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A FAMILY OF ITERATION FUNCTIONS WITH INFINITELY MANY MEMBERS OF FIFTH AND SIXTH ORDER

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ABSTRACT. We present a family of methods for solving nonlinear equations. Convergence analysis shows that our family contains methods of convergence order from 3 to 6, one method of order 3, three methods of order 4 and infinitely many methods of order 5 and 6. The methods require only four function evaluations per iteration. In this regard, we consider a subset of sixth order methods, the efficiency index of which is $\sqrt[4]{6} \approx 1.56508$. Numerical examples, obtained using Mathematica with high precision arithmetic, are included to demonstrate convergence and efficacy of our methods. For some tested examples the new sixth order methods produce very good results, compared to the results produced by some of the sixth order methods existing in the related literature.

1. INTRODUCTION

Let α be a simple zero of a sufficiently smooth function f, let J be an open interval containing α and assume that f' does not vanish in the interval of interest J. It is known that the classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, ...$$

converges quadratically to a simple zero for some starting values $x_0 \in J$. To get a method with a higher order of convergence, some new variants of Newton's method have been proposed. We observe that many of the sixth order methods have been obtained by extension of another method, [4], [7], [8], [12], [15]. In [14] a sixth order method is given which is extending a third order method from [13]. In [4] sixth order method is developed by extending third order modifications of Newton's method based on Stolarsky and Gini means from [5]. Some well known methods belong to the family of methods from [5], for example, the Arithmetic mean Newton's method from [13], the Harmonic mean Newton's method [1] and the Power mean Newton's method from [16]. The aim

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of this paper is to develop a family of methods by using Padé approximations. The family of methods we consider is

$$x_{n+1} = \Phi_{k,m,p,q}(x_n), \quad n = 0, 1, ... \tag{1.1}$$

with

$$\Phi_{k,m,p,q}(x) = F_{k,m}(x) - \frac{f(F_{k,m}(x))}{f'(x)}(2\psi_{p,q}(z(x)) - 1), \quad k,m,p,q = 0,1,\dots \tag{1.2}$$

and

$$F_{k,m}(x) = x - \frac{f(x)}{f'(x)}\varphi_{k,m}(z(x)), \quad k,m = 0,1,\dots \tag{1.3}$$

where functions $\varphi_{k,m}$ and $\psi_{p,q}$ are Padé approximations of order (k,m) and (p,q) to the function $2/\left(1+\sqrt{1-2z}\right)$ at 0 and z is a function for which (1.4) holds true.

$$\begin{aligned} z(\alpha) &= 0, \\ z'(\alpha) &= 2c_2, \\ z''(\alpha) &= 4\left(-3c_2^2 + 2c_3\right), \\ z'''(\alpha) &= 12\left(8c_2^3 - 10c_2c_3 + \varepsilon c_4\right), \end{aligned} \tag{1.4}$$

where ε is a real number and

$$c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}, \quad j = 1,2,\dots$$

Let us note that Padé approximations in (1.2) and (1.3) can be different. We can organize the considered Padé approximations into a table. Here we give Padé approximations for $k,m = 0,1,2,3$ only in Table 1.

TABLE 1. Padé approximations for $k,m = 0,1,2,3$

$k\backslash m$	0	1	2	3
0	1	$\frac{-2}{z-2}$	$\frac{-4}{z^2+2z-4}$	$\frac{-4}{z^3+z^2+2z-4}$
1	$\frac{z}{2}+1$	$\frac{z-2}{2(z-1)}$	$\frac{4-4z}{z^2-6z+4}$	$\frac{16-20z}{z^3+6z^2-28z+16}$
2	$\frac{z^2}{2}+\frac{z}{2}+1$	$\frac{z^2+6z-8}{10z-8}$	$\frac{z^2-6z+4}{3z^2-8z+4}$	$\frac{-2\left(3z^2-8z+4\right)}{z^3-12z^2+20z-8}$
3	$\frac{5z^3}{8}+\frac{z^2}{2}+\frac{z}{2}+1$	$\frac{3z^3+8z^2+36z-40}{8(7z-5)}$	$\frac{z^3+12z^2-44z+24}{28z^2-56z+24}$	$\frac{z^3-12z^2+20z-8}{4(z^3-5z^2+6z-2)}$

In Table 2 we present convergence orders of methods defined by (1.1), (1.2) and (1.3) depending on the parameters k, m, p, q if $k+m \leq 1$. In cases where $k+m > 1$ all methods have order 5 if $p+q = 0$, and order 6 if $p+q > 0$.

TABLE 2. Convergence order

	$p+q=0$		$p+q>0$	
$k\backslash m$	0	1	0	1
0	3	4	4	5
1	4		5	

In terms of computational cost, our methods require only four function evaluations per iteration. A common proof of convergence, convergence order and a common asymptotic error constant are provided for all methods. Numerical results are also provided.

2. Main results

Let us consider the family of iterative methods defined by (1.1), (1.2) and (1.3). For the sake of simplicity, let us omit the indices k, m, p and q by φ, ψ and F. Convergence analysis shows that our family contains methods of convergence order from 3 to 6, one method of order 3, 3 methods of order 4 and infinitely many methods of order 5 and 6.

Theorem 2.1. *Let $f : J \to R$ and let f have a sufficient number of continuous derivatives in J. Assume that α is a simple zero of f in J. Then the family of methods* (1.1) *has order of convergence*

$$ord_{k,m,p,q} = \begin{cases} \min\{k+m+3,5\}, & p+q=0 \\ \min\{k+m+4,6\}, & p+q>0 \end{cases} \tag{2.1}$$

and the corresponding asymptotic error constant is

$$E_{k,m,p,q} = E_{k,m} \begin{cases} R_{p,q}, & k+m=0 \\ Q_{p,q}, & k+m>0 \end{cases} \tag{2.2}$$

where $E_{k,m}$ for $k+m>2$ is

$$\delta = -c_2c_3 + (3-\varepsilon)\, c_4 \tag{2.3}$$

for $k+m \le 2$ $E_{k,m}$ is given in Table 3, and $R_{p,q}$ and $Q_{p,q}$ are given in Table 4.

TABLE 3. Coefficients $E_{k,m}$ for $k+m \le 2$

$k\backslash m$	0	1	2
0	c_2	c_2^2	$2c_2^3+\delta$
1	$2c_2^2$	$c_2^3+\delta$	
2	$5c_2^3+\delta$		

TABLE 4. Coefficients $R_{p,q}, Q_{p,q}$

	$R_{p,q}$	$Q_{p,q}$
$p+q=0$	$2c_2$	$2c_2$
$p=1,\ q=0$	$5c_2^2-c_3$	$6c_2^2-c_3$
$p=0,\ q=1$	$3c_2^2-c_3$	$4c_2^2-c_3$
$p+q>1$	$c_2^2-c_3$	$2c_2^2-c_3$

Proof. The proof will be given by observing particular cases for $p+q=0$ and $p+q>0$. In each case, we observe the subcases where $k+m=0$ and $k+m>0$. Firstly we present some properties of the functions F and φ (the functions ψ have the same properties as φ) which are easy to perform and apply in all cases that we observe. So, as shown in [6], the relations (2.4)– (2.11) are true:

$$\varphi(0)=1,\quad \varphi'(0)=\frac{1}{2}\begin{cases}0, & k+m=0\\ 1, & k+m>0\end{cases},\quad \varphi''(0)=\frac{1}{2}\begin{cases}0, & k=0,1,\ \ m=0\\ 1, & k=0,\ \ m=1\\ 2, & k+m>1\end{cases}\tag{2.4}$$

$$\varphi'''(0)=\frac{1}{4}\begin{cases}0 & k=0,1,2,\ \ m=0\\ 3 & k=0,\ \ m=1\\ 12 & k=1,\ \ m=1\\ 9 & k=0,\ \ m=2\\ 15 & k+m>2\end{cases}\tag{2.5}$$

$$\begin{aligned}&F(\alpha)=\alpha,\quad F'(\alpha)=0,\quad F''(\alpha)=2c_2\left(1-2\varphi'(0)\right)\\ &F'''(\alpha)=12\left(c_3\left(1-2\varphi'(0)\right)+c_2^2\left(-1+4\varphi'(0)-\varphi''(0)\right)\right)\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}F^{(4)}(\alpha)&=24c_4\left(3-2\varepsilon\varphi'(0)\right)+24c_2c_3\left(-7+28\varphi'(0)-8\varphi''(0)\right)\\ &+16c_2^3\left(6-39\varphi'(0)+21\varphi''(0)-2\varphi'''(0)\right)\end{aligned}\tag{2.7}$$

From (2.4) and (2.6) we obtain

$$\frac{F''(\alpha)}{2}=c_2\begin{cases}1, & k+m=0\\ 0, & k+m>0\end{cases}\tag{2.8}$$

and

$$\frac{F'''(\alpha)}{6}=c_2^2\begin{cases}2, & k=1,\ \ m=0\\ 1, & k=0,\ \ m=1\\ 0, & k+m>1\end{cases}\tag{2.9}$$

The case $k+m>1$ is considered separately. From (2.4) and (2.7) we obtain

$$\frac{F^{(4)}(\alpha)}{24}=\delta+c_2^3\left(5-\frac{4}{3}\varphi'''(0)\right)\tag{2.10}$$

and depending on $\varphi'''(0)$, using (2.5),

$$\frac{F^{(4)}(\alpha)}{24}=\delta+c_2^3\begin{cases}5 & k=2,\ \ m=0\\ 1 & k=1,\ \ m=1\\ 2 & k=0,\ \ m=2\\ 0 & k+m>2\end{cases}\tag{2.11}$$

Now we give the proof of the theorem.

Case I $p+q=0$, i.e. $p=q=0$. In this case $\psi(z)=1$, $\psi'(0)=0$ and

$$\Phi_{k,m,0,0}(x)=F_{k,m}(x)-\frac{f\left(F_{k,m}(x)\right)}{f'(x)},\quad k,m=0,1,\ldots$$

Using (2.6) we obtain

$$\Phi(\alpha)=\alpha,\qquad \Phi'(\alpha)=0,\qquad \Phi''(\alpha)=2c_2\left(2-F'(\alpha)\right)F'(\alpha)=0\tag{2.12}$$

$$\Phi'''(\alpha) = 6c_2F''(\alpha), \qquad \Phi^{(4)}(\alpha) = 8c_2F^{(3)}(\alpha), \qquad \Phi^{(5)}(\alpha) = 10c_2F^{(4)}(\alpha) \tag{2.13}$$

Subcase I $k+m=0$, i.e. $k=m=0$. In this subcase $\varphi(z)=1$, and it holds that $\varphi'(0)=\varphi''(0)=\varphi'''(0)=0$, thus from (2.13) and (2.8) it follows

$$\frac{\Phi'''(\alpha)}{6} = c_2F''(\alpha) = 2c_2^2.$$

Therefore, the order of convergence in this subcase is 3, i.e. (2.1) holds true. The asymptotic error constant is $E_{0,0,0,0} = 2c_2^2 = c_2 \cdot 2c_2 = E_{0,0}R_{0,0}$, i.e. (2.2) holds true. The theorem is proved in this subcase.

Subcase II $k+m>0$. From (2.13) and (2.8) it follows

$$\frac{\Phi'''(\alpha)}{6} = c_2F''(\alpha) = 0.$$

Since

$$\Phi^{(4)}(\alpha) = 8c_2F^{(3)}(\alpha) \ ,$$

because of (2.9) it follows

$$\frac{\Phi^{(4)}(\alpha)}{24} = 2c_2^3 \begin{cases} 2, & k=1, \ m=0 \\ 1, & k=0, \ m=1 \\ 0, & k+m>1 \end{cases} .$$

So, for $k=1, \ m=0$ and for $k=0, \ m=1$ the order of convergence is 4, i.e. (2.1) holds true, and asymptotic error constant is $E_{1,0,0,0} = 4c_2^3 = E_{1,0}Q_{0,0} = 2c_2^2 \cdot 2c_2$, i.e. $E_{0,1,0,0} = 2c_2^3 = c_2^2 \cdot 2c_2 = E_{0,1}Q_{0,0}$, i.e. (2.2) holds. For $k+m>1$ it holds $\Phi^{(4)}(\alpha)=0$ and the order of convergence is at least 5. Since

$$\frac{\Phi^{(5)}(\alpha)}{120} = 2c_2\frac{F^{(4)}(\alpha)}{24} \tag{2.14}$$

because of (2.1) we have order of convergence 5, i.e. (2.1) is true and (2.2) holds.

Case II $p+q>0$.

Subcase I $k+m=0$, i.e. $k=m=0$. Using (2.6) we obtain

$$\begin{gathered} \Phi(\alpha)=\alpha, \quad \Phi'(\alpha)=0, \quad \Phi''(\alpha) = -2(-1+\psi(0))F''(\alpha), \\ \Phi'''(\alpha) = 2\left(3c_2(-1+2\psi(0)-2\psi'(0))F''(\alpha) - (-1+\psi(0))F'''(\alpha)\right) \end{gathered} . \tag{2.15}$$

Because of (2.4) it holds that $\psi(0)=1$ and $\psi'(0)=\frac{1}{2}$. Now from (2.15) we obtain

$$\Phi(\alpha)=\alpha, \quad \Phi'(\alpha)=\Phi''(\alpha)=\Phi'''(\alpha)=0 \tag{2.16}$$

and

$$\Phi^{(4)}(\alpha) = -6F''(\alpha)\left(2c_3 + c_2F''(\alpha) + 4c_2^2(-3+2\psi''(0))\right) \tag{2.17}$$

For $k+m=0$ because of (2.8) it holds that $F''(\alpha)=2c_2$ and

$$\Phi^{(4)}(\alpha) = 24c_2\left(-c_3 + c_2^2(5-4\psi''(0))\right) .$$

Therefore, for $k+m=0$ the order of convergence is 4, i.e. (2.1) holds true, and asymptotic error constant is $E_{0,0}R_{p,q} = c_2R_{p,q}$, i.e., to attribute (2.4) that applies to the ψ we obtain

$$\frac{\Phi^{(4)}(\alpha)}{24} = c_2 \begin{cases} 5c_2^2 - c_3, & p=1, \ q=0 \\ 3c_2^2 - c_3, & p=0, \ q=1 \\ c_2^2 - c_3 & p+q>1 \end{cases} \tag{2.18}$$

i.e. (2.2) holds true.

Subcase II $k+m>0.$ Since for $k+m>0$ $F''(\alpha)=0$ it follows from (2.17) that $\Phi^{(4)}(\alpha)=0$ and the order of convergence is at least 5. Now, we have

$$\frac{\Phi^{(5)}(\alpha)}{120}=\left(-c_3+2c_2^2\left(3-2\psi''(0)\right)\right)\frac{F'''(\alpha)}{6} \tag{2.19}$$

Because of (2.9) we obtain

$$\frac{\Phi^{(5)}(\alpha)}{120}=c_2^2\left(-c_3+2c_2^2\left(3-2\psi''(0)\right)\right)\begin{cases}2, & k=1,\ m=0\\ 1, & k=0,\ m=1\\ 0, & k+m>1\end{cases}$$

So, for $k=1,\ m=0$ and for $k=0,\ m=1$ the order of convergence is 5, i.e. (2.1) is true, and asymptotic error constant is $E_{0,0}Q_{p,q}=2c_2^2\cdot Q_{p,q}$, i.e. $E_{0,1}Q_{p,q}=c_2^2\cdot Q_{p,q}$ so (2.2) holds true.

For $k+m>1$ because of (2) it follows that the order of convergence is at least 6, since $\Phi^{(5)}(\alpha)=0$ and

$$\frac{\Phi^{(6)}(\alpha)}{720}=\left(-c_3+2c_2^2\left(3-2\psi''(0)\right)\right)\frac{F^{(4)}(\alpha)}{24}$$

Since (2.4) holds for $\psi''(0)$, we obtain

$$\frac{\Phi^{(6)}(\alpha)}{720}=\frac{F^{(4)}(\alpha)}{24}\begin{cases}6c_2^2-c_3, & p=1,\ q=0\\ 4c_2^2-c_3, & p=0,\ q=1\\ 2c_2^2-c_3, & p+q>1\end{cases}$$

Bearing (2.11) in mind we see that for $k+m>1$ the statement of the theorem is true.

When selecting a function z it is important that it satisfies the conditions (1.4) and in addition does not require more than one additional evaluation of the function f or some of its derivatives. Then we would have a total of four function or derivative evaluations, as in (1.2) and (1.3) two functions evaluations and a computation of its first derivative are required.

The functions σ, λ and μ defined by

$$\sigma(x)=2\frac{f(x-u(x))}{f(x)}$$

$$\lambda(x)=\frac{3}{2}\left(1-\frac{f'\left(x-\frac{2}{3}u(x)\right)}{f'(x)}\right)$$

$$\mu(x)=\frac{f(x)}{f'(x)^2}f''\left(x-\frac{1}{3}u(x)\right)$$

where

$$u(x)=\frac{f(x)}{f'(x)}$$

can be seen as z since they meet (1.4) with $\varepsilon=3$, for $z=\sigma$, $\varepsilon=26/9$ for $z=\lambda$ and $\varepsilon=8/3$ for $z=\mu$, as shown in [6]. □

3. Numerical examples

The equation $f(x) = 0$ was solved using the following test functions with corresponding starting values x_0:

$f_1(x) = \frac{1}{2} - \sin x$, [8], $\alpha_1 * \approx 0.5235987755982988731$, $x_0 = 0.7$,
$f_2(x) = x^3 - 10$, [14], $\alpha_2 * \approx 2.1544346900318837218$, $x_0 = 2$,
$f_3(x) = 3x^2 - e^x$, [11], $\alpha_3 * \approx 0.9100075724887090607$, $x_0 = 2$,
$f_4(x) = x^3 + 4x^2 - 10$, [1], [11], [14], [16], $\alpha_4 * \approx 1.3652300134140968458$, $x_0 = 2$,
$f_5(x) = (x-1)^3 - 1$, [1], [14], [16], $\alpha_5 = 2$, $x_0 = 1.8$,
$f_6(x) = (x-1)^3 - 2$, [1], [11], [16], $\alpha_6^* \approx 2.259921049894873$, $x_0 = 2$,
$f_7(x) = \frac{x}{2} - \sin x$, $\alpha_7 * \approx 1.8954942670339809471$, $x_0 = 1.5$,
$f_8(x) = e^{x^2+7x-30} - 1$, [1], [8], [14], [16], $\alpha_8 = 3$, $x_0 = 2.8$,
$f_9(x) = x - \cos x$, [16], $\alpha_9^* \approx 0.7390851332151606$, $x_0 = 2$,
$f_{10}(x) = x^2 \sin x - \cos x$, [10], $\alpha_{10}^* \approx 0.8952060453842319$, $x_0 = 1.5$,
$f_{11}(x) = e^{-x} \sin x + \ln\left(x^2+1\right)$, [10], [11], $\alpha_{11} = 0$, $x_0 = 2$,
$f_{12}(x) = \arctan x$, [10], $\alpha_{12} = 0$, $x_0 = 1$,
$f_{13}(x) = \sin^2 x - x^2 + 1$, [14], $\alpha_{13} = 1.4044916482153412260$, $x_0 = 1$,
$f_{14}(x) = x^2 - e^x - 3x + 2$, [14], $\alpha_{14} = 0.25753028543986076046$, $x_0 = 3$,
$f_{15}(x) = \cos x - xe^x + x^2$, [14], $\alpha_{15} = 0.63915409633200758106$, $x_0 = 1$,
$f_{16}(x) = e^x - 1.5 + \arctan x$, [14], $\alpha_{16} = 0.23693335723885061990$, $x_0 = 1$,
$f_{17}(x) = 8x - \cos x - 2x^2$, [4], $\alpha_{17} = 0.12807710275379877853$, $x_0 = 1$,
$f_{18}(x) = x^{10} - 1$, [14], $\alpha_{18} = 1$, $x_0 = 0.8$,
$f_{19}(x) = (x-2)^{23} - 1$, [14], $\alpha_{19} = 3$, $x_0 = 3.5$,
$f_{20}(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, [14], $\alpha_{20} = -1.2076478271309189270$, $x_0 = -2$,
$f_{21}(x) = x^3 + 4x^2 - 10$, [1], [7], [11], [14], $\alpha_{21}^* \approx 1.3652300134140968458$, $x_0 = 1$,
$f_{22}(x) = \sin x - \frac{1}{3}$, [7], $\alpha_{22}^* \approx 2.2788626600758283127$, $x_0 = 2$,
$f_{23}(x) = e^{-x} + \cos x$, [7], $\alpha_{23}^* \approx 1.7461395304080124176$, $x_0 = 2$.

All computations were carried out in *Mathematica* 8. The precision of all numerical values was increased to 20000 digits with the `SetPrecision` function. The following stopping criterion was used: $|x_k - \alpha| < \varepsilon$ and $|f(x_k)| < \varepsilon$ where α is the exact solution of the considered equation. In cases where the exact solution was not available, we used the approximation α^*, which was also calculated with 20000 digits. For simplicity, only 20 digits are displayed.

The computational order of convergence (COC) was calculated for all test equations as

$$COC = \frac{\ln |(x_{n+1} - \alpha) / (x_n - \alpha)|}{\ln |(x_n - \alpha) / (x_{n-1} - \alpha)|}.$$

Results of numerical experiments confirm the theoretical results in terms of the rate of convergence and asymptotic error constant. Here we present only the results for the methods of order 6 and compare them with the results from [4]. In all cases $|COC - 6| \leq 10^{-5}$, for $n = 3, 4, \ldots$.

In [4], the same set of test functions and 17 methods were observed. In this paper we present the results obtained for some members of our family (1.1), denoted by PM, and the following methods: Newton's method (NM), the sixth order methods BN, given by [2], KM, given by Kou and Li [9], CM given by [3], the sixth order methods, RM1, RM2, RM3, RM4, RM5, and RM6 from [7]. All methods are considered with appropriate parameters, given in parentheses.

TABLE 5. Comparison of various iterative methods for f_{21}

Method	$\|x_n - \alpha\|$	$\|f(x_n)\|$
NM	$2.41(-44)$	$3.98(-43)$
KM	$6.33(-200)$	$1.05(-198)$
CM $(a = 1)$	$2.41(-211)$	$3.99(-210)$
RM1 $(\alpha = 1.66,\ \beta = 0.1,\ \gamma = 0.1)$	$1.31(-209)$	$2.16(-208)$
RM2 $(\alpha = -0.68,\ \beta = 0.1,\ \gamma = 0.1)$	$5.80(-248)$	$9.58(-247)$
RM3 $(\alpha = 0.08,\ \beta = 0.2,\ \gamma = 0.1)$	$1.84(-208)$	$3.04(-207)$
RM4 $(\alpha = 1.66,\ \beta = 0.1)$	$2.59(-212)$	$4.28(-211)$
RM5 $(\alpha = -0.72,\ \beta = 0.5)$	$5.02(-224)$	$8.29(-223)$
RM6 $(\alpha = 0.1,\ \beta = 0.1,\ \gamma = 0.2,\ \delta = 0.2)$	$1.71(-260)$	$2.83(-259)$
BN $(a = -0.5)$	$7.41(-179)$	$1.22(-177)$
PM $k = 3,\ m = 1,\ p = 0,\ q = 2$	$8.80(-214)$	$1.45(-212)$

TABLE 6. Comparison of various iterative methods for f_{22}

Method	$\|x_n - \alpha\|$	$\|f(x_n)\|$
NM	$4.27(-57)$	$4.21(-57)$
KM	$4.11(-202)$	$4.04(-202)$
CM $(a = 1)$	$2.52(-175)$	$2.48(-175)$
RM1 $(\alpha = 2,\ \beta = 1.2,\ \gamma = 1)$	$5.99(-223)$	$5.89(-223)$
RM2 $(\alpha = -0.7,\ \beta = 0.2,\ \gamma = 0)$	$4.21(-228)$	$4.14(-228)$
RM3 $(\alpha = 1,\ \beta = 0.2,\ \gamma = -1)$	$4.87(-240)$	$4.79(-240)$
RM4 $(\alpha = 1.7,\ \beta = 0.2)$	$1.17(-237)$	$1.15(-237)$
RM5 $(\alpha = -0.7,\ \beta = -0.1)$	$2.58(-239)$	$2.54(-239)$
RM6 $(\alpha = 1,\ \beta = 1.2,\ \gamma = 0.1,\ \delta = 0.1)$	$7.76(-235)$	$7.64(-235)$
BN $(a = -0.5)$	$2.02(-288)$	$1.99(-288)$
PM $k = 3,\ m = 0,\ p = 2,\ q = 0$	$5.00(-240)$	$4.62(-240)$

In Tables 5, 6 and 7 we present the difference of between the root α and the approximation x_n to α for test functions $f_{21},\ f_{22},\ f_{23}$ respectively. The approximation x_n is calculated using twelve function evaluations for all methods. The absolute values of the function values $|f(x_n)|$ are also shown.

4. CONCLUSIONS

A family of methods with the order of convergence of 3 to 6 is defined. Iterative functions are defined using Padé approximations of order (k, m) and (p, q) to the function $2/\left(1 + \sqrt{1 - 2z}\right)$ at 0 and z is a function for which (1.4) holds true. It is proved that the family contains one member of the order of convergence of three, three members of the order four and infinitely many members of the order of convergence 5 and 6. For the observed family the asymptotic error constants are obtained. For the function z three choices are proposed that are frequently encountered in the literature when the order of convergence of the Newton's method is increased. The first two choices (32) and (33) are free from the second derivative.

TABLE 7. Comparison of various iterative methods for f_{23}

Method	$\lvert x_n - \alpha \rvert$	$\lvert f(x_n) \rvert$
NM	$7.97(-85)$	$9.24(-85)$
KM	$5.02(-235)$	$5.82(-235)$
CM $(a = 1)$	$2.07(-227)$	$2.40(-227)$
RM1 $(\alpha = 1.66,\ \beta = 0.1,\ \gamma = 0.1)$	$4.91(-242)$	$5.69(-242)$
RM2 $(\alpha = -0.7,\ \beta = 0.3,\ \gamma = -0.2)$	$5.85(-268)$	$6.78(-268)$
RM3 $(\alpha = 0.08,\ \beta = -0.1,\ \gamma = 0.1)$	$5.86(-270)$	$6.79(-270)$
RM4 $(\alpha = 1.6,\ \beta = -0.2)$	$1.47(-291)$	$1.70(-291)$
RM5 $(\alpha = -0.72,\ \beta = 0.2)$	$2.27(-245)$	$2.63(-245)$
RM6 $(\alpha = 0.5,\ \beta = 1.2,\ \gamma = 0.2,\ \delta = 0.2)$	$1.12(-243)$	$1.30(-243)$
BN $(a = -1.5)$	$1.26(-273)$	$1.40(-273)$
PM $k = 0,\ m = 5,\ p = 2,\ q = 0$	$1.58(-238)$	$8.06(-238)$

A single proof of Theorem 1 applies to the whole family. The asymptotic error constant for each of the methods is expressed as a single function of four parameters k, m, p and q. The computational order of convergence of our methods was very close to those we described, which supports the theoretical results obtained in this paper.

Computational results for test functions f_{21}, f_{22}, f_{23} from [4] and [7], presented in Tables 5, 6 and 7, show that our methods are efficient and show similar or better performance as compared with the other methods of the same order: RM1, RM2, RM3, RM4, RM5, RM6 given in [7], NM, KM [9], CM [3], BN [2].

Our methods PM show similar good performance for all considered test functions. Numerical examples show that no single method of our family can be considered generally better than the others.

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