{-(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\alpha\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}}{+(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\alpha\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}} \right\}
$$

$$
= \mathrm{Re}\left\{ \frac{\frac{z^{\beta(\lambda+1)}-\alpha z^{\beta(\lambda+1)}-\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}.(2(k-1)+j+\lambda)|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}{+(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}.(2(k-1)+j+\lambda)|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}}{z^{\beta(\lambda+1)}-\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}} \right.
$$

$$
\left. \frac{\frac{+\sum_{k=1}^{2}\sum_{j=1}^{\infty}\alpha\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}{-(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\alpha\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}}{+(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}\overline{b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}} \right\}
$$

$$
= \mathrm{Re}\left\{ \frac{\frac{(1-\alpha)z^{\beta(\lambda+1)}-\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}.(2(k-1)+j+\lambda-\alpha)|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}{-(-1)^{2n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}.(2(k-1)+j+\lambda+\alpha)|z|^{2(k-1)}b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}}{\frac{z^{\beta(\lambda+1)}-\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}a_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}{+(-1)^{n}\sum_{k=1}^{2}\sum_{j=1}^{\infty}\alpha\beta(2(k-1)+j+\lambda)^{n}|z|^{2(k-1)}\overline{b_{j,k}z^{(\beta-1)(\lambda+1)+(\lambda+j)}}}} \right\}
$$

$$
\geq 0\,.
$$

The above required condition must hold for all values of z in U. If we choose the values of z on the positive real axis, where $0 \leq z = r < 1$, we must have:

$$
\frac{\frac{(1-\alpha)-\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}.(2(k-1)+j+\lambda-\alpha)a_{j,k}r^{2(k-1)+j-1}}{-\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}(2(k-1)+j+\lambda+\alpha)b_{j,k}r^{2(k-1)+j-1}}}{\frac{1-\sum_{k=1}^{2}\sum_{j=2}^{\infty}\beta(2(k-1)+j+\lambda)^{n}a_{j,k}r^{2(k-1)+j-1}}{+\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta(2(k-1)+j+\lambda)^{n}b_{j,k}r^{2(k-1)+j-1}}} \geq 0
$$

If the condition (3.1) does not hold, then the numerator in the last inequality is negative for r sufficiently close to 1. Hence, there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in the last inequality is negative.

This contradicts the required condition for $F^{\beta} \in TBH_{\lambda}^{0}(n, \beta, \alpha)$ and the proof is complete.

□

Theorem 3.3. *Let $F^\beta = H^\beta + G^\beta$ be given by (2.5). Then $F^\beta \in TBH^0_\lambda(n,\beta,\alpha)$ if and only if*

$$F^\beta = \sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(X_{j,k}H_{j,k}(z)^\beta + Y_{j,k}G_{j,k}(z)^\beta\right) ,$$

where

$$\begin{aligned}
H_{1,1}(z)^\beta &= z^{\beta(\lambda+1)} \\
H_{j,k}(z)^\beta &= z^{\beta(\lambda+1)} - \frac{(1-\alpha)}{(2(k-1)+j+\lambda-\alpha)(2(k-1)+j+\lambda)^n}|z|^{2(k-1)}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& \quad j = 2,3,\cdots \quad 1 \le k \le 2 \\
G_{j,k}(z)^\beta &= z^{\beta(\lambda+1)} + \frac{(-1)^n(1-\alpha)}{(2(k-1)+j+\lambda+\alpha)(2(k-1)+j+\lambda)^n}|z|^{2(k-1)}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
& \quad j = 1,2,\cdots \quad 1 \le k \le 2
\end{aligned}$$

Proof.

$$\begin{aligned}
F(z)^\beta &= X_{1,1}H_{1,1}(z)^\beta + Y_{1,1}G_{1,1}(z)^\beta + \sum_{k=1}^{2}\sum_{j=2}^{\infty}\left(X_{j,k}H_{j,k}(z)^\beta + Y_{j,k}G_{j,k}(z)^\beta\right) \\
&= X_{1,1}z^{\beta(\lambda+1)} + Y_{1,1}[z^{\beta(\lambda+1)} + (-1)^n(\frac{1-\alpha}{(\lambda+1+\alpha)(\lambda+1)^n}z^{\beta(\lambda+1)}] \\
&+\sum_{k=1}^{2}\sum_{j=2}^{\infty}X_{j,k}[z^{\beta(\lambda+1)} - \frac{(1-\alpha)}{(2(k-1)+j+\lambda-\alpha)(2(k-1)+j+\lambda)^n}|z|^{2(k-1)}z^{(\beta-1)(\lambda+1)+(\lambda+j)}] \\
&+\sum_{k=1}^{2}\sum_{j=2}^{\infty}Y_{j,k}[z^{\beta(\lambda+1)} + (-1)^n\frac{(1-\alpha)}{(2(k-1)+j+\lambda+\alpha)(2(k-1)+j+\lambda)^n}|z|^{2(k-1)}z^{(\beta-1)(\lambda+1)+(\lambda+j)}] \\
&= X_{1,1}z^{\beta(\lambda+1)} + Y_{1,1}z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=2}^{\infty}X_{j,k}z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=2}^{\infty}Y_{j,k}z^{\beta(\lambda+1)} \\
&\quad -\sum_{k=1}^{2}\sum_{j=2}^{\infty}X_{j,k}\frac{(1-\alpha)}{(2(k-1)+j+\lambda-\alpha).(2(k-1)+j+\lambda)^n}|z|^{2(k-1)}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
&\quad +(-1)^n\frac{(1-\alpha)}{(\lambda+1+\alpha).(\lambda+1)^n}Y_{1,1}z^{\beta(\lambda+1)} \\
&\quad +(-1)^n\sum_{k=1}^{2}\sum_{j=2}^{\infty}Y_{j,k}(\frac{1-\alpha}{2(k-1)+j+\lambda+\alpha})(2(k-1)+j+\lambda)^n|z|^{2(k-1)}\overline{z^{(\beta-1)(\lambda+1)+(\lambda+j)}} \\
&= \sum_{k=1}^{2}\sum_{j=1}^{\infty}(X_{j,k}+Y_{j,k}) + (-1)(\frac{1-\alpha}{\lambda+1+\alpha})Y_{1,k}z^{\beta(\lambda+1)} \\
&\quad +(-1)^n(\frac{1-\alpha}{2(k-1)+j+\lambda+\alpha})(2(k-1)+j+\lambda)^n \\
&\quad \times\sum_{k=1}^{2}\sum_{j=2}^{\infty}(2(k-1)+j+\lambda)^n|z|^{2(k-1)}\overline{z^{(\beta-1)(\lambda+1)+(\lambda+j)}}Y_{j,k} .
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta\left\{\frac{2(k-1)+j+\lambda-\alpha}{1-\alpha}\right\}(2(k-1)+j+\lambda)^n|a_{j,k}|+\\
&+\left\{\frac{2(k-1)+j+\lambda+\alpha}{1-\alpha}\right\}(2(k-1)+j+\lambda)^n|b_{j,k}|\\
=\;&\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta\left\{\frac{2(k-1)+j+\lambda-\alpha}{1-\alpha}\right\}(2(k-1)+j+\lambda)^n\\
&\times\left|\frac{(1-\alpha)X_{j,k}}{(2(k-1)+j+\lambda-\alpha)\beta(2(k-1)+j+\lambda)^n}\right|\\
&+\left\{\frac{2(k-1)+j+\lambda+\alpha}{1-\alpha}\right\}(2(k-1)+j+\lambda)^n\\
&\times\left|\frac{(1-\alpha)X_{j,k}}{(2(k-1)+j+\lambda-\alpha)\beta(2(k-1)+j+\lambda)^n}\right|\\
=\;&\sum_{k=1}^{2}\sum_{j=2}^{\infty}X_{j,k}+\sum_{k=1}^{2}\sum_{j=1}^{\infty}Y_{j,k}=1-X_{1,1}\leq 1
\end{aligned}$$

Conversely, suppose $F^{\beta}\in TBH_{\lambda}^{0}(n,\beta,\alpha)$, by setting:

$$\begin{aligned}
X_{j,k} &= \left(\frac{2(k-1)+j+\lambda-\alpha}{1-\alpha}\right)\beta(2(k-1)+j+\lambda)^n|a_{j,k}|;0\leq X_{j,k}\leq 1;j=2,3,\cdots\\
Y_{j,k} &= \left(\frac{2(k-1)+j+\lambda+\alpha}{1-\alpha}\right)\beta(2(k-1)+j)^n|b_{j,k}|;0\leq Y_{j,k}\leq 1;j=1,2,\cdots
\end{aligned}$$

and

$$X_{1,1}=1-\sum_{k=1}^{2}\sum_{j=2}^{\infty}X_{j,k}-\sum_{k=1}^{2}\sum_{j=1}^{\infty}Y_{j,k}\,.$$

Therefore,

$$\begin{aligned}
F(z)^{\beta} &= z^{\beta(\lambda+1)}-\beta\sum_{k=1}^{2}\sum_{j=2}^{\infty}|z|^{2(k-1)}|a_{j,k}|z^{(\beta-1)(\lambda+1)+(\lambda+j)}\\
&\quad+(-1)^n\beta\sum_{k=1}^{2}\sum_{j=1}^{\infty}|z|^{2(k-1)}|b_{j,k}|\overline{z^{(\beta-1)(\lambda+1)+(\lambda+j)}}
\end{aligned}$$

$$
\begin{aligned}
&= z^{\beta(\lambda+1)} - \sum_{k=1}^{2}\sum_{j=2}^{\infty}\left(\frac{1-\alpha}{2(k-1)+j+\lambda-\alpha}\right)\frac{X_{j,k}}{(2(k-1)+j+\lambda)^n}|z|^{2(k-1)}z^{(\beta-1)(\lambda+1)+(\lambda+j)} \\
&\quad +(-1)^n\sum_{k=1}^{2}\sum_{j=1}^{\infty}\beta\left(\frac{1-\alpha}{2(k-1)+j+\lambda+\alpha}\right)\frac{Y_{j,k}}{(2(k-1)+j+\lambda)^n}|z|^{2(k-1)}\overline{z^{(\beta-1)(\lambda+1)+(\lambda+j)}} \\
&= z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=2}^{\infty}(H_{j,k} - z^{\beta(\lambda+1)})X_{j,k} + \sum_{k=1}^{2}\sum_{j=1}^{\infty}(G_{j,k} - z^{\beta(\lambda)})Y_{j,k} \\
&= \sum_{k=1}^{2}\sum_{j=2}^{\infty}H_{j,k}(z)^{\beta}X_{j,k} + \sum_{k=1}^{2}\sum_{j=1}^{\infty}G_{j,k}(z)^{\beta}Y_{j,k} + z^{\beta(\lambda+1)}[1 - \sum_{k=1}^{2}\sum_{j=2}^{\infty}X_{j,k} - \sum_{k=1}^{2}\sum_{j=1}^{\infty}Y_{j,k}] \\
&= X_{1,1}z^{\beta(\lambda+1)} + \sum_{k=1}^{2}\sum_{j=2}^{\infty}\left(H_{j,k}(z)^{\beta}X_{j,k} + G_{j,k}(z)^{\beta}Y_{j,k}\right) .
\end{aligned}
$$

as required. □

Corollary 3.1. *Let $F = H + \overline{G}$. If $\sum_{j=1}^{\infty}[(j-\alpha)|a_j| + (j+\alpha)|b_j|] \leq 2(1-\alpha)$ where $a_1 = 1$, $n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, then $f(z) \in BH_0^0(0,1,\alpha) \equiv S_H^*(\alpha)$.*

Proof. The proof of the corollary follows if we put $\beta = 1$, $k = 1$, $\lambda = 0$ and $n = 0$ in Theorem 3.1. □

Remark 3.1. *This is the result obtained by Jahangiri,* [3].

Corollary 3.2. *Let $F(z)^{\beta} = H(z)^{\beta} + \overline{G(z)^{\beta}}$, where $H(z)^{\beta}$, $G(z)^{\beta}$ are given by (2.2). If $\sum_{j=1}^{\infty}[(j-\alpha)|a_j| + (j+\alpha)|b_j|]\beta j^n \leq (1+\beta)(1-\alpha)$ where $a_1 = 1$, $n \in \mathbb{N}_0$, $\beta \geq 1$ and $0 \leq \alpha < 1$, then $F(z)^{\beta} \in BH_0^0(n,\alpha,\beta) \equiv \Phi(n,\beta,\alpha)$.*

Proof. The proof of the corollary follows if we put $k = 1$, $\lambda = 0$ in Theorem 3.1. □

Remark 3.2. *This is the result obtained by Al-shaqsi et.al.* [4].

Corollary 3.3. *Let $F(z)^{\beta} = H(z)^{\beta} + \overline{G(z)^{\beta}}$, where $H(z)^{\beta}$, $G(z)^{\beta}$ are given by (2.5). If $\sum_{j=1}^{\infty}[(j-\alpha)|a_j| + (j+\alpha)|b_j|]\beta j^n \leq (1+\beta)(1-\alpha)$ where $a_1 = 1$, $n \in \mathbb{N}_0$, $\beta \geq 1$ and $0 \leq \alpha < 1$, then $F(z)^{\beta} \in TBH_0^0(n,\alpha,\beta) \equiv \overline{\Phi(n,\beta,\alpha)}$.*

Proof. Put $k = 1$, $\lambda = 0$ in Theorem 3.2. □

Remark 3.3. *This is the result obtained by Al-shaqsi et.al.* [4].

Corollary 3.4. *Let $F(z)^{\beta} = H(z)^{\beta} + \overline{G(z)^{\beta}}$, where$H(z)^{\beta}$, $G(z)^{\beta}$ are given by (2.2). If $\sum_{k=1}^{2}\sum_{j=1}^{\infty}[j+\lambda-\alpha)|a_{j,k}| + (j+\lambda+\alpha)|b_{j,k}|]\beta(j+\lambda)^n \leq (1+\beta)(1-\alpha)$, then $f(z)$ is univalent, sense preserving and $f \in BH_{\lambda}^0(n,\alpha,\beta) \equiv \Phi_{\lambda}(n,\alpha,\beta)$.*

Proof. The proof of the corollary follows if we put $k = 1$ in Theorem 3.1. □

Remark 3.4. *This is the result obtained by Fadipe-Joseph et. al.*[2].

Corollary 3.5. *Let $F(z)^{\beta} = H(z)^{\beta} + \overline{G(z)^{\beta}}$, where $H(z)^{\beta}$, $G(z)^{\beta}$ are given by (2.5). If $\sum_{k=1}^{2}\sum_{j=1}^{\infty}[j+\lambda-\alpha)|a_{j,k}| + (j+\lambda+\alpha)|b_{j,k}|]\beta(j+\lambda)^n \leq (1+\beta)(1-\alpha)$, then $f \in TBH_{\lambda}^0(n,\alpha,\beta) \equiv \overline{\Phi_{\lambda}(n,\alpha,\beta)}$.*

Proof. The proof of the corollary follows if we put $k = 1$ in Theorem 3.2. □

Remark 3.5. *This is the result obtained by Fadipe-Joseph et.al.*[2].

Corollary 3.6. *Let* $F^\beta = H^\beta + \overline{G^\beta}$. *If*

$$\sum_{k=1}^{2}\sum_{j=1}^{\infty}[(2(k-1)+j+\lambda-\alpha)|a_{j,k}|+(2(k-1)+j+\lambda+\alpha)|b_{j,k}|](2(k-1)+j+\lambda)^n \leq 2(1-\alpha)$$

where $a_{1,1} = 1$, $n \in \mathbb{N}_0$ *and* $0 \leq \alpha < 1$, *then F(z) is in the class* $BH^0_\lambda(n, 1, \alpha)$.

Proof. The proof of the corollary follows if we put $\beta = 1$ in Theorem 3.1. □

Corollary 3.7. *Let* $F = H + \overline{G}$. *If*

$$\sum_{k=1}^{2}\sum_{j=1}^{\infty}[(2(k-1)+j)\,(|a_{j,k}| + (2(k-1)+j)|b_{j,k}|)] \leq 2$$

where $a_{1,1} = 1$, $n \in \mathbb{N}_0$, *then* $F(z)$ *is univalent and sense preserving in* U *and* $F(z) \in BH^0_0(0, 1, 0) \equiv BH^0(U)$.

Proof. The proof of the corollary follows if we put $\beta = 1$ and $\alpha = 0$ and $\lambda = 0$ and $n = 0$ in Theorem 3.1. □

Remark 3.6. *This is the result obtained by Qiao and Wang,* [7].

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