ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **6** (2017), no.2, 57–61
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

# REMARK ON TRIGONOMETRIC FUNCTIONS

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ABSTRACT. Noting the derivatives for functions $\sin z$ and $\cos z$, we assume the fractional derivatives for $\sin z$ and $\cos z$. Applying the fractional derivatives, we consider generalized expansions for functions $\sin z$ and $\cos z$. Further, the generalized expansion for $f(z) = e^{iz}$ is also discussed.

## 1. INTRODUCTION

Let $\mathcal{A}(\alpha)$ be the class of functions $f(z)$ of the form

$$f(z) = a_0 z^{\alpha} + a_1 z^{\alpha+1} + a_2 z^{\alpha+2} + \cdots = \sum_{n=0}^{\infty} a_n z^{\alpha+n}$$

for $0 \leq \alpha < 1$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $\alpha = 0$ in (1.1), then $f(z) \in \mathcal{A}(0)$ becomes

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n,$$

and that

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

This is Taylor expansion for $f(z) \in \mathcal{A}(0)$. Therefore, for $f(z) \in \mathcal{A}(\alpha)$, we need to consider the generalization for Taylor expansion of $f(z)$.

To discuss Taylor expansion for $f(z)$ in the class $\mathcal{A}(\alpha)$, we have to introduce the fractional calculus (fractional integrals and fractional derivatives) defined by Owa [1], Owa and Srivastava [2], and Srivastava and Owa [3].

**Definition 1.1.** *The fractional integral of order $\alpha$ is defined, for an analytic function $f(z)$ in $\mathbb{U}$, by*

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt \qquad (\alpha > 0),$$

*where the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.*

2010 *Mathematics Subject Classification.* 30C45.
*Key words and phrases.* Analytic function, Taylor expansion, fractional calculus, trigonometric function.

**Definition 1.2.** *The fractional derivative of order $\alpha$ is defined, for an analytic function $f(z)$ in $\mathbb{U}$, by*

$$D_z^{\alpha} f(z) = \frac{d}{dz}\left(D_z^{\alpha-1} f(z)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\left(\int_0^z \frac{f(t)}{(z-t)^{\alpha}}dt\right),$$

*where $0 \leq \alpha < 1$ and the multiplicity of $(z-t)^{-\alpha}$ is removed as Definition 1.1 above.*

**Definition 1.3.** *Under the hypotheses of Definition 1.2, the fractional derivative of order $n+\alpha$ is defined by*

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n}\left(D_z^{\alpha} f(z)\right),$$

*where $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.*

By means of Definition 1.2, we have that

$$\begin{aligned} D_z^{\alpha} z^{\alpha+n} &= \frac{d}{dz}\left(D_z^{\alpha-1} z^{\alpha+n}\right) = \frac{d}{dz}\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^z \frac{t^{\alpha+n}}{(z-t)^{\alpha}}dt\right\} \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\left\{z^{n+1}\int_0^1 \frac{(1-\zeta)^{\alpha+n}}{\zeta^{\alpha}}d\zeta\right\} \qquad (z-t=z\zeta) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\left(z^{n+1}B(1-\alpha,\alpha+n+1)\right) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}z^n, \end{aligned}$$

where $B(x, y)$ is the beta function. Thus, we obtain, for $f(z) \in \mathcal{A}(\alpha)$, that

$$D_z^{\alpha} f(z) = D_z^{\alpha}\left(\sum_{n=0}^{\infty} a_n z^{\alpha+n}\right) = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}a_n z^n.$$

This gives us that

$$a_0 = \frac{D_z^{\alpha} f(0)}{\Gamma(\alpha+1)}.$$

Since

$$D_z^{\alpha+1} f(0) = \Gamma(\alpha+2)a_1, \qquad a_1 = \frac{D_z^{\alpha+1} f(0)}{\Gamma(\alpha+2)}.$$

Further, we obtain that

$$a_n = \frac{D_z^{\alpha+n} f(0)}{\Gamma(\alpha+n+1)}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$. With the above, we can write that

$$f(z) = \sum_{n=0}^{\infty}\frac{D_z^{\alpha+n} f(0)}{\Gamma(\alpha+n+1)}z^{\alpha+n} \tag{1.1}$$

for $f(z) \in \mathcal{A}(\alpha)$ with $z \neq 0$. Therefore, we use this expansion (1.1) for $f(z) \in \mathcal{A}(\alpha)$.

## 2. Expansions for trigonometric functions

Let us consider a function $f(z) = \sin z$ for all $z \in \mathbb{U}$. Then it is easy to write that

$$\begin{aligned} f'(z) &= \cos z = \sin\left(z+\frac{\pi}{2}\right), \\ f''(z) &= \cos\left(z+\frac{\pi}{2}\right) = \sin(z+\pi), \end{aligned}$$

and that

$$f^{(n)}(z) = \sin\left(z + \frac{n}{2}\pi\right) \qquad (n \in \mathbb{N}_0). \tag{2.1}$$

With (2.1), we may assume that

$$D_z^{\alpha} f(z) = \sin\left(z + \frac{\alpha}{2}\pi\right) \qquad (0 \leq \alpha < 1) \tag{2.2}$$

and

$$D_z^{\alpha+n} f(z) = \sin\left(z + \frac{\alpha+n}{2}\pi\right) \tag{2.3}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$.

**Remark 2.1.** *Using the formula* (2.2)*, we have*

$$f^{(n)}(z) = D_z^{n-\alpha}\left(D_z^{\alpha} f(z)\right) = D_z^{n-\alpha}\left(\sin\left(z + \frac{\alpha}{2}\pi\right)\right) = \sin\left(z + \frac{n}{2}\pi\right)$$

*for* $0 \leq \alpha < 1$ *and* $n \in \mathbb{N}_0$.

Now, we derive

**Theorem 2.1.** *If the equation* (2.2) *is true for* $f(z) = \sin z$*, then*

$$\sin z = \sum_{n=0}^{\infty} \frac{\sin\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{U} - \{0\}) \tag{2.4}$$

*where* $0 \leq \alpha < 1$.

*Proof.* By (2.2), we see that

$$D_z^{\alpha} f(0) = \sin\left(\frac{\alpha}{2}\pi\right) \qquad (0 \leq \alpha < 1)$$

and by (2.3), we have that

$$D_z^{\alpha+n} f(0) = \sin\left(\frac{\alpha+n}{2}\pi\right) \qquad (0 \leq \alpha < 1, n \in \mathbb{N}_0).$$

This shows us (2.4) with (1.1). □

**Corollary 2.1.** *If the equation* (2.2) *is true for* $f(z) = \sin z$ *with* $\alpha = \frac{1}{2}$*, then*

$$\begin{aligned}\sin z &= \sum_{n=0}^{\infty} \frac{\sin\left(\frac{2n+1}{4}\pi\right)}{\Gamma\left(n+\frac{3}{2}\right)} z^{n+\frac{1}{2}} \\ &= \frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}}\left(1 + \frac{2}{3}z - \frac{2^2}{3\cdot 5}z^2 - \frac{2^3}{3\cdot 5\cdot 7}z^3 + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}z^4 + \frac{2^5}{3\cdot 5\cdot 7\cdot 9\cdot 11}z^5 - \cdots\right)\end{aligned}$$

*for* $z \in \mathbb{U} - \{0\}$.

Next, we try to consider for $f(z) = \cos z$ $(z \in \mathbb{U})$. It is clear that

$$f'(z) = -\sin z = \cos\left(z + \frac{\pi}{2}\right),$$
$$f''(z) = -\sin\left(z + \frac{\pi}{2}\right) = \cos(z + \pi),$$

and that

$$f^{(n)}(z) = \cos\left(z + \frac{n}{2}\pi\right) \qquad (n \in \mathbb{N}_0). \tag{2.5}$$

With the above, we can assume that

$$D_z^{\alpha} f(z) = \cos\left(z + \frac{\alpha}{2}\pi\right) \tag{2.6}$$

and

$$D_z^{\alpha+n} f(z) = \cos\left(z + \frac{\alpha+n}{2}\pi\right) \tag{2.7}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$.

**Remark 2.2.** *With the formula (2.6), we see that*

$$f^{(n)}(z) = D_z^{n-\alpha}\left(D_z^{\alpha} f(z)\right) = D_z^{n-\alpha}\left(\cos\left(z + \frac{\alpha}{2}\pi\right)\right) = \cos\left(z + \frac{n}{2}\pi\right)$$

*for* $0 \leq \alpha < 1$ *and* $n \in \mathbb{N}_0$.

For a function $f(z) = \cos z$, we have

**Theorem 2.2.** *If the equation (2.6) is true for* $f(z) = \cos z$*, then*

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{U} - \{0\}) \tag{2.8}$$

*with* $0 \leq \alpha < 1$.

*Proof.* Using (2.6), we have

$$D_z^{\alpha} f(0) = \cos\left(\frac{\alpha}{2}\pi\right) \qquad (0 \leq \alpha < 1). \tag{2.9}$$

Also, by (2.7), we see

$$D_z^{\alpha+n} f(0) = \cos\left(\frac{\alpha+n}{2}\pi\right) \qquad (0 \leq \alpha < 1), z \in \mathbb{U} - \{0\}). \tag{2.10}$$

Putting (2.9) and (2.10) in (2.5), we prove the equation (2.8). □

Making $\alpha = \frac{1}{2}$ in Theorem 2.2, we give

**Corollary 2.2.** *If the equation (2.6) is true for* $\alpha = \frac{1}{2}$*, then*

$$\begin{aligned}\cos z &= \sum_{n=0}^{\infty} \frac{\cos\left(\frac{2n+1}{4}\pi\right)}{\Gamma\left(n+\frac{3}{2}\right)} z^{n+\frac{1}{2}} \\ &= \frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}}\left(1 - \frac{2}{3}z - \frac{2^2}{3\cdot 5}z^2 + \frac{2^3}{3\cdot 5\cdot 7}z^3 + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}z^4 - \frac{2^5}{3\cdot 5\cdot 7\cdot 9\cdot 11}z^5 - \cdots\right)\end{aligned}$$

*for* $z \in \mathbb{U} - \{0\}$.

Finally, we derive for $f(z) = e^{iz}$.

**Theorem 2.3.** *If the equation* (2.2) *and (2.6) are satisfied, then we have*

$$e^{iz} = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\alpha+n}{2}\pi\right) + i\sin\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{N} - \{0\}) \tag{2.11}$$

*for* $0 \leq \alpha < 1$.

Letting $\alpha = \frac{1}{2}$ in (2.11), we see

**Corollary 2.3.** *If the equations* (2.2) *and* (2.6) *are satisfied for* $\alpha = \frac{1}{2}$, *then*

$$e^{iz} = \frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}} \left\{ (1+i) - \frac{2}{3}(1-i)z - \frac{2^2}{3\cdot 5}(1+i)z^2 + \frac{2^3}{3\cdot 5\cdot 7}(1-i)z^3 \right.$$
$$\left. + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}(1+i)z^4 - \frac{2^5}{3\cdot 5\cdot 7\cdot 9\cdot 11}(1-i)z^5 - \cdots \right)$$

*for* $z \in \mathbb{U} - \{0\}$.

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