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ON THE DETERMINATION OF JUMP BY THE DIFFERENTIATED CONJUGATE FOURIER-JACOBI SERIES

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ABSTRACT. In the present paper we prove a new result on determination of jump discontinuities by the differentiated conjugate Fourier-Jacobi series. Further, we establish Cesàro summability of the sequence of partial sums of the conjugate Fourier-Chebyshev series, a special type of Fourier-Jacobi series which are obtained for $\alpha = \beta = -\frac{1}{2}$.

1. INTRODUCTION AND PRELIMINARIES

Conjugate Fourier-Jacobi series was introduced by B. Muckenhoupt and E. M. Stein, see [6], when $\alpha = \beta$, and by Zh.-K. Li, see [4], for general α and β. "Conjugacy" is an important concept in classical Fourier analysis which links the study of the more fundamental properties of harmonic functions to that of analytic functions and is used to study the mean convergence of Fourier series, see [11].

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n and order $(\alpha,\beta), \alpha,\beta > -1$, normalized so that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. They are orthogonal on the interval $(-1,1)$ with respect to the measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$.

Define $R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$, and denote by $L_p(\alpha,\beta), (1 \le p < \infty)$ the space of functions $f(x)$ for which $\|f\|_{p(\alpha,\beta)} = \{\int_{-1}^1 |f(x)|^p d\mu_{\alpha,\beta}(x)\}^{\frac{1}{p}}$ is finite.

For functions $f \in L_1(\alpha,\beta)$, its Fourier-Jacobi expansion is

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x),$$

where

$$\hat{f}(n) = \int_{-1}^{1} f(y) R_n^{(\alpha,\beta)}(y) d\mu_{\alpha,\beta}(y)\,,$$

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are the Fourier coefficients and

$$\omega_n^{(\alpha,\beta)} = \{\int_{-1}^{1} [R_n^{(\alpha,\beta)}(y)]^2 d\mu_{\alpha,\beta}(y)\}^{-1} \sim n^{2\alpha+1}\,.$$

With $x = \cos\theta, \theta \in (0,\pi)$, in an equivalent way Fourier-Jacobi expansion is given by

$$f(\theta) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta), \tag{1.1}$$

where

$$\hat{f}(n) = \int_0^{\pi} f(\varphi) R_n^{(\alpha,\beta)}(\cos\varphi) d\mu_{\alpha,\beta}(\varphi),$$

$$\omega_n^{(\alpha,\beta)} = \{\int_0^{\pi} [R_n^{(\alpha,\beta)}(\cos\varphi)]^2 d\mu_{\alpha,\beta}(\varphi)\}^{-1} \sim n^{2\alpha+1}, \tag{1.2}$$

and correspondingly $d\mu_{\alpha,\beta}(\theta) = 2^{\alpha+\beta+1}\sin^{2\alpha+1}\frac{\theta}{2}\cos^{2\beta+1}\frac{\theta}{2}d\theta$.

To the Fourier-Jacobi series of the form (1.1), its conjugate series is defined by

$$\tilde{f}(\theta) \sim \frac{1}{2\alpha+2}\sum_{n=1}^{\infty} n\hat{f}(n)\omega_n^{(\alpha,\beta)} R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta. \tag{1.3}$$

Denote by $S_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (1.1), and by $\tilde{S}_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (1.3), where $x = \cos\theta$. If $\alpha = \beta = -\dfrac{1}{2}$, the corresponding Fourier-Jacobi series becomes Fourier-Chebyshev series, so by $S_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)$ we denote the n-th partial sum of the Fourier-Chebyshev series of f.

Also, throughout this paper we use the following general notations: $L[a,b]$ is the space of integrable functions on $[a,b]$ and $C[a,b]$ is the space of continuous function on $[a,b]$ with the uniform norm $\|\cdot\|_{C[a,b]}$. $W[a,b]$ is the space of functions on $[a,b]$ which may have discontinuities only of the first kind and which are normalized by the condition $f(x) = \frac{1}{2}(f(x+) + f(x-))$.

In this paper first we give a review of the results on determination of jump discontinuities for functions of generalized bounded variation by the differentiated Fourier series, and then we prove new results on the determination of jump discontinuities by the differentiated conjugate Fourier-Jacobi series. Further, we prove that the sequence of the conjugate partial sums of Fourier-Chebyshev series is Cesàro summable to 0.

2. Jump of a function and differentiated Fourier series

The knowledge of the precise location of the discontinuity points is essential for many of the methods aiming at obtaining exponential convergence of the Fourier series of a piecewise smooth function, avoiding the well-known Gibbs phenomenon: the oscillatory behavior of the Fourier partial sums of a discontinuous function.

If a function f is integrable on $[-\pi,\pi]$, then it has a Fourier series with respect to the trigonometric system $\{1, \cos nx, \sin nx\}_{n=1}^{\infty}$, and we denote the n-th partial sum of the Fourier series of f by $S_n(x,f)$, i.e.,

$$S_n(x,f) = \frac{a_0(f)}{2} + \sum_{k=1}^{n}(a_k(f)\cos kx + b_k(f)\sin kx),$$

where $a_k(f) = \dfrac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\cos\, kt\, dt$ and $b_k(f) = \dfrac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\sin\, kt\, dt$ are the $k-$th Fourier coefficients of the function f. By $\tilde{S}_n(x,f)$ we denote the $n-$th partial sum of the conjugate series, i.e.,

$$\tilde{S}_n(x,f) = \sum_{k=1}^{n}(a_k(f)\sin\, kx - b_k(f)\cos\, kx).$$

The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time. Let $f(x)$ be a function of bounded variation with period 2π, and $S_n(x,f)$ be the partial sum of order n of its Fourier series. By the classical theorem of Fejer [11] the identity

$$\lim_{n\to\infty}\frac{S_n'(x,f)}{n} = \frac{1}{\pi}(f(x+0)-f(x-0)) \tag{2.1}$$

holds at any point x.

Obviously, Fejér's identity (2.1) is a statement about Cesàro summability of the sequence $\{kb_k\cos kx - ka_k\sin kx\}$, $a_k = a_k(f)$ and $b_k = b_k(f)$ being the k-th cosine and sine coefficient, respectively. As it is well-known, a sequence s_n is Cesàro or $(C,1)$ summable to s if the sequence σ_n of its arithmetical means converges to s, i.e. $\sigma_n = \dfrac{s_0+s_1+\ldots+s_n}{n+1} \to s, n\to\infty$.

Analogously, the sequence s_n is $(C,\alpha), \alpha > -1$, summable to s, if the sequence

$$\sigma_n^{(\alpha)} = \frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}s_k,$$

converges to s.

The concept of higher variation was firstly introduced by N. Wiener, see [10].

A function f is said to be of bounded p-variation, $p\geq 1$, on the segment $[a,b]$ and to belong to the class $\mathcal{V}_p[a,b]$ if

$$V_{a\ p}^{b}(f) = \sup_{\Pi_{a,b}}\Big\{\sum_i |f(x_i)-f(x_{i-1})|^p\Big\}^{\frac{1}{p}} < \infty,$$

where $\Pi_{a,b} = \{a = x_0 < x_1 < ... < x_n = b\}$ is an arbitrary partition of the segment $[a,b]$. $V_{a\ p}^{b}(f)$ is the p-variation of f on $[a,b]$.

B. I. Golubov, see [2], has shown that identity (2.1) is valid for classes $\mathcal{V}_p$.

Theorem 2.1. *Let $f(x)\in\mathcal{V}_p$, $(1\leq p<\infty)$ and $r\in\mathbb{N}_0$. Then for any point x one has the equation*

$$\lim_{n\to\infty}\frac{S_n^{(2r+1)}(x,f)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(f(x+0)-f(x-0)).$$

Another type of generalization of the class BV on everywhere convergence of Fourier series, for every change of variable, was introduced by D. Waterman in [9].

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum\dfrac{1}{\lambda_n}$ diverges and $\{I_n\}$ be a sequence of nonoverlapping segments $I_n = [a_n,b_n]\subset[a,b]$. A function f is said to be of Λ-bounded variation on $I=[a,b]$ $(f\in\Lambda BV)$ if

$$\sum\frac{|f(b_n)-f(a_n)|}{\lambda_n} < \infty$$

for every choice of $\{I_n\}$. The supremum of these sums is called the Λ-variation of f on I. In the case $\Lambda = \{n\}$, one speaks of harmonic bounded variation (HBV).

The class HBV contains all Wiener classes. M. Avdispahić has shown in [1] that HBV is the limiting case for validity of the identity (2.1).

G. Kvernadze in [3] generalized Theorem 2.1 for ΛBV classes:

Theorem 2.2. *Let $r \in \mathbb{Z}_+$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then*

(a) *the identity*

$$\lim_{n\to\infty} \frac{((S_n(g;\theta))^{(2r+1)}}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(g(\theta+) - g(\theta-)).$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi,\pi]$ if and only if $\Lambda BV \subseteq HBV$.

(b) *there is no way to determine the jump at the point $\theta \in [-\pi,\pi]$ of an arbitrary function $g \in \Lambda BV$ by means of the sequence $((S_n(g;\theta))^{(2r)}, n \in \mathbb{N}_0$.*

Here we also note the result from [3] for the conjugate Fourier series:

Theorem 2.3. *Let $r \in \mathbb{N}$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then*

(a) *the identity*

$$\lim_{n\to\infty} \frac{(\widetilde{S}_n(g;\theta))^{(2r)}}{n^{2r}} = \frac{(-1)^{(r+1)}}{2r\pi}(g(\theta+) - g(\theta-)).$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi,\pi]$ if and only if $\Lambda BV \subseteq HBV$.

(b) *there is no way to determine the jump at the point $\theta \in [-\pi,\pi]$ of an arbitrary function $g \in \Lambda BV$ by means of the sequence $((\widetilde{S}_n(g;\theta))^{(2r+1)}, n \in \mathbb{N}$.*

3. Main Results

Theorem 3.1. *Let $r \in \mathbb{N}$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$, and $\alpha \geq -\frac{1}{2}$,*
$\beta \geq -\frac{1}{2}$. Then the identity

$$\lim_{n\to\infty} \frac{[\widetilde{S}_n^{(\alpha,\beta)}(f,x)]^{(2r)}}{n^{2r}} = \frac{(-1)^{(r+1)}}{2r\pi}(1-x^2)^{-r-\frac{1}{2}}[f(x+0) - f(x-0)],$$

is valid for every $f \in \Lambda BV$ and each $x \in (-1,1)$, where $\widetilde{S}_n^{(\alpha,\beta)}(f,x)$ is the n-th partial sum of the conjugate Fourier-Jacobi series, if and only if $\Lambda BV \subseteq HBV$.

Proof. Differentiating an obvious identity, see [8]

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = S_n(g,\theta),$$

where $x = \cos\theta,\ g(\theta) = f(\cos\theta)$ one has

$$\left(S_n^{(-1/2,-1/2)}(f,x)\right)' = S_n'(g,\theta) \cdot \frac{-1}{\sqrt{1-x^2}}.$$

Continuing the differentiation of the last identity with respect to x ($x = \cos\theta$), we obtain by induction the following representation ($r \in \mathbb{N}$) :

$$[S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(2r+1)} =$$

$$= (1-x^2)^{-r-\frac{1}{2}}(S_n(g;\theta))^{(2r+1)} - \sum_{i=1}^{2r} d_i(x)(S_n(g;\theta))^{(i)} \tag{3.1}$$

for $\theta \in [0,\pi]$, where d_i, $i = 1,2,\ldots,2r$, are infinitely differentiable functions on $(-1,1)$. In addition,

$$\|S_n(g;\cdot))^{(i)}\|_{C[-\pi,\pi]} = o(n^{2r+1}) \tag{3.2}$$

for $i = 1,2,\ldots,2r$, $r \in \mathbb{N}$, since $g \in W \subset L$, see [3].

By Theorem 4.1 in [7] we have for $\alpha = \beta = -\dfrac{1}{2}$

$$\lim_{n\to\infty}[\frac{-1}{n}\left(S_n^{(-1/2,-1/2)}(f,x)\right)' - \tilde{S}_n^{(-1/2,-1/2)}(f,x)] = 0$$

thus taking that into account, dividing (3.1) by n^{2r+1} and letting $n \to \infty$ we get

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = \lim_{n\to\infty}\frac{1}{n^{2r+1}}[(1-x^2)^{-r-\frac{1}{2}}(S_n(g;\theta))^{(2r+1)} - \sum_{i=1}^{2r} d_i(x)(S_n(g;\theta))^{(i)}].$$

Using the well-known relation $\widetilde{S}_n(g,\theta) = \dfrac{-1}{n}S_n'(g,\theta)$, we have

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = \lim_{n\to\infty}[\frac{-1}{n^{2r}}(1-x^2)^{-r-\frac{1}{2}}(\widetilde{S}_n(g;\theta))^{(2r)} - \frac{1}{n^{2r}}\sum_{i=1}^{2r} d_i(x)(S_n(g;\theta))^{(i)}].$$

By Theorem 2.3 and (3.2) we have further

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = -(1-x^2)^{-r-\frac{1}{2}}\frac{(-1)^{(r+1)}}{2r\pi}(g(\theta+) - g(\theta-)).$$

Taking into account that $f(x\pm) = g(\theta\mp)$, $\theta \in [0,\pi]$, we get

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = (1-x^2)^{-r-\frac{1}{2}}\frac{(-1)^{(r+1)}}{2r\pi}[f(x+0) - f(x-0)].$$

Finally, using the equiconvergence formula

$$\|\tilde{S}_n^{(\alpha,\beta)}(f,x) - \tilde{S}_n^{(-1/2,-1/2)}(f,x)\|_{C[\Delta(\nu,\varepsilon)]} = o(1),$$

where $\alpha \geq -\dfrac{1}{2}$ and $\beta \geq -\dfrac{1}{2}$, proved in [7] (for an arbitrary function $f \in HBV$ and a fixed $\varepsilon \in (0, \frac{x_{\nu+1}-x_\nu}{2})$, $\nu = 0,1,2,...,M$, where it is assumed that $x_0 = -1$, $x_{M+1} = 1$ and $\Delta(\nu;\varepsilon) = [x_\nu + \varepsilon; x_{\nu+1} - \varepsilon]$,) we prove the result. □

For $\alpha = \beta = -\dfrac{1}{2}$ the corresponding Fourier-Jacobi series becomes Fourier-Chebyshev series, so by $\tilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)$ we denote the n-th partial sum of the conjugate Fourier-Chebyshev series of f. Further, we prove that the sequence of the conjugate partial sums of Fourier-Chebyshev series is Cesàro summable to 0.

Theorem 3.2.
$$\lim_{n\to\infty}\frac{\tilde{S}_1^{(-1/2,-1/2)}(f,x)+\tilde{S}_2^{(-1/2,-1/2)}(f,x)+\ldots+\tilde{S}_{n-1}^{(-1/2,-1/2)}(f,x)}{n}=0,$$
for every $f\in L_1(-1/2,-1/2)$ *and each* $-1<x<1$.

Proof. According to (1.3)
$$\tilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)=\sum_{k=1}^{n}k\cdot\hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})}\cdot R_{k-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)\sin\theta.$$
The sum
$$\tilde{S}_1^{(-1/2,-1/2)}(f,x)+\tilde{S}_2^{(-1/2,-1/2)}(f,x)+\ldots+\tilde{S}_{n-1}^{(-1/2,-1/2)}(f,x)$$
can be written as
$$1(n-1)a_1+2(n-2)a_2+\ldots+(n-1)1a_{n-1},$$
where $a_i=\hat{f}(i)\omega_i^{(-\frac{1}{2},-\frac{1}{2})}R_{i-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)\sin\theta$. First we will use the Stolz-Cesàro theorem, so
$$\lim_{n\to\infty}\frac{1(n-1)a_1+2(n-2)a_2+\ldots+(n-1)\cdot 1\cdot a_{n-1}}{n}=\lim_{n\to\infty}na_n.$$
In order to prove the equiconvergence we use (1.2), the approximation [8, Theorem 8.21.8]
$$\begin{aligned}
P_n^{(\alpha,\beta)}(\cos\theta) &= n^{-1/2}k(\theta)cos(N\theta+\gamma)+O(n^{-3/2}),\\
k(\theta) &= \pi^{-\frac{1}{2}}(sin\frac{\theta}{2})^{-\alpha-\frac{1}{2}}(cos\frac{\theta}{2})^{\beta-\frac{1}{2}},\\
N &= n+\frac{\alpha+\beta+1}{2},\\
\gamma &= -(\alpha+\frac{1}{2})\frac{\pi}{2}, 0<\theta<\pi,
\end{aligned}$$
and [5, Lemma 2.3.]
$$\lim_{n\to\infty}n^{\alpha+\frac{1}{2}}\int_{-1}^{1}f(y)R_n^{(\alpha,\beta)}(y)d\mu_{\alpha,\beta}(y)=0,$$
for $\alpha,\beta>-1$, $f\in L_1(\min(\alpha,\alpha/2-1/4),\min(\beta,\beta/2-1/4))$, which is a direct generalization of the Riemann-Lebesgue theorem. Finally we get
$$\begin{aligned}
\lim_{n\to\infty}na_n &= \lim_{n\to\infty}n\hat{f}(n)\omega_n^{(-\frac{1}{2},-\frac{1}{2})}R_{n-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)\sin\theta\\
&= \lim_{n\to\infty}n\hat{f}(n)\omega_n^{(-\frac{1}{2},-\frac{1}{2})}\frac{P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)}{P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(1)}\sin\theta\\
&= 0,
\end{aligned}$$
as $P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(1)\sim(n-1)^{1/2}$.

□

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