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ON GENERAL COSINE OPERATOR FUNCTION IN BANACH SPACE

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ABSTRACT. In this paper we explain our motivation for introducing a new definition of cosine operator function in Banach space and then, along with an appropriate explanation, we introduce a new definition of cosine operator function and its infinitesimal generator. We give relevant comparison and draw a parallel with an earlier definition of these functions as solutions of D'Alembert equation.

1. INTRODUCTION

Many mathematicians, motivated by ordinary cosine and hyperbolic cosine function in the set of real numbers $\mathbb{R}$, have studied possible generalizations of these functions and their properties in Banach space. Ordinary cosine functions and hyperbolic cosine functions are solutions of D'Alembert equation

$$f(t+s)+f(t-s)=2f(t)f(s), \tag{1.1}$$

which fulfill an additional condition

$$f(0)=1. \tag{1.2}$$

Therefore, many mathematicians defined the family of general cosine functions in Banach spaces as the solutions of D'Alembert equation (1.1) which also satisfy the condition

$$f(0)=I, \tag{1.3}$$

where I is an identity operator on Banach space (see [1, 2, 3, 4, 5]). In this paper we want to define on another way, the general cosine operator function in Banach space and its infinitesimal generator. The reason for our approach arises from the natural connection between the solutions of considering functional equations with the solutions of the appropriate second order Cauchy problem.

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2. Motivation for introducing a new definition of general cosine operator function

Let us show that all real functions $y = \cos at$ and $y = \cosh at$ $(a \in \mathbb{R}; a \neq 0)$, $(t \in \mathbb{R})$ and $f(t) \equiv 1\,(t \in \mathbb{R})$ are the only solutions of functional equations

$$2f^2(t) = 1 + f(2t) \tag{2.1}$$

and

$$f(t+s) = f(t)f(s) \pm a^2 \int_0^t f(u)\,du \cdot \int_0^s f(v)\,dv, \quad (t, s \in \mathbb{R}) \tag{2.2}$$

where in (2.2) sign $+$ matches to a solution $y = \cosh at$, and sign $-$ matches to a solution $y = \cos at$. We are going to find two times continuously differentiable and even functions $f(t)$ defined on $\mathbb{R}$, that are the solutions of equations (2.1) and (2.2). Since

$$\int_0^{-s} f(v)\,dv = -\int_0^s f(r)\,dr \quad (s \in \mathbb{R})$$

and because the function $f(t)$ is even, from (2.2) it follows

$$f(t-s) = f(t)f(s) \mp a^2 \int_0^t f(u)\,du \cdot \int_0^s f(v)\,dv. \tag{2.3}$$

After summing the equations (2.2) and (2.3), we get

$$f(t+s) + f(t-s) = 2f(t)f(s)$$

and by setting $s = 0$ in (2.2) we find $f(t) = f(t)f(0)$ for every $t \in \mathbb{R}$, so $f(0) = 1$ and this also satisfies the equation (2.1). Hence, the solutions $f(t)$ of equations (2.1) and (2.2), which are even functions on $\mathbb{R}$, are at the same time the solutions of D'Alembert functional equation (1.1) that satisfy the condition (1.2).

From (2.2) for $s = t$ we obtain

$$f(2t) = f^2(t) \pm a^2 \left(\int_0^t f(v)\,dv\right)^2,$$

which putting in (2.1) gives

$$f^2(t) - 1 = \pm a^2 \left(\int_0^t f(v)\,dv\right)^2, \tag{2.4}$$

Differentiating in (2.4) with respect to t yields

$$2f(t) \cdot f'(t) = \pm 2a^2 \left(\int_0^t f(v)\,dv\right) \cdot f(t),$$

and for $f(t) \neq 0$, it follows

$$f'(t) = \pm a^2 \left(\int_0^t f(v)\,dv\right). \tag{2.5}$$

When we differentiate with respect to t both sides of equation (2.5) we obtain

$$f''(t) = \pm a^2 \cdot f(t). \tag{2.6}$$

General solution of the differential equation (2.6) for the sign $+$ is

$$f(t) = C_1 e^{at} + C_2 e^{-at}$$

and for the sign $-$ it is

$$f(t) = C_1 e^{i\cdot at} + C_2 e^{-i\cdot at}$$

($C_1, C_2 \in \mathbb{R}$- arbitrary constants; i- imaginary unit). Now from (2.1) we find $C_1 = C_2 = \frac{1}{2}$, thus the solutions of the differential equation (2.6), i.e. functional equations (2.1) and (2.2), are all real functions $y = \cosh at$ and $f(t) = \cos at$ ($a \in \mathbb{R}; a \neq 0$, $t \in \mathbb{R}$). Notice that $f(t) \equiv 1$, ($t \in \mathbb{R}$) which corresponds to mentioned functions for $a = 0$, is also a solution of the equations (2.1) and (2.2), as well as a solution of the equation (2.6) for $a = 0$.

3. A new definition of general cosine operator function in Banach space

Let E be a Banach space and let $C(t)\,(t \in \mathbb{R})$ be a family of operators from E to E, that are all even functions, at least two times continuously differentiable at the variable t on E. Suppose that there exists a linear, bounded and closed operator $A : E \to E$ with domain $D(A)$ which commutes with all operators $C(t)\,(t \in \mathbb{R})$ (i.e. $AC(t)x = C(t)Ax$ for every $x \in D(A)$) and such that for every $x \in E$ it holds:

$$2C^2(t)x = x + C(2t)x \quad (t \in \mathbb{R}) \tag{3.1}$$

and

$$C(t+s)x = C(t)C(s)x + A\int_0^t C(u)\,du \int_0^s C(v)x\,dv \quad (t, s \in \mathbb{R}). \tag{3.2}$$

From (3.2) for every $x \in E$ we have $C(2t)x = C^2(t)x + A\left(\int_0^t C(u)\,du\right)^2 x$, which together with (3.1) for every $x \in E$, gives

$$C^2(t)x - x = A\left(\int_0^t C(u)\,du\right)^2 x. \tag{3.3}$$

Let us denote:

$$E^k = \{x \in E : C(\cdot)x \text{ is } k\text{ -times continuously differentiable function from } \mathbb{R} \text{ to } E\}$$

($k = 1, 2$). Then by differentiating left and right side of the equation (3.3) with respect to t, we find that for all $x \in E^1$ it holds

$$2C(t)C'(t)x = 2AC(t)\left(\int_0^t C(u)x\,du\right). \tag{3.4}$$

Since, by assumption, we have $AC(t) = C(t)A$, we may (3.4) rewrite in the following form

$$2C(t)\left[C'(t)x - A\left(\int_0^t C(u)x\,du\right)\right] = 0 \quad (x \in E^1). \tag{3.5}$$

A trivial solution of the equation (3.5) is $C(t) \equiv 0\,(t \in \mathbb{R})$, but it is not the solution of the equation (3.1) and nontrivial solutions of the equation (3.5) are got from the equation

$$C'(t)x - A\left(\int_0^t C(u)x\,du\right) = 0 \quad (x \in E^1), \tag{3.6}$$

from which, by differentiating with respect to t, we get

$$C''(t)x = AC(t)x \quad (x \in E^2). \tag{3.7}$$

Because of linearity of an operator A, from (3.6) it also follows that $C'(0)x = 0\,(x \in E^1)$. As the operator $C(t)\,(t \in \mathbb{R})$ is even, from (3.2) it follows that for every $x \in E$,

$$C(t-s)x = C(t)C(s)x - A\int_0^t C(u)\,du\int_0^s C(v)x\,dv$$

and summing this with (3.2) yields

$$C(t+s)x + C(t-s)x = 2C(t)C(s)x, \tag{3.8}$$

for every $x \in E$, i.e. the family of operators $C(t)\,(t \in \mathbb{R})$ satisfies D'Alembert functional equation. Specially for $t = s = 0$ from (3.8) we have $2C(0)x = 2C^2(0)x$ for every $x \in E$. Since the solution $C(0) = 0$ does not satisfy the equation (3.1), we find $C(0) = I$. Therefore, the solution of the functional equations which satisfies the functional equations (3.1) and (3.2), with other above mentioned assumptions for the family of the operators $C(t)\,(t \in \mathbb{R})$ and operator A, satisfies also the condition $C(0) = I$, where I is an identity operator on Banach space E. Hence now from (3.7) it follows

$$C''(0)x = Ax \quad (x \in E^2). \tag{3.9}$$

From (3.9) we infer

$$D(A) = D(C''(0)) = E^2 \quad \text{and} \quad A = C''(0). \tag{3.10}$$

In paper [4] the cosine operator functions $C(t)\,(t \in \mathbb{R})$ have been studied as solutions of the equations (1.1) and (1.3) and there is defined that for every $x \in D(A)$ it holds $Ax = 2\lim_{h\to 0}\frac{C(h)-I}{h^2}x$ and it is proved that $A = C''(0)$, i.e. (3.10) holds. It is also proved that operator A is a linear and closed operator and it commutes with operators $C(t)\,(t \in \mathbb{R})$. Therefore, operator A satisfies above mentioned assumptions. This operator is called infinitesimal generator of cosine family of operators $C(t)\,(t \in \mathbb{R})$. Hence, we may give the following definition of such family of operators.

Definition 1. *An one-parameter family of linear and bounded operators $C(t)\,(t \in \mathbb{R})$, which are even and at least two times continuously differentiable functions in Banach space E and satisfying on E the functional equations*

$$2f^2(t) = 1 + f(2t)\,(t \in \mathbb{R})$$

and

$$f(t+s) = f(t)f(s) + A\int_0^t f(u)\,du \cdot \int_0^s f(v)\,dv\,(t, s \in \mathbb{R}),$$

where $A : E \to E$ is an operator with domain $D(A) = D(C''(0)) = E^2$ and $A = C''(0)$, is called cosine operator function in Banach space E and operator A is called its infinitesimal generator. If in addition, it values $\lim_{t\to 0} C(t)x = x$ for every $x \in E$, where we assume that limit is taken in a strong operator topology, then we say that $C(t)\,(t \in \mathbb{R})$ is a strongly continuous or C_0 cosine operator function in E.

Remark 1. *If we compare the equations* (2.2) *and* (3.2) *we may see that instead of number* $\pm a^2$ *in* (2.2)*, there is an operator* A *in* (3.2)*. That is so because* $Ax = C''(0)x = 2\lim_{h\to 0}\frac{C(h)-I}{h^2}x$*, specially for real functions* $y = \cos at$ *and* $y = \cosh at$ $(a \in \mathbb{R}; a \neq 0, t \in \mathbb{R})$*, it gives*

$$2\lim_{h\to 0}\frac{\cos ah - 1}{h^2} = -a^2 \quad \textit{and} \quad 2\lim_{h\to 0}\frac{\cosh ah - 1}{h^2} = a^2.$$

Remark 2. *One may notice from above deductions that cosine operator function* $C(t)$ $(t \in \mathbb{R})$ *in Banach space* E*, defined by Definition 1, is a solution of the Cauchy problem*

$$\begin{array}{lll} C''(t)x & = AC(t)x & (x \in E^2) \\ C(0)x & = x & (x \in E) \\ C'(0)x & = 0 & (x \in E^1), \end{array}$$

where $A = C''(0)$ *is a linear, bounded and closed operator on* $D(A) = E^2 \subseteq E$*. In* [1] *it is proved that* $D(A)$ *is everywhere dense set in* E*.*

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