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PROPERTIES FOR ANALYTIC FUNCTIONS DEFINED BY FRACTIONAL CALCULUS

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ABSTRACT. There are many results for analytic functions in the open unit disk $\mathbb{U}$ concerning with fractional calculus of $f(z)$. A subclass $\mathcal{P}(\alpha, \lambda)$ of analytic functions in $\mathbb{U}$ is introduced using fractional calculus of $f(z)$. The object of the present paper is to consider some interesting properties of functions $f(z)$ belonging to this class. Further, some partial sums for $f(z)$ are also considered.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $f(z)$ and $g(z)$ be alalytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisftying $w(0) = 0, |w(z)| < 1$ $(z \in \mathbb{U})$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

The subordinations are applied for many papers for the univalent function theory by Breaz, Owa and Breaz [1], Rogosinski ([7], [8]), and Singh and Gupta [9].
Let us consider a function $g(z)$ given by

$$g(z) = \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in \mathbb{U}) \tag{1.1}$$

for some real α $(0 < \alpha < 1)$. Then, $g(z)$ is analytic in $\mathbb{U}$ and $g(0) = 1$. If we take $z = re^{i\theta} \in \mathbb{U}$ for $g(z)$, then we have that

$$\frac{\alpha - r}{\alpha(1-r)} \leqq \mathrm{Re}g(z) \leqq \frac{\alpha + r}{\alpha(1+r)} < \frac{\alpha+1}{2\alpha} \qquad (0 < \alpha < 1).$$

Therefore, if $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and satisfies

$$\mathrm{Re}p(z) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U}) \tag{1.2}$$

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for some real α $(0 < \alpha < 1)$, then

$$p(z) \prec \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in \mathbb{U}). \tag{1.3}$$

Conversely, if $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and satisfies the subordination (1.3), then $p(z)$ satisfies (1.2).

In view of the above, we say that if a function $p(z)$ which is analytic in $\mathbb{U}$ with $p(0) = 1$ satisfies the subordination (1.3), then $p(z) \in \mathcal{P}(\alpha)$. We note that if $p(z)$ which is analytic in $\mathbb{U}$ with $p(0) = 1$ satisfies $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$, then $p(z)$ is said to be Carathéodory function in $\mathbb{U}$ (see [2], [10]).

For $f(z) \in \mathcal{A}$, Owa [5], Owa and Srivastava [6] consider the following fractional calculus (fractional integrals and fractional derivatives).

Definition 1.1. *The fractional integral of order λ is defined, for a function $f(z) \in \mathcal{A}$, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt,$$

where $\lambda > 0$ and the multiplicity of $(z-t)^{\lambda - 1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.

Definition 1.2. *The fractional derivative of order λ is defined, for a function $f(z) \in \mathcal{A}$, by*

$$D_z^{\lambda} f(z) = \frac{d}{dz}\left(D_z^{\lambda-1} f(z)\right) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt,$$

where $0 \leqq \lambda < 1$ and the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.1 above.

Definition 1.3. *Under the hypotheses of Definition 1.2, the fractional derivative of order $n + \lambda$ is defined by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z),$$

where $0 \leqq \lambda < 1$ and $n = 0, 1, 2, \cdots$.

From Definition 1.2, if $f(z)$ is given by (1.1), then we have that

$$D_z^{\lambda} f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda}$$

for $0 \leqq \lambda < 1$. Therefore, we say that $f(z) \in \mathcal{P}(\alpha, \lambda)$ if $f(z) \in \mathcal{A}$ satisfies

$$\frac{z}{\Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z)} \prec \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in \mathbb{U})$$

for some real α $(0 < \alpha < 1)$ and λ $(0 \leqq \lambda < 1)$.

2. Some properties for the class $\mathcal{P}(\alpha, \lambda)$

We first derive the following coefficient inequalities for the class $\mathcal{P}(\alpha, \lambda)$.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| \leqq \frac{1-\alpha}{1+\alpha} \tag{2.1}$$

for some α $(0 < \alpha < 1)$ *and* λ $(0 \leqq \lambda < 1)$*, then* $f(z) \in \mathcal{P}(\alpha, \lambda)$*. The equality holds true for* $f(z)$ *given by*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\Gamma(n+1-\lambda)\epsilon}{(1+\alpha)n(n-1)\Gamma(2-\lambda)\Gamma(n+1)} z^n , \tag{2.2}$$

where $|\epsilon| = 1$.

Proof. Let us consider the function

$$g(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n$$

for $f(z) \in \mathcal{A}$. Then we see that $f(z) \in \mathcal{P}(\alpha, \lambda)$ if $g(z)$ satisfies

$$\alpha \left| 1 - \frac{z}{g(z)} \right| < \left| 1 - \alpha \frac{z}{g(z)} \right| \qquad (z \in \mathbb{U}), \tag{2.3}$$

that is, that

$$|g(z) - z| < \left| \frac{1}{\alpha} g(z) - z \right| \qquad (z \in \mathbb{U}).$$

This gives us that

$$\left| \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right| < \left| \left(\frac{1}{\alpha} - 1 \right) + \frac{1}{\alpha} \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right| .$$

Therefore, if $f(z)$ satisfies that

$$\sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| \leqq \frac{1-\alpha}{\alpha} - \frac{1}{\alpha} \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| ,$$

that is, that

$$(1+\alpha) \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| \leqq 1 - \alpha ,$$

then $g(z)$ satisfies (2.3). This implies that if $f(z)$ satisfies the coefficient inequality (2.1), then $f(z) \in \mathcal{P}(\alpha, \lambda)$.

Furthermore, we cinsider $f(z)$ given by (2.2). Then we have that

$$\sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| = \sum_{n=2}^{\infty} \frac{1-\alpha}{n(n-1)(1+\alpha)}$$

$$= \frac{1-\alpha}{1+\alpha} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \frac{1-\alpha}{1+\alpha} .$$

Thus, $f(z)$ given by (2.2) satisfies the equality in (2.1). □

Letting $\lambda = 0$ in Theorem 2.1, we have

Corollary 2.1. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} |a_n| \leqq \frac{1-\alpha}{1+\alpha}$$

for some α $(0 < \alpha < 1)$*, then* $f(z) \in \mathcal{P}(\alpha, 0)$*, that is*

$$\frac{z}{f(z)} \prec \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in \mathbb{U}).$$

3. Applications of Miller and Mocanu lemma

For considering the next properties of $f(z)$ for the class $\mathcal{P}(\alpha,\lambda)$, we have to recall here the following lemma due to Miller and Mocanu [4] (also, due to Jack [3]).

Lemma 3.1. *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0)=0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathbb{U}$, then we have that*

$$z_0w'(z_0)=kw(z_0),$$

where $k\geqq 1$.

With the above lemma, we derive

Theorem 3.1. *If $f(z)\in\mathcal{A}$ satisfies $z^\lambda D_z^\lambda f(z)\neq 0$ $(z\neq 0)$ and*

(3.1)

$$\operatorname{Re}\left\{1-\lambda+\frac{z^{1-\lambda}}{\Gamma(2-\lambda)D_z^\lambda f(z)}\left(1-\Gamma(2-\lambda)z^\lambda D_z^{1+\lambda}f(z)\right)\right\}<\frac{1+3\alpha}{2\alpha(1+\alpha)}\qquad(z\in\mathbb{U})$$

for some α $(0<\alpha<1)$ and λ $(0\leqq\lambda<1)$, then $f(z)\in\mathcal{P}(\alpha,\lambda)$.

Proof. Let us define a function $w(z)$ by

$$\frac{z^{1-\lambda}}{\Gamma(2-\lambda)D_z^\lambda f(z)}=\frac{\alpha-w(z)}{\alpha(1-w(z))}\qquad(z\in\mathbb{U}).$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. It follows that

$$1-\lambda-\frac{zD_z^{1+\lambda}f(z)}{D_z^\lambda f(z)}=\frac{zw'(z)}{1-w(z)}-\frac{zw'(z)}{\alpha-w(z)}.$$

Therefore, we have that

$$\begin{aligned}1-\lambda+\frac{z^{1-\lambda}}{\Gamma(2-\lambda)D_z^\lambda f(z)}&\left(1-\Gamma(2-\lambda)z^\lambda D_z^{1+\lambda}f(z)\right)\\&=\frac{\alpha-w(z)}{\alpha(1-w(z))}+\frac{zw'(z)}{1-w(z)}-\frac{zw'(z)}{\alpha-w(z)}.\end{aligned}$$

We assume that there exists a point $z_0\in\mathbb{U}$ such that

$$Max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=1.$$

Then, Lemma 3.1 gives us that

$$z_0w'(z_0)=kw(z_0)\qquad(k\geqq 1).$$

Writing that $w(z_0)=e^{i\theta}$ $(0\leqq\theta<2\pi)$, we have that

$$\begin{aligned}\operatorname{Re}&\left\{(1-\lambda)+\frac{z_0^{1-\lambda}}{\Gamma(2-\lambda)D_z^\lambda f(z_0)}\left(1-\Gamma(2-\lambda)z_0^\lambda D_z^{1+\lambda}f(z_0)\right)\right\}\\&=\operatorname{Re}\left\{\frac{\alpha-w(z_0)}{\alpha(1-w(z_0))}+\frac{z_0w'(z_0)}{1-w(z_0)}-\frac{z_0w'(z_0)}{\alpha-w(z_0)}\right\}\\&=\operatorname{Re}\left\{\frac{\alpha-e^{i\theta}}{\alpha(1-e^{i\theta})}+\frac{ke^{i\theta}}{1-e^{i\theta}}-\frac{ke^{i\theta}}{\alpha-e^{i\theta}}\right\}\\&=\frac{1+\alpha}{2\alpha}+k\left(\frac{1-\alpha\cos\theta}{1+\alpha^2-2\alpha\cos\theta}-\frac{1}{2}\right).\end{aligned}$$

Let us define

$$h(t)=\frac{1-\alpha t}{1+\alpha^2-2\alpha t}\qquad(t=\cos\theta).$$

Then $h(t)$ satisfies that

$$h'(t) = \frac{\alpha(1-\alpha^2)}{(1+\alpha^2-2\alpha t)^2} \qquad (-1 \leqq t \leqq 1),$$

that is, that $h'(t) > 0$ for $0 < \alpha < 1$. Therefore, we see that

$$\operatorname{Re}\left\{(1-\lambda) + \frac{z_0^{1-\lambda}}{\Gamma(2-\lambda)D_z^{\lambda} f(z_0)}\left(1-\Gamma(2-\lambda)z_0^{\lambda}D_z^{1+\lambda}f(z_0)\right)\right\}$$

$$= \frac{1+\alpha}{2\alpha} + k\left(\frac{1}{1+\alpha} - \frac{1}{2}\right) \geqq \frac{1+3\alpha}{2\alpha(1+\alpha)}$$

for $0 < \alpha < 1$. Since, this contradicts our condition (3.1) of the theorem, we say that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. Thus, $w(z)$ satisfies $|w(z)| < 1$ for all $z \in \mathbb{U}$. This implies that

$$|w(z)| = \left|\frac{\alpha\left(1 - \dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)D_z^{\lambda}f(z)}\right)}{1 - \alpha\dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)D_z^{\lambda}f(z)}}\right| < 1 \qquad (z \in \mathbb{U}),$$

that, is that

$$\operatorname{Re}\left(\frac{z}{\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)}\right) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U}).$$

Consequently, we obtain that $f(z) \in \mathcal{P}(\alpha, \lambda)$. □

Taking $\lambda = 0$ in Theorem 3.1, we have

Corollary 3.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{z}{f(z)}(1 - f'(z))\right) < \frac{1+3\alpha}{2\alpha(1+\alpha)} \qquad (z \in \mathbb{U})$$

for some α $(0 < \alpha < 1)$, then $f(z) \in \mathcal{P}(\alpha, 0)$, so that

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U}).$$

If we take $\alpha = \frac{1}{2}$ in Theorem 3.1, then we have

Corollary 3.2. *If $f(z) \in \mathcal{A}$ satisfies $z^{\lambda}D_z^{\lambda}f(z) \neq 0$ $(z \neq 0)$ and*

$$\operatorname{Re}\left\{1 - \lambda + \frac{z^{1-\lambda}}{\Gamma(2-\lambda)D_z^{\lambda}f(z)}\left(1 - \Gamma(2-\lambda)z^{\lambda}D_z^{1+\lambda}f(z)\right)\right\} < \frac{5}{3} \qquad (z \in \mathbb{U})$$

for some λ $(0 \leqq \lambda < 1)$, then

$$\operatorname{Re}\left(\frac{z}{\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)}\right) < \frac{3}{2} \qquad (z \in \mathbb{U}).$$

4. Partial sums

Let a function $g(z)$ be given by (1.1). Then $g(z)$ has

$$g(z) = \frac{\alpha - z}{\alpha(1-z)} = 1 + \left(1 - \frac{1}{\alpha}\right) \sum_{n=1}^{\infty} z^n \qquad (0 < \alpha < 1).$$

For this function $g(z)$, we consider a partial sum $g_2(z)$ of $g(z)$ which is given by

$$g_2(z) = 1 + \left(1 - \frac{1}{\alpha}\right) z + \left(1 - \frac{1}{\alpha}\right) z^2.$$

Taking $z = re^{i\theta}$ $(0 \leqq r < 1,\ 0 \leqq \theta < 2\pi)$, we obtain that

$$\mathrm{Re} g_2(z) = 1 + \frac{1-\alpha}{\alpha} r - \frac{1-\alpha}{\alpha} r \cos\theta \,(2r\cos\theta + 1).$$

Consider a function $h(t)$ given by

$$h(t) = t(2rt + 1) \qquad (t = \cos\theta).$$

If $0 \leqq r \leqq \frac{1}{4}$, then

$$2r - 1 \leqq h(t) \leqq 2r + 1.$$

This shows us that

$$\frac{1}{\alpha}\left(1 - (1-\alpha)r - 2(1-\alpha)r^2\right) \leqq \mathrm{Re} g_2(z) \leqq \frac{1}{\alpha}\left(1 + (1-\alpha)r - 2(1-\alpha)r^2\right).$$

Let us consider the radius r such that

$$\frac{1}{\alpha}\left(1 + (1-\alpha)r - 2(1-\alpha)r^2\right) \geqq \frac{1+\alpha}{2\alpha}.$$

It follows that

$$4r^2 - 2r - 1 \leqq 0.$$

Therefore, if we consider r such that

$$0 \leqq r \leqq \frac{1+\sqrt{5}}{4} = 0.8090\ldots,$$

then

$$\mathrm{Re} g_2(z) \not< \frac{1+\alpha}{2\alpha}.$$

Therefore, we see that

$$\mathrm{Re} g(z) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U})$$

for $g(z)$ given by (1.1), but

$$\mathrm{Re} g_2(z) \not< \frac{1+\alpha}{2\alpha} \qquad \left(0 \leqq r \leqq \frac{1+\sqrt{5}}{4}\right)$$

With this fact, we consider a function $f(z) \in \mathcal{P}(\alpha, \lambda)$.

Since

$$\frac{z}{\Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z)} = 1 - \frac{2}{2-\lambda} a_2 z + \frac{2}{2-\lambda}\left(\frac{2}{2-\lambda} a_2^2 - \frac{3}{3-\lambda} a_3\right) z^2 + \cdots,$$

we consider a function $F_1(z)$ given by

$$F_1(z) = 1 - \frac{2}{2-\lambda} a_2 z \qquad (z \in \mathbb{U}).$$

For this function $F_1(z)$, we have

Theorem 4.1. *If a_2 satisfies*

$$|a_2| < \frac{(1-\alpha)(2-\lambda)}{4\alpha}$$

with $0 < \alpha < 1$ and $0 \leqq \lambda < 1$, then

$$\text{Re}F_1(z) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U}). \tag{4.1}$$

Proof. Writing that $z = re^{i\theta}$ and $a_2 = |a_2|e^{i\phi}$, we see that

$$\text{Re}F_1(z) = 1 - \frac{2}{2-\lambda}|a_2|r\cos(\theta+\phi) \leqq 1 + \frac{2}{2-\lambda}|a_2|r.$$

Therefore, if $F_1(z)$ satisfies

$$|a_2| < \frac{(1-\alpha)(2-\lambda)}{4\alpha},$$

then $f_1(z)$ satisfies (4.1). □

Next, we consider $F_2(z)$ which is given by

$$F_2(z) = 1 - \frac{2}{2-\lambda}a_2 z + \frac{2}{2-\lambda}\left(\frac{2}{2-\lambda}a_2^2 - \frac{3}{3-\lambda}a_3\right)z^2.$$

Then, we have

Theorem 4.2. *If $F_2(z)$ satisfies*

$$|a_2| + \frac{2}{2-\lambda}|a_2|^2 + \frac{3}{3-\lambda}|a_3| < \frac{(1-\alpha)(2-\lambda)}{4\alpha} \tag{4.2}$$

with $0 < \alpha < 1$ and $0 \leqq \lambda < 1$, then

$$\text{Re}F_2(z) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U}). \tag{4.3}$$

Proof. Taking $z = re^{i\theta}, a_2 = |a_2|e^{i\phi}$ and $a_3 = |a_3|e^{i\rho}$, we obtain that

$$\begin{aligned}\text{Re}F_2(z) &= 1 - \frac{2}{2-\lambda}|a_2|r\cos(\theta+\phi) + \left(\frac{2}{2-\lambda}\right)^2|a_2|^2r^2\cos 2(\theta+\phi)\\ &\qquad - \frac{2\cdot 3}{(2-\lambda)(3-\lambda)}|a_3|r^2\cos(2\theta+\rho)\\ &\leqq 1 + \frac{2}{2-\lambda}|a_2|r + \left(\frac{2}{2-\lambda}\right)^2|a_2|^2r^2\left(2\cos^2(\theta+\phi)-1\right) + \frac{2\cdot 3}{(2-\lambda)(3-\lambda)}|a_3|r^2\\ &< 1 + \frac{2}{2-\lambda}|a_2| + \left(\frac{2}{2-\lambda}\right)^2|a_2|^2 + \frac{2\cdot 3}{(2-\lambda)(3-\lambda)}|a_3|.\end{aligned}$$

Therefore, if a_2 and a_3 safisfy the inequality (4.2), then we obtain (4.3). □

Letting $a_2 = 0$ in Theorem 4.2, we see that

Corollary 4.1. *If $F_2(z)$ satisfies $a_2 = 0$ and*

$$|a_3| < \frac{(1-\alpha)(2-\lambda)(3-\lambda)}{12\alpha}$$

with $0 < \alpha < 1$ and $0 \leqq \lambda < 1$, then

$$\text{Re}F_2(z) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U}).$$

Finally, we derive

Theorem 4.3. *If* $F_{n-1}(z)$ *satisfies* $a_2 = a_3 = \cdots = a_{n-1} = 0$ *and*

$$|a_n| < \frac{(1-\alpha)\prod_{j=2}^{n}(j-\lambda)}{2\alpha n!}$$

with $0 < \alpha < 1$ *and* $0 \leqq \lambda < 1$, *then*

$$\text{Re}F_{n-1}(z) < \frac{1+\alpha}{2\alpha} \qquad (z \in \mathbb{U}). \tag{4.4}$$

Proof. For $a_2 = a_3 = \cdots = a_{n-1} = 0$, we can write that

$$F_{n-1}(z) = 1 - \frac{n!}{\prod_{j=2}^{n}(j-\lambda)} a_n z^{n-1} \qquad (z \in \mathbb{U}).$$

Thus, it is clear that $F_{n-1}(z)$ satisfies (4.4) if a_n satisfies

$$\frac{n!}{\prod_{j=2}^{n}(j-\lambda)} |a_n| < \frac{1-\alpha}{2\alpha}.$$

This completes the proof of the theorem. □

From Theorem 4.3, we give

Problem 4.1. *Find some conditions for* $a_2, a_3, \cdots, a_n$ *such that* $F_{n-1} \in \mathcal{P}(\alpha, \lambda)$.

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