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FURTHER RESULTS ON CONVOLUTIONS INVOLVING GAUSSIAN ERROR FUNCTION $erf(|x|^{1/2})$

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ABSTRACT. In the paper we consider some convolutions, which are related to results of [7], of the Gassuian error function and pseudo-functions.

1. INTRODUCTION

Recall definition of the error function (also called Gaussian error function) $erf(x)$ [16] defined for $x \in \mathbb{R}$ by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)} x^{2i+1}.$$

The error function $erf(x)$ is odd, convex on $(-\infty, 0]$, concave on $[0, \infty)$ and strictly increasing on $\mathbb{R}$. The reader refer to [1, 2] for the other properties of the error function.

The error function is a special function of sigmoid shape that occurs in probability, statistics and partial differential equations describing diffusion. It plays an important roles in problems from mathematical physics, in analytic solutions for problems of thermomechanics and mass flow due to diffusion.

Dirschmid and Fischer defined the generalized Gaussian error function by

$$erf_i(x) = \frac{2}{\sqrt{\pi}} \int_0^x u^i e^{-u^2}\, du,$$

for $i \in \mathbb{N}$, [4] and it follows from the definition that

$$\begin{aligned}
&\lim_{x\to 0} erf_i(x) = 0,\\
&\lim_{x\to\infty} erf_i(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} u^i e^{-u^2}\, du = \frac{1}{\sqrt{\pi}}\Gamma\big(\frac{i+1}{2}\big) = \frac{1}{\sqrt{\pi}}\Gamma\big(\frac{i+1}{2}\big),\\
&\lim_{x\to-\infty} erf_i(x) = \frac{2}{\sqrt{\pi}} \int_0^{-\infty} u^i e^{-u^2}\, du = \frac{(-1)^{i+1}}{\sqrt{\pi}}\Gamma\big(\frac{i+1}{2}\big).
\end{aligned}$$

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The locally summable functions $erf(x_+)$ and $erf(x_-)$ are defined by

$$erf(x_+) = H(x)erf(x), \quad erf(x_-) = H(-x)erf(x),$$

where H denotes Heaviside's function.

The functions $erf(|x|^{1/2}),\ erf(x_+^{1/2})$ and $erf(x_-^{1/2})$ are similarly defined by

$$erf(|x|^{1/2}) = \frac{2}{\sqrt{\pi}} \int_0^{|x|^{1/2}} \exp(-u^2)\, du,$$
$$erf(x_+^{1/2}) = H(x)erf(|x|^{1/2}), \quad erf(x_-^{1/2}) = H(-x)erf(|x|^{1/2}),$$

and the functions $erf_{2i}(x_+^{1/2})$ and $erf_{2i}(x_-^{1/2})$ by

$$erf_{2i}(x_+^{1/2}) = H(x)erf_{2i}(|x|^{1/2}), \quad erf_{2i}(x_-^{1/2}) = H(-x)erf_{2i}(|x|^{1/2}),$$

for $i = 0, 1, 2, \ldots$.

It is easy to prove the following equations, which we need for our results, by induction:

$$\begin{aligned} erf_{2i}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x u^{2i} e^{-u^2}\, du \\ &= -\sum_{j=0}^{i-1} \frac{(2i)!(i-j)!}{\sqrt{\pi}2^{2j}i!(2i-2j)!} x^{2i-2j-1} \exp(-x^2) + \frac{(2i)!}{2^{2i}i!} erf(x) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} erf_{2i+1}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x u^{2i+1} e^{-u^2}\, du \\ &= -\sum_{j=0}^{i} \frac{i!}{\sqrt{\pi}(i-j)!} x^{2i-2j} \exp(-x^2) + \frac{i!}{\sqrt{\pi}}, \end{aligned} \tag{1.2}$$

for $i = 0, 1, 2, \ldots$, where the sum in (1.2) is empty when $i = 0$.

In classical analysis one often deals with the convolution of two functions $f(x)$ and $g(x)$ defined as follows:

Definition 1.1. *Let f and g be functions. Then the convolution $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dt = \int_{-\infty}^{\infty} f(x-t)g(t)\, dt$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f;$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{1.3}$$

The following result was proved in [10]:

$$x_+^r * erf_+(x) = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i erf_i(x) x_+^{r-i+1},$$

for $r = 0, 1, 2, \ldots$.

Further, the next two results were proved in [7]:

$$x_+^r * erf(x_+^{1/2}) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i erf_{2i}(|x|^{1/2})x_+^{r-i+1} \tag{1.4}$$

and

$$x_+^r * [x_+^{-1/2}\exp(-|x|)] = \sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^i erf_{2i}(x_+^{1/2})x_+^{r-i}, \tag{1.5}$$

for $r = 0, 1, 2, \ldots$.

2. Results on Convolution

Consider Poission's classical formula in the theory of heat conduction

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} exp(-\frac{(x-\xi)^2}{4t})\mu(\xi)\,d\xi. \tag{2.1}$$

It is well known that $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{2.2}$$

with the initial condition $u(x,0) = \mu(x)$. Equation (2.1) can be written in the convolution form

$$u(x,t) = \mu(x) * \frac{1}{2\sqrt{\pi t}}exp(-\frac{x^2}{4t}),$$

which is a solution of the heat equation (2.2).

As seen in generalized functional analysis, the convolution is of great importance. In order to define the convolution of two distributions, we first let $\mathbb{D}$ be the space of infinitely differentiable functions with compact support and $\mathbb{D}'$ be the space of distributions defined on $\mathbb{D}$.

Definition 2.1. *The* convolution $f * g$ *of two distributions* f *and* g *in* $\mathbb{D}'$ *is defined by the equation*

$$\langle (f*g)(x), \varphi\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle$$

for arbitrary φ *in* $\mathbb{D}$*, provided* f *and* g *satisfy either of the conditions:*

(B1) *either* f *or* g *has bounded support,*

(B2) *the supports of* f *and* g *are bounded on the same side.*

See Gel'fand and Shilov [11] *(or* [12]*).*

Note that if f and g are locally summable functions satisfying either of the above conditions and the classical convolution $f*g$ exists, then it is in agreement with Definition 1.1.

Some attempts to define the convolution product of distributions have been made. The convolution product of distributions may be defined in a more general way without any restriction on the supports. The most well-known are given by Vladimirov and Jones, see [17, 14]. However, there still exist many pairs of distributions such that the convolution products do not exist in the sense of these definitions.

Using the concepts of the neutrix and the neutrix limit due to van der Corput [3], Fisher gave the general principle for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in context of distributions, particularly in connection with convolution product and distributional multiplication. See [5, 6, 8, 13, 15].

To recall the definition of neutrix convolution product given by Fisher, we first of all let τ be a function in $\mathbb{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The infinitely differentiable function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

Definition 2.2. *Let f and g be distributions in $\mathbb{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense that*

$$\underset{n\to\infty}{\mathrm{N-lim}}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle,$$

for all φ in $\mathbb{D}$, where N is the neutrix (see van der Corput [3]*) having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In this definition the convolution product $f_n * g$ exists since the distribution f_n has the bounded support and $\mathrm{supp}(\tau_n) \subset [-n - n^{-n}, n + n^{-n}]$. Note that because of the lack of symmetry in the definition of $f \circledast g$, the neutrix convolution is in general non-commutative.

Now let f and g be distributions in $\mathbb{D}'$ satisfying either condition $(B1)$ or condition $(B2)$ of Definition 2.1. Then it was proved in [6] that the neutrix convolution $f \circledast g$ exists and

$$f \circledast g = f * g.$$

This shows that the neutrix convolution is a generalization of the convolution.

It was also proved that if f and g are distributions in $\mathbb{D}'$ and $f \circledast g$ exists, then the neutrix convolution $f \circledast g'$ exists and

$$(f \circledast g)' = f \circledast g'.$$

Note however that equation (1.3) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$. However we have the following lemma which was proved in [6].

Lemma 2.1. *Let f and g be distributions in $\mathbb{D}'$ and suppose that $f \circledast g$ exists. If $\underset{n\to\infty}{\mathrm{N-lim}}\langle (f\tau_n') * g, \varphi\rangle$ exists and equals $\langle h, \varphi\rangle$ for all $\varphi \in \mathbb{D}$, then the neutrix convolution $f' \circledast g$ exists and*

$$(f \circledast g)' = f' \circledast g + h.$$

In order to prove our next results we need to extend our set of negligible functions given in Definition 2.2 to also include finite linear sums of the function

$$n^r erf[(x+n)^{1/2})], \qquad r = 1, 2, \ldots.$$

The following results were proved in [7]:

$$x^r \circledast erf(x_+^{1/2}) = \frac{1}{(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(-1)^i (2i)!}{2^{2i} i!} x^{r-i+1},$$

$$x^r \circledast [x_+^{-1/2} \exp(-|x|)] = \frac{\sqrt{\pi}}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(-1)^i (r-i+1)(2i)!}{2^{2i} i!} x^{r-i}$$

for $r = 0, 1, 2, \ldots$.

We now prove

Theorem 2.1. *The neutrix convolution* $erf(x_+^{1/2}) \circledast x^r$ *exists and*

$$erf(x_+^{1/2}) \circledast x^r = \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} \frac{(-1)^i (2i)!}{2^{2i} i!} x^{r-i+1} \tag{2.3}$$

for $r = 0, 1, 2, \ldots$.

Proof. We put $[erf(x_+^{1/2})]_n = erf(x_+^{1/2})\tau_n(x)$ for $n = 0, 1, 2, \ldots$. Since $[erf(x_+^{1/2})]_n$ has compact support, the convolution $[erf(x_+^{1/2})]_n * x^r$ exists and

$$\begin{aligned} \frac{\sqrt{\pi}}{2}[erf(x_+^{1/2})]_n * x^r &= \int_0^n (x-t)^r erf(t^{1/2})\, dt \\ &+ \int_n^{n+n^{-n}} (x-t)^r \tau_n(t) erf(t^{1/2})\, dt \\ &= I_1 + I_2. \end{aligned} \tag{2.4}$$

Now

$$\Big| \int_n^{n+n^{-n}} (x-t)^r \tau_n(t) erf(t^{1/2})\, dt \Big| \leq (x+n)^r n^{-n} e^{-n}$$

and so

$$\lim_{n\to\infty} I_2 = 0. \tag{2.5}$$

Further

$$\begin{aligned} I_1 &= \int_0^n (x-t)^r \int_0^{t^{1/2}} \exp(-u^2)\, du\, dt \\ &= \int_0^{n^{1/2}} \exp(-u^2) \int_{u^2}^n (x-t)^r\, dt\, du \\ &= \frac{1}{r+1} \int_0^{n^{1/2}} \big[(x-u^2)^{r+1} - (x-n)^{r+1}\big] \exp(-u^2)\, du \\ &= \frac{\sqrt{\pi}}{2(r+1)} \sum_{i=1}^{r+1} \binom{r+1}{i} (-1)^i x^{r-i+1} erf_{2i}(n^{1/2}) \\ &- \frac{\sqrt{\pi}}{2(r+1)} \sum_{i=1}^{r+1} \binom{r+1}{i} (-1)^i x^{r-i+1} n^i erf(n^{1/2}). \end{aligned}$$

It follows from equation (1.1) and the $n^i erf(n^{1/2})$ being negligible functions that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} I_1 &= \frac{\sqrt{\pi}}{2(r+1)}\sum_{i=1}^{r+1}\binom{r+1}{i}(-1)^i x^{r-i+1} erf_{2i}(\infty) \\ &= \frac{\sqrt{\pi}}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}\frac{(-1)^i(2i)!}{2^{2i+1}i!}x^{r-i+1}, \end{aligned} \tag{2.6}$$

on noting that $erf(\infty) = 1$. Equation (2.3) now follows from equations (2.4) to (2.6). □

Replacing x by $-x$ in equation (2.3), we get

Corollary 2.1. *The neutrix convolution* $erf(x_-^{1/2}) \circledast x^r$ *exists and*

$$erf(x_-^{1/2}) \circledast x^r = -\frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}\frac{(2i)!}{2^{2i}i!}x^{r-i+1} \tag{2.7}$$

for $r = 0, 1, 2, \ldots$.

Noting that

$$erf(|x|^{1/2}) = erf(x_+^{1/2}) + erf(x_-^{1/2})$$

and using equations (2.3) and (2.7), we get

Corollary 2.2. *The neutrix convolution* $erf(|x|^{1/2}) \circledast x^r$ *exists and*

$$erf(|x|^{1/2}) \circledast x^r = \frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}\frac{[(-1)^i - 1](2i)!}{2^{2i}i!}x^{r-i+1} \tag{2.8}$$

for $r = 0, 1, 2, \ldots$.

Corollary 2.3. *The neutrix convolution* $erf(|x|^{1/2}) \circledast x_+^r$ *exists and*

$$\begin{aligned} erf(|x|^{1/2}) \circledast x_+^r &= \frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}(-1)^i erf_{2i}(|x|^{1/2})[x_+^{r-i+1} - (-1)^r x_-^{r-i+1}] \\ &\quad - \frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}\frac{(2i)!}{2^{2i}i!}x^{r-i+1} \end{aligned} \tag{2.9}$$

for $r = 0, 1, 2, \ldots$.

Proof. Note that

$$\begin{aligned} erf(|x|^{1/2}) \circledast x_+^r &= erf(x_+^{1/2}) * x_+^r + erf(x_-^{1/2}) \circledast x_+^r \\ &= erf(x_+^{1/2}) * x_+^r + erf(x_-^{1/2}) \circledast x^r - (-1)^r erf(x_-^{1/2}) \circledast x_-^r. \end{aligned}$$

Then replacing x by $-x$ in equation (2.8), we get

$$x_-^r * erf(x_-^{1/2}) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i erf_{2i}(|x|^{1/2})x_-^{r-i+1} \tag{2.10}$$

and using equations (1.4), (2.7) and (2.10), we get

$$\begin{aligned} erf(|x|^{1/2}) \circledast x_+^r = {} & \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i erf_{2i}(|x|^{1/2})x_+^{r-i+1} \\ & - \frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}\frac{(2i)!}{2^{2i}i!}x^{r-i+1} \\ & - \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r+i}erf_{2i}(|x|^{1/2})x_-^{r-i+1}. \end{aligned}$$

Equation (2.9) follows. □

Replacing x by $-x$ in equation (2.9), we get

Corollary 2.4. *The neutrix convolution* $erf(|x|^{1/2}) \circledast x_-^r$ *exists and*

$$\begin{aligned} erf(|x|^{1/2}) \circledast x_-^r = {} & \frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}(-1)^i erf_{2i}(|x|^{1/2})[x_-^{r-i+1} - (-1)^r x_+^{r-i+1}] \\ & + \frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}\frac{(-1)^{r-i}(2i)!}{2^{2i}i!}x^{r-i+1} \end{aligned}$$

for $r = 0, 1, 2, \ldots$.

Theorem 2.2. *The neutrix convolution* $[x_+^{-1/2}\exp(-|x|)] \circledast x^r$ *exists and*

$$[x_+^{-1/2}\exp(-|x|)] \circledast x^r = \frac{\sqrt{\pi}r}{r+1}\sum_{i=1}^{r}\binom{r+1}{i}\frac{(-1)^i(r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \tag{2.11}$$

for $r = 1, 2, \ldots$ *and*

$$[x_+^{-1/2}\exp(-|x|)] \circledast 1 = 0. \tag{2.12}$$

Proof. Differentiating the convolution $[erf(x_+^{1/2})]_n * x^r$, we get

$$\begin{aligned} \frac{1}{\sqrt{\pi}}[x_+^{-1/2}\exp(-|x|)\tau_n(x)] * x^r + [erf(|x|^{1/2})\tau_n'(x)] * x^r & \\ = r[erf(x_+^{1/2})]_n * x^{r-1}, & \end{aligned} \tag{2.13}$$

for $r = 1, 2, \ldots$.

Now

$$\begin{aligned} \left|[erf(|x|^{1/2})\tau_n'(x)] * x^r\right| &= \left|\int_n^{n+n^{-n}}(x-t)^r erf(|t|^{1/2})\tau_n(t)\,dt\right| \\ &\le (x+n)^r n^{-n}e^{-n} \end{aligned}$$

and so

$$\lim_{n\to\infty}[erf(|x|^{1/2})\tau_n'(x)] * x^r = 0. \tag{2.14}$$

It now follows from equations (2.3), (2.13) and (2.14) that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}[x_+^{-1/2}\exp(-|x|)\tau_n(x)]*x^r &= \sqrt{\pi}rerf(x_+^{1/2})\circledast x^{r-1}\\ &= \frac{\sqrt{\pi}r}{r+1}\sum_{i=1}^{r}\binom{r+1}{i}\frac{(-1)^i(r-i+1)(2i)!}{2^{2i}i!}x^{r-i},\end{aligned}$$

for $r=1,2,\ldots$, proving equation (2.11).

When $r=0$, equation (2.13) is replaced by

$$\frac{1}{\sqrt{\pi}}[x_+^{-1/2}\exp(-|x|)\tau_n(x)]*1+[erf(|x|^{1/2})\tau_n'(x)]*1=0,$$

and it follows that

$$\frac{1}{\sqrt{\pi}}\operatorname*{N-lim}_{n\to\infty}[x_+^{-1/2}\exp(-|x|)\tau_n(x)]*1=\frac{1}{\sqrt{\pi}}[x_+^{-1/2}\exp(-|x|)]\circledast 1=0,$$

proving equation (2.12). □

Replacing x by $-x$ in equations (2.11) and (2.12), we get

Corollary 2.5. *The neutrix convolution* $[x_-^{-1/2}\exp(-|x|)]\circledast x^r$ *exists and*

$$[x_-^{-1/2}\exp(-|x|)]\circledast x^r=\frac{\sqrt{\pi}r}{r+1}\sum_{i=1}^{r}\binom{r+1}{i}\frac{(r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \tag{2.15}$$

for $r=1,2,\ldots$ *and*

$$[x_-^{-1/2}\exp(-|x|)]\circledast 1=0. \tag{2.16}$$

Noting that

$$|x|^{-1/2}\exp(-|x|)=x_+^{-1/2}\exp(-|x|)+x_-^{-1/2}\exp(-|x|)$$

and using equations (2.11), (2.12), (2.15) and (2.16), we get

Corollary 2.6. *The neutrix convolution* $[|x|^{-1/2}\exp(-|x|)]\circledast x^r$ *exists and*

$$[|x|^{-1/2}\exp(-|x|)]\circledast x^r=\frac{\sqrt{\pi}r}{r+1}\sum_{i=1}^{r}\binom{r}{i}\frac{[1+(-1)^i](r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \tag{2.17}$$

for $r=1,2,\ldots$ *and*

$$[|x|^{-1/2}\exp(-|x|)]\circledast 1=0. \tag{2.18}$$

Corollary 2.7. *The neutrix convolution* $[|x|^{-1/2}\exp(-|x|)]\circledast x_+^r$ *exists and*

$$\begin{aligned}[|x|^{-1/2}\exp(-|x|)]\circledast x_+^r &= \sqrt{\pi}\sum_{i=1}^{r}\binom{r}{i}\frac{(2i)!}{2^{2i}i!}x^{r-i}\\ &\quad+\sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^i erf_{2i}(|x|^{1/2})x_+^{r-i}\\ &\quad-\sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r+i}erf_{2i}(|x|^{1/2})x_-^{r-i}\end{aligned} \tag{2.19}$$

for $r=1,2,\ldots$ *and*

$$[|x|^{-1/2}\exp(-|x|)]\circledast H(x)=0. \tag{2.20}$$

Proof. Note that

$$\begin{aligned}[|x|^{-1/2}\exp(-|x|)] \circledast x_+^r &= [x_+^{-1/2}\exp(-|x|)] * x_+^r + [x_-^{-1/2}\exp(-|x|)] \circledast x_+^r \\ &= [x_+^{-1/2}\exp(-|x|)] * x_+^r + [x_-^{-1/2}\exp(-|x|)] \circledast x^r \\ &- (-1)^r[x_-^{-1/2}\exp(-|x|)] * x_-^r\end{aligned}$$

and replacing x by $-x$ in equation (1.5), we get

$$x_-^r * [x_-^{-1/2}\exp(-|x|)] = \sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^i erf_{2i}(x_-^{1/2})x_-^{r-i}. \tag{2.21}$$

Then

$$\begin{aligned}[|x|^{-1/2}\exp(-|x|)] \circledast x_+^r &= \sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^i erf_{2i}(x_+^{1/2})x_+^{r-i} \\ &+ \frac{\sqrt{\pi}r}{r+1}\sum_{i=1}^{r}\binom{r+1}{i}\frac{(r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \\ &- \sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i} erf_{2i}(x_-^{1/2})x_-^{r-i},\end{aligned}$$

on using equations (1.5), (2.15) and (2.21). Equation (2.19) follows for $r = 1, 2, \ldots$.
Equation (2.20) follows similarly. □

In the particular case $r = 0$, we have

$$[|x|^{-1/2}\exp(-|x|)] \circledast H(x) = \sqrt{\pi}\,\mathrm{sgn}(x) erf(|x|^{1/2}),$$

on using equations (2.17), (2.18) and (2.19).

For further related results, see [9, 10].

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