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PROXIMAL PLANAR SHAPE SIGNATURES. HOMOLOGY NERVES AND DESCRIPTIVE PROXIMITY

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Dedicated to J.H.C. Whitehead and Som Naimpally

ABSTRACT. This article introduces planar shape signatures derived from homology nerves, which are intersecting 1-cycles in a collection of homology groups endowed with a proximal relator (set of nearness relations) that includes a descriptive proximity. A 1-cycle is a closed, connected path with a zero boundary in a simplicial complex covering a finite, bounded planar shape. The *signature of a shape* sh$A$ (denoted by sig(sh$A$)) is a feature vector that describes sh$A$. A signature sig(sh$A$) is derived from the geometry, homology nerves, Betti number, and descriptive CW topology on the shape sh$A$. Several main results are given, namely, (a) every finite, bounded planar shape has a signature derived from the homology group on the shape, (b) a homology group equipped with a proximal relator defines a descriptive Leader uniform topology and (c) a description of a homology nerve and union of the descriptions of the 1-cycles in the nerve have same homotopy type.

1. INTRODUCTION

This paper introduces shape signatures restricted to the Euclidean plane. A finite, bounded *planar shape* A (denoted by sh$A$) is a finite region of the Euclidean plane bounded by a simple closed curve and with a nonempty interior [36].

After covering a shape with a simplicial complex, the signature of a shape is derived from the characteristics of the simple closed connected paths derived from connections between vertices in the covering. A *path* in a simplicial complex is a sequence of connected simplexes. A *closed path* is a connected path in which one can start at any vertex v in the path and traverse the path to reach

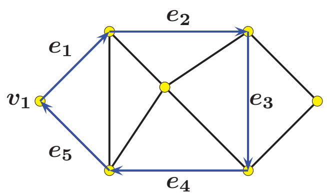

FIGURE 1. Path

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v. A *simple closed path* contains no self intersections (loops). A pair of adjacent simplexes σ_1, σ_2 are *connected*, provided σ_1, σ_2 have a common part [10, §IV.1, p. 169].

A path is oriented, provided the path can be traversed in either forward (positive) or reverse (negative) direction. In other words, for any pair of adjacent edges in an oriented path, we can choose one of the edges and the direction to take in traversing the edges (*cf.*, M. Berger and G. Gostiaux [8, §0.1.3] and J.W. Ulrich [42, §2, p. 364] on oriented graphs).

Example 1. Sample Connected 1-simplexes in a Simple Closed Path.
Let e_1, e_2, e_3, e_4, e_5 be a sequence of oriented path containing 1-simplexes (edges) as shown in Fig. 1. The ordering of the 0-simplexes (vertices) is suggested by the directed edges. For example, $e_1 \to e_2 \to e_3 \to e_4 \to e_5 \to e_1$ defines a path. This path is closed, since $e_5 \to e_1$ at the end of a traversal of the edges, starting at v_1. This closed path is simple, since it has no loops.

A triangulated shape A (also denoted by sh$A$) is connected, provided there is an edgewise simple closed path between each pair of vertices in sh$A$. Let K be a simplicial complex covering shape sh$A$. A 1-chain is a formal sum of edges leading from one vertex to another vertex on K. A 1-cycle is a 1-chain with an empty boundary. Also let σ_i denote the ith edge in a path in K, $C_1(K)$ be a set of cycles on edges on K and let $C_0(K)$ be a set of cycles on vertices on K. Let σ be a simplex spanned by the vertices $v_0, \dots, v_n$ in K. For $p \geq 1$, the homomorphic mapping $\partial_p : C_1(K) \longrightarrow C_0(K)$ is defined by

$$\partial_1 \sigma = \sum_{i=0}^{n} (-1)^i [v_0, \dots, v_n] = \sum_{i=0}^{n} \sigma_i.$$

The alternating signs on the terms indicate the simplexes are oriented, which means that for each positive term $+v_j$, there is a corresponding $-v_j, 0 \leq j \leq n$. The signs are inserted to take path orientation into account, so that all faces of a simplex are coherently oriented [19, §2.1].

The maps ∂_n are called *chain maps* (or *simplicial boundary maps*). Each chain map ∂_n is a *homomorphism*. The sum of the connected, oriented paths is called a *chain*. For a path with n edges in a triangulated planar shape, ∂_n defines a 1-chain. The vertices on a 1-simplex (edge) σ_i are the boundaries on σ_i. In other words, the boundary of n vertices $[v_0, \dots, v_n]$ is the $(n-1)$-chain formed by the sum of the faces [19, §2.1]. For a 1-chain $c = \sum \lambda_i \sigma_i, \lambda_i \in \mathbb{Z}\text{mod } 2$ (*i.e.*, for an integer coefficient λ_i in a 1-chain summand, λ_i mod 2 = 0 or 1), the *boundary* of the 1-chain is the sum of the boundaries of its 1-simplexes, namely,

$$\partial c = \lambda_1 \partial \sigma_1 + \cdots + \lambda_n \partial \sigma_n = \sum_{i=1}^{n} \lambda_i \partial \sigma_i.$$

Let K be a simplicial complex and let $C_2(K), C_1(K), C_0(K)$ be an additive Abelian group of 2-chains, 1-chains and 0-chains, respectively. Consider a sequence of homomorphisms (boundary maps) of Abelian groups, namely,

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Elements of img∂_2 are called boundaries. The quotient group $H_1 = Z_1/B_1 = \ker\partial_1/\text{img}\partial_2$ isolates those cycles in Z_1 with empty boundaries. Elements of H_1 are called 1-cycles, *i.e.*, those cycles in Z_1 that are not boundaries. From a quotient group perspective, elements of H_1 are cosets of img$\partial_2 = B_1$.

Let C_1 be a group of 1-chains of edges and let C_0 be a group of 0-chains of vertices. In general, p-chains under addition form an Abelian group (denoted by $(C_p, +)$ or $C_p = C_p(K)$, when addition is understood). Each member of C_0 is a 0-chain (a linear combination of vertices) on the boundary of a 1-chain in C_1. The kernel $\partial_1 : C_1(K) \longrightarrow C_0(K)$ is a group denoted by Z_1. Elements of ker∂_1 are called cycles. The image of ∂_2 is the group $B_1 = B_1(K)$, which is a subgroup of Z_1.

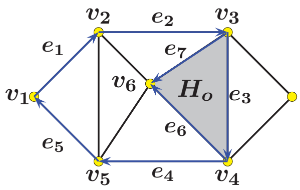

FIGURE 2. **1-cycle**

Example 2. Sample Cycles.
For example, let edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and vertices $v_1, v_2, v_3, v_4, v_5, v_6$ on a triangulated shape (not shown) be represented in Fig. 2. Then, we have

- $\boldsymbol{B_1}$**:** *collection of boundaries written as 1-chains,* e.g.,
  - $\partial(e_3, e_6, e_7) = \partial H_o = v_3 + v_4 - v_6$ *is the boundary of the hole H_o in Fig. 2.*
- $\boldsymbol{Z_1}$**:** *collection of cycles written as 1-chains. For simplicity, we consider only three cycles in Z_1 based on the labelled edges in Fig. 2, namely,*
  - $\partial\,(e_1, e_2, e_3, e_4, e_5) = v_1 + v_2 + v_3 + v_4 + v_5 - v_5 - v_4 - v_3 - v_2 - v_1 = 0.$
  - $\partial\,(e_1, e_2, e_7, e_6, e_4, e_5) = v_1+v_2+v_3+v_6+v_4+v_5-v_5-v_4-v_6-v_4-v_3-v_2-v_1 = 0.$
  - $\partial(e_3, e_6, e_7) = \partial H_o = v_3 + v_4 - v_6$ *(appears in B_1).*

Remark 1.1. *With the quotient group H_1, we factor out of Z_1 the chains that are the hole boundaries in B_1. From the features of the 1-cycles in homology groups H_1, we define a signature of a shape based on the description of 1-cycles, which is easily compared with the signatures of other shapes.*

Let $(\mathcal{H}_1, \delta_\Phi)$ be a collection of 1-cycles on shape complexes equipped with a descriptive proximity δ_Φ [12, §4], [32, §1.8], based on the descriptive intersection $\underset{\Phi}{\cap}$ of nonempty sets A and B [28, §3]. With respect to 1-cycle sets of connected, oriented edges e_1, e_2 in H_1, for example, we consider $e_1 \underset{\Phi}{\cap} e_2$. For each given 1-cycle A (denoted by cyc$A$), find all 1-cycles cyc$B$ in $\mathcal{H}_1$ that have nonempty descriptive intersection with cyc$A$, *i.e.*, cyc$A \underset{\Phi}{\cap}$ cyc$B \neq \emptyset$. This results in a Leader uniform topology on H_1 [23] and a main result in this paper.

Let $A \overset{\wedge\!\!\wedge}{\delta} B$ be a strong proximity between nonempty sets A and B, *i.e.*, A and B have nonempty intersection.

Theorem 1.1. *Let $\left(\mathcal{H}_1, \left\{\overset{\wedge\!\!\wedge}{\delta}, \delta_\Phi\right\}\right)$ be a collection of 1-dimensional homology groups H_1 equipped with a proximal relator $\left\{\overset{\wedge\!\!\wedge}{\delta}, \delta_\Phi\right\}$ and which is a collection of 1-cycles on a simplicial complex covering a finite, bounded planar shape and let*

$$\Phi(\mathcal{H}_1) = \{\Phi(\text{cyc}A) : \text{1-cycle cyc}A \in \mathcal{H}_1\} \quad \textbf{(Set of descriptions of cyc}\boldsymbol{A} \in \boldsymbol{\mathcal{H}_1}\textbf{)}$$

be a set of descriptions $\Phi(\text{cyc}A)$ of 1-cycles cyc$A$ in $\mathcal{H}_1$. A Leader uniform topology is derivable from $\Phi(\mathcal{H}_1)$.

2. Preliminaries

This section briefly presents the basic approach to defining finite, bounded planar shape barcodes based on two useful proximities (strong spatial proximity $\overset{\wedge\wedge}{\delta}$ and descriptive proximity δ_Φ). A *shape barcode* is a feature vector that describes a specific shape in terms of 1-cycle geometry, rank of H_1, characteristics of a homology nerve on H_1, closure finiteness and 1-cycle arc characteristics based on a descriptive weak topology on H_1. By *proximity* of a pair of sets, we mean spatial closeness of the sets. For a complete introduction to spatial proximity, see A. Di Concilio [13] and the earlier overview of proximity by S.A. Naimpally and B.D. Warrack [26]. A proximal hit-and-miss topology is a natural outcome of the traditional forms of proximity (see, *e.g.*, G. Beer [5, §2.2, p. 45]). By descriptive proximity of a pair of sets, we mean the closeness of the descriptions of the sets. For a complete study of descriptive proximity, see A. Di Concilio, C. Guadagni, J.F. Peters and S. Ramanna [12]. In Section 2.5, a descriptive CW topology (Closure finite Weak topology) is defined for a collection $\mathcal{H}_1$ of homology groups H_1 equipped the descriptive proximity δ_Φ.

2.1. **Basic Approach.** The basic approach in homology in classifying a finite, bounded planar shape sh$A$ covered with a simplicial complex K is to analyze a collection $\mathcal{H}_1$ of homology groups H_1 on sh$A$, which is a set of 1-cycles. A *1-cycle* A in $\mathcal{H}_1$ (denoted by cyc$A$) is a simple, closed, connected path containing 1-simplexes (edges) that are not boundaries of holes in sh$A$. The story starts by identifying 1-dimensional homology groups Z_1 (*i.e.*, groups whose members are cycles that are closed, connected paths on 1-simplexes) and 1-dimensional groups B_1 containing cycles that are boundaries of holes. From Z_1 and B_1, we then derive a homology group $H_1 = Z_1/B_1$ (a quotient group which factors out the cycle boundaries in Z_1) containing 1-cycles.

Notice that every planar shape has a distinguished 1-cycle, namely, the contour of a shape. The features (distinguishable characteristics) of 1-cycles in H_1 provide a barcode for a particular shape sh$A$, which is a feature vector in an n-dimensional Euclidean space $\mathbb{R}^n$. A shape sh$A$ barcode describes sh$A$ and is an instance of the signature of the shape (denoted by sig(sh$A$)). In the study of a shape sh$A$ that *persists* and yet changes over time, the rank of H_1 is an important shape characteristic to include in the signature sig(sh$A$). In simple terms, the rank of H_1 is the number of 1-cycles in H_1 [6, §2.2, p. 96] on complex K on a shape sh$A$. The *rank* of H_1 (denoted by rH_1) is also called the Betti number of H_1. Viewing the rank of H_1 in another way, the Betti number of H_1 is the number $\mathbb{Z}$ summands, when H_1 is written as the direct sum of its cyclic subgroups [19, §2.1, p. 1390]. For example, the rank of Z_1 for Example 2 is 2.

2.2. **Framework for Two Recent Proximities.** This section briefly presents a framework for two recent types of proximities, namely, *strong proximity* and the more recent *descriptive proximity* in the study of computational proximity [32].

Let A be a nonempty set of vertices, $p \in A$ in a bounded region X of the Euclidean plane. An *open ball* $B_r(p)$ with radius r is defined by

$$B_r(p) = \{q \in X : \|p - q\| < r\} \text{ (Open ball with center } p\text{, radius } r).$$

The *closure* of A (denoted by cl$A$) is defined by

$$\text{cl}A = \{q \in X : B_r(q) \subset A \text{ for some } r\} \text{ (Closure of set } A).$$

The *boundary* of A (denoted by bdy$A$) is defined by

$$\text{bdy}A = \{q \in X : B(q) \subset A \cap X \setminus A\} \text{ (Boundary of set } A).$$

Of great interest in the study of shapes is the interior of a shape, found by subtracting the boundary of a shape from its closure. In general, the *interior* of a nonempty set $A \subset X$ (denoted by int$A$) defined by

$$\text{int}A = \text{cl}A - \text{bdy}A \text{ (Interior of set } A).$$

Proximities are nearness relations. In other words, a *proximity* between nonempty sets is a mathematical expression that specifies the closeness of the sets. A *proximity space* results from endowing a nonempty set with one or more proximities. Typically, a proximity space is endowed with a common proximity such as the proximities from Čech [41], Efremovič [15], Lodato [24], and Wallman [44], or the more recent descriptive proximity [29].

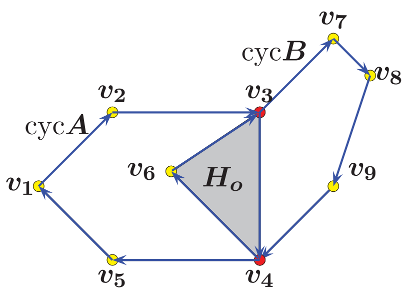

FIGURE 3. cyc$\boldsymbol{A} \stackrel{\wedge\wedge}{\delta}$ cyc$\boldsymbol{B}$

2.3. **Strong Proximity.** Nonempty sets A, B in a space X equipped with the strong proximity $\stackrel{\wedge\wedge}{\delta}$ are *strongly near* [*strongly contacted*] (denoted $A \stackrel{\wedge\wedge}{\delta} B$), provided the sets have at least one point in common.L The strong contact relation $\stackrel{\wedge\wedge}{\delta}$ was introduced in [31] and axiomatized in [38], [18, §6 Appendix] (see, also, [32, §1.5], [31, 37]) and elaborated in [32].

Let $A, B, C \subset X$ and $x \in X$. The relation $\stackrel{\wedge\wedge}{\delta}$ on the family of subsets 2^X is a *strong proximity*, provided it satisfies the following axioms.

(snN0): $\emptyset \stackrel{\wedge\wedge}{\not\delta} A, \forall A \subset X$, and $X \stackrel{\wedge\wedge}{\delta} A, \forall A \subset X$.

(snN1): $A \stackrel{\wedge\wedge}{\delta} B \Leftrightarrow B \stackrel{\wedge\wedge}{\delta} A$.

(snN2): $A \stackrel{\wedge\wedge}{\delta} B$ implies $A \cap B \neq \emptyset$.

(snN3): If $\{B_i\}_{i \in I}$ is an arbitrary family of subsets of X and $A \stackrel{\wedge\wedge}{\delta} B_{i^*}$ for some $i^* \in I$ such that int$(B_{i^*}) \neq \emptyset$, then $A \stackrel{\wedge\wedge}{\delta} (\bigcup_{i \in I} B_i)$

(snN4): int$A \cap$ int$B \neq \emptyset \Rightarrow A \stackrel{\wedge\wedge}{\delta} B$.

When we write $A \stackrel{\wedge\wedge}{\delta} B$, we read A is *strongly near* B (A *strongly contacts* B). The notation $A \stackrel{\wedge\wedge}{\not\delta} B$ reads A is not strongly near B (A does not *strongly contact* B). For each *strong proximity* (*strong contact*), we assume the following relations:

(snN5): $x \in \text{int}(A) \Rightarrow x \stackrel{\wedge\wedge}{\delta} A$

(snN6): $\{x\} \stackrel{\wedge\wedge}{\delta} \{y\} \Leftrightarrow x = y$

For strong proximity of the nonempty intersection of interiors, we have that $A \stackrel{\wedge\wedge}{\delta} B \Leftrightarrow$ int$A \cap$ int$B \neq \emptyset$ or either A or B is equal to X, provided A and B are not singletons; if $A = \{x\}$, then $x \in \text{int}(B)$, and if B too is a singleton, then $x = y$. It turns out that if $A \subset X$ is an open set, then each point that belongs to A is strongly near A. The bottom line is that strongly near sets always share points, which is another way of saying that sets with strong contact have nonempty intersection.

Example 3. *Assume that a finite, bounded shape sh$A$ is covered by a simplicial complex containing 1-cycles cyc$A$, cyc$B$. Let 1-cycle cyc$A$ be represented by a sequence of vertices*

$$v_1 \to v_2 \to v_3 \to v_6 \to v_4 \to v_5 \to v_1 \text{ (cyc}A\text{)},$$

and let 1-cycle cyc$B$ be represented by a sequence of vertices

$$v_3 \to v_7 \to v_8 \to v_9 \to v_4 \to v_6 \to v_3 \text{ (cyc}B\text{)},$$

as shown in Fig. 3. Notice, for example, that the interior of 1-cycle cyc$A$ includes the arc $\widehat{v_3v_6}$, which is also in the interior of 1-cycle $cycB$. In this case, int$(\text{cyc}A)$, int$(\text{cyc}B)$ have $\widehat{v_3v_6}$ in common. Hence, from axiom (snN4), cyc$A$ $\overset{\wedge\!\!\!\wedge}{\delta}$ cyc$B$.

Definition 2.1. *Let K be a simplicial complex covering a shape sh$A$ and let $\mathcal{H}_1$ be the collection of 1-cycles in the homology groups on K. A* homology nerve *on $\mathcal{H}_1(K)$ (denoted by Nrv$\mathcal{H}_1$) is defined by*

$$Nrv\mathcal{H}_1 = \left\{cycA \in \mathcal{H}_1 : \bigcap cycA \neq \emptyset\right\} \text{ (\textbf{Homology Nerve})}.$$

The assumption made here is that every finite planar shape is bounded by a simple closed curve and has a nonempty interior.

Conjecture 2.1. *Every finite, bounded, planar shape with a decomposition and with at least one hole contains a homology nerve that intersects with the boundary of a hole.*

Conjecture 2.2. *Every finite, bounded, planar shape with a decomposition and with at least one hole contains a homology nerve that does not intersect with the boundary of any hole.*

Remark 2.1. Short History of Topological Nerves.
In topology, a nerve structure first appeared in 1926 in a paper on simplicial approximation by P. Alexandroff [3] *and in 1932 in a monograph by P. Alexandroff* [2, §, p. 39]*, elaborated by C. Kuratowski in 1933* [22]*. Let the system of sets $F_1, \ldots, F_s$ and system of vertices $v_1, \ldots, v_s$ of a complex K be related in such a way that the sets $F_{i_1}, \ldots, F_{r_F}$ have nonempty intersection if and only if the vertices $v_{i_0}, \ldots, v_{i_r}$ belong to K. Then the complex K is called the nerve of the system of sets in K. A fundamental theorem concerning simplicial nerve complexes is given by B. Grünbaum in 1970* [17]*, namely,*

Theorem 2.1. *Each simplicial complex has the same homotopy type as its nerve.*

Earlier, K. Borsuk obtained the following result in 1948.

Theorem 2.2. [9, Cor. 2, p. 233] *Finite dimensional spaces admitting similar regular decompositions have necessarily the same homotopy type.*

As a result of Theorem 2.2, K. Borsuk observed that (i) for every finite dimensional space with a regular decomposition, there exists a polytope with the same homotopy type and (ii) the notion of an Alexandroff nerve makes it possible to construct such a polytope [9, p. 233]*, leading to*

Corollary 2.1. [9, Cor. 3, p. 234] *If the simplicial complex K is a geometrical realization of the nerve of a regular decomposition of a finite dimensional space A, then the space A and the polytope $|K|$ have the same homotopy type.*

A more tractable view of a nerve, more amenable for computational topology, is given by H. Edelsbrunner and J.L. Harer [14, §III.2, p. 59]. *Let F be a finite collection of sets. A nerve* *consists of all nonempty subcollections of F (denoted by $NrvF$) whose sets have nonempty intersection,* i.e.,

$$NrvF = \left\{X \subseteq F : \bigcap X \neq \emptyset\right\} \text{ (Edelsbrunner-Harer Nerve)}.$$

A nerve is an example of an abstract simplicial complex, regardless of the sets in F. Strongly proximal Edelsbrunner-Harer nerves were introduced in 2016 by J.F. Peters and E. İnan [39]. *Nerve spoke complexes (useful for nerves on Voronoï tessellations) are introduced in J.F. Peters* [35]. *An overview of recent work on nerve complexes is given by H. Dao, J. Doolittle, K. Duna, B. Goeckner, B. Holmes and J. Lyle* [11].

Example 4. *Assume that a shape shA is covered by a simplicial complex with homology groups H_1 containing 1-cycles $cycA, cycB$ from Example 3. Hence,*

$$Nrv\mathcal{H}_1 = \{cycA, cycB\} \ \textbf{(Sample homology nerve)}.$$

Lemma 2.1. *Let homology groups H_1 contain 1-cycles $cycA, cycB$ on complex K covering shape shA. Then $cycA \overset{\wedge\wedge}{\delta} cycB \Rightarrow cycA \cap cycB \neq \emptyset$, if and only if $cycA, cycB \in Nrv\mathcal{H}_1$ for homology nerve complex $Nrv\mathcal{H}_1 \in 2^{\mathcal{H}_1}$.*

Proof. $\mathrm{cyc}A \overset{\wedge\wedge}{\delta} \mathrm{cyc}B \Rightarrow \mathrm{cyc}A \cap \mathrm{cyc}B \neq \emptyset$ (from (snN2)) $\Leftrightarrow \mathrm{cyc}A, \mathrm{cyc}B \in \mathrm{Nrv}\mathcal{H}_1$ (from Def. 2.1) for at least one nerve complex $\mathrm{Nrv}\mathcal{H}_1 \in 2^{\mathcal{H}_1}$. □

Lemma 2.2. *Let $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1$ be homology nerves for homology groups H_1 on complex K covering shape shA. Then $Nrv_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_2\mathcal{H}_1$ implies $Nrv_1\mathcal{H}_1 \cap Nrv_2\mathcal{H}_1 \neq \emptyset$ for some $cycA \in Nrv_1\mathcal{H}_1$ and $cycB \in Nrv_2\mathcal{H}_1$.*

Proof. $\mathrm{Nrv}_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_2\mathcal{H}_1 \Rightarrow \mathrm{Nrv}_1\mathcal{H}_1 \cap \mathrm{Nrv}_2\mathcal{H}_1 \neq \emptyset$ (from (snN2)). Consequently, $\mathrm{cyc}A \overset{\wedge\wedge}{\delta} \mathrm{cyc}B \rightarrow \mathrm{cyc}A \cap \mathrm{cyc}B \neq \emptyset$ (from Lemma 2.1) for at least one $\mathrm{cyc}A \in \mathrm{Nrv}\mathcal{H}_1$ and for at least one $\mathrm{cyc}B \in \mathrm{Nrv}\mathcal{H}_2$, since a homology nerve is a set of 1-cycles (from Def. 2.1). □

Theorem 2.3. *Let $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1$ be homology nerves for homology groups H_1 on a simplicial complex covering shape shA. $Nrv_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_2\mathcal{H}_1$ if and only if $cycA \overset{\wedge\wedge}{\delta} cycB$ for some $cycA \in Nrv_1\mathcal{H}_1$ and $cycB \in Nrv_2\mathcal{H}_1$.*

Proof. Immediate from Lemma 2.2. □

Corollary 2.2. *Let $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1, Nrv_3\mathcal{H}_1$ be homology nerves for homology groups H_1 on a simplicial complex covering shape shA. If $(Nrv_1\mathcal{H}_1 \cup Nrv_2\mathcal{H}_1) \overset{\wedge\wedge}{\delta} Nrv_3\mathcal{H}_1$, then*

$$Nrv_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_3\mathcal{H}_1$$

or

$$Nrv_2\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_3\mathcal{H}_1$$

for the three homology nerves $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1, Nrv_3\mathcal{H}_1$ on the homology groups H_1.

Proof. From Lemma 2.2, $(\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1) \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1 \Rightarrow (\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1) \cap \mathrm{Nrv}_2\mathcal{H}_1 \neq \emptyset$. And, from Theorem 2.3, $(\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1) \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1$ if and only if $\mathrm{cyc}A \overset{\wedge\wedge}{\delta} \mathrm{cyc}B$ for some $\mathrm{cyc}A \in (\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1)$ and $\mathrm{cyc}B \in \mathrm{Nrv}_3\mathcal{H}_1$. Hence, $\mathrm{Nrv}_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1$ or $\mathrm{Nrv}_2\mathcal{H}_1 \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1$. □

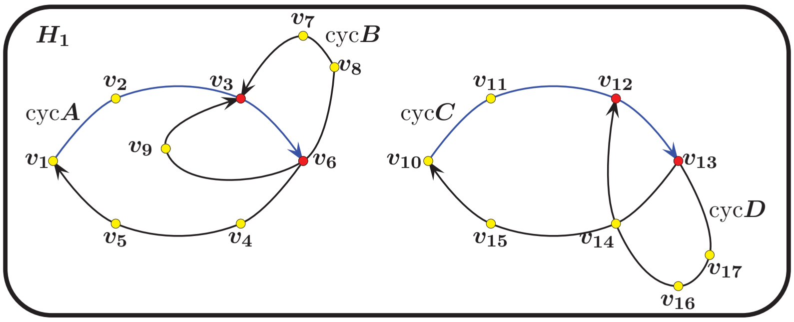

FIGURE 4. cyc$A \ \overset{\wedge\!\!\wedge}{\delta}\ $ cyc$B$ and cyc$A\ \delta_\Phi$ cyc$C$

2.4. **Descriptive Proximity.** In the run-up to a close look at extracting features from shape complexes, we first consider descriptive proximities introduced in [29], fully covered in [12] and briefly introduced, here. Descriptive proximities resulted from the introduction of the descriptive intersection of pairs of nonempty sets [29], [25, §4.3, p. 84].

(Φ): $\Phi(A) = \{\Phi(x) \in \mathbb{R}^n : x \in A\}$, set of feature vectors.
($\underset{\Phi}{\cap}$): $A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \Phi(A) \& \in \Phi(x) \in \Phi(B)\}$.

Let $\Phi(x)$ be a feature vector for an arc x in a simplicial complex on a planar shape. For example, let $\Phi(x)$ be a feature vector representing single arc feature such as a Fourier descriptor in measuring the difference between arcs in a complex [27] or uniform iso-curvature of arc along a curved edge [7, §2.2]. For simplicity, we limit the description of an arc to the uniform iso-curvature of the arc between vertices in the curved edges of a 1-cycle such as those shown in Fig. 4. $A\ \delta_\Phi\ B$ reads A is descriptively near B, provided $\Phi(x) = \Phi(y)$ for at least one pair of points, $x \in A, y \in B$. The proximity δ in the Čech, Efremovič, and Wallman proximities is replaced by δ_Φ, which satisfies the following Descriptive Lodato Axioms from [30, §4.15.2].

(dP0): $\emptyset \not\delta_\Phi A, \forall A \subset X$.
(dP1): $A\ \delta_\Phi\ B \Leftrightarrow B\ \delta_\Phi\ A$.
(dP2): $A \underset{\Phi}{\cap} B \neq \emptyset \Rightarrow\ A\ \delta_\Phi\ B$.
(dP3): $A\ \delta_\Phi\ (B \cup C) \Leftrightarrow A\ \delta_\Phi\ B$ or $A\ \delta_\Phi\ C$.
(dP4): $A\ \delta_\Phi\ B$ and $\{b\}\ \delta_\Phi\ C$ for each $b \in B\ \Rightarrow A\ \delta_\Phi\ C$ (**Descriptive Lodato**).

Proposition 2.1. [36, §2.2] *Let (X, δ_Φ) be a descriptive proximity space, $A, B \subset X$. Then $A\ \delta_\Phi\ B \Rightarrow A \underset{\Phi}{\cap} B \neq \emptyset$.*

Proof. See [36, §2.2] for the proof. $\square$

Next, consider a proximal form of a Száz relator [40]. A *proximal relator* $\mathcal{R}$ is a set of relations on a nonempty set X [33]. The pair $(X, \mathcal{R})$ is a proximal relator space. The connection between $\overset{\wedge\!\!\wedge}{\delta}$ and δ is summarized in Prop. 2.3.

Lemma 2.3. [36, §2.2] *Let $\left(X, \left\{\delta_\Phi, \overset{\wedge\!\!\wedge}{\delta}\right\}\right)$ be a proximal relator space, $A, B \subset X$. Then $A\ \overset{\wedge\!\!\wedge}{\delta}\ B \Rightarrow A\ \delta_\Phi\ B$.*

Proof. See [36, §2.2] for the proof. $\square$

Example 5. Descriptively Near 1-Cycles in H_1. *Let $cycA, cycB, cycC, cycD$ in Fig. 4 be 1-cycles in a collection of homology groups $\mathcal{H}_1$ on a simplicial complex covering a planar shape. Further, for example, let*

$$\Phi(cycA) = \left\{\Phi(\widehat{vv'}) \in cycA : \Phi(\widehat{vv'}) = \text{ uniform iso-curvature of } \widehat{vv'}\right\}.$$

Let $\mathcal{H}_1$ be equipped with the relator $\left\{\overset{\wedge\!\!\wedge}{\delta}, \delta_\Phi\right\}$. Then observe

1^o *In Fig. 4,*

$$cycA \text{ has } \overbrace{[v_1, v_2, v_3, v_4, v_5, v_6]}^{cycA \text{ vertices}} : v_1 \to v_2 \to v_3 \to v_6 \to v_4 \to v_5 \to v_1.$$

$$cycB \text{ has } \overbrace{[v_3, v_6, v_7, v_8]}^{cycB \text{ vertices}} : v_3 \to v_6 \to v_8 \to v_7 \to v_3.$$

$$\text{edge } \widehat{v_3v_6} \in int(cycA) \text{ and } \widehat{v_3v_6} \in int(cycB), \text{i.e.}, int(cycA) \cap int(cycB) \neq \emptyset.$$

Consequently, from Axiom (snN4), $cycA \overset{\wedge\!\!\wedge}{\delta} cycB$ and from Lemma 2.3, $cycA\ \delta_\Phi\ cycB$. Hence, from Proposition 2.1, $cycA \underset{\Phi}{\cap} cycB \neq \emptyset$.

2^o *In Fig. 4,*

$$cycC \text{ has } \overbrace{[v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}]}^{cycC \text{ vertices}} :$$
$$v_{10} \to v_{11} \to v_{12} \to v_{13} \to v_{14} \to v_{15} \to v_{10}.$$

$$cycD \text{ has } \overbrace{[v_{12}, v_{13}, v_{17}, v_{16}, v_{14}]}^{cycD \text{ vertices}} : v_{12} \to v_{13} \to v_{17} \to v_{16} \to v_{14} \to v_{12}.$$

$$cycA\ \delta_\Phi\ cycC \text{ , since } \overbrace{\Phi(\widehat{v_3v_6}) = \Phi(\widehat{v_{12}, v_{13}})}^{\text{arcs have matching uniform iso-curvature}}.$$

Consequently, $cycA \underset{\Phi}{\cap} cycC \neq \emptyset$. From Axiom (dP2), $cycA\ \delta_\Phi\ cycC$. Hence, from Proposition 2.1, the converse also holds, i.e.*,*

$$cycA\ \delta_\Phi\ cycC \Rightarrow cycA \underset{\Phi}{\cap} cycC \neq \emptyset.$$

In other words, the 1-cycles $cycA, cycC$ in homology groups $\mathcal{H}_1$ represented in Fig. 4 have descriptive proximity, since $cycA, cycC$ have curved edges with the same uniform iso-curvature.

Let $2^{2^{\mathcal{H}_1}}$ denote a collection of sub-collections of 1-cycles $\mathcal{H}_1$.

Theorem 2.4. *Let $\left(\mathcal{H}_1, \left\{\delta_\Phi, \overset{\wedge\!\!\wedge}{\delta}\right\}\right)$ be a collection of homology groups endowed with a proximal relator and let 1-cycles $cycA, cycB \in \mathcal{H}_1$, homology nerves $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1 \in 2^{2^{\mathcal{H}_1}}$. Then*

1^o *$Nrv_1\mathcal{H}_1 \overset{\wedge\!\!\wedge}{\delta} Nrv_2\mathcal{H}_1$ implies $Nrv_1\mathcal{H}_1\ \delta_\Phi\ Nrv_2\mathcal{H}_1$.*
2^o *A 1-cycle $cycA \in Nrv_1\mathcal{H}_1 \cap Nrv_2\mathcal{H}_1$ implies $cycA \in Nrv_1\mathcal{H}_1 \underset{\Phi}{\cap} Nrv_2\mathcal{H}_1$.*
3^o *$cycA \overset{\wedge\!\!\wedge}{\delta} cycB \to cycA\ \delta_\Phi\ cycB$.*

Proof.
1^o: Immediate from Lemma 2.3.
2^o: Let $cycA \in \mathcal{H}_1$. cyc$A \in \text{Nrv}_1\mathcal{H}_1 \cap \text{Nrv}_2\mathcal{H}_1$, provided $\text{Nrv}_1\mathcal{H}_1 \overset{\wedge\!\wedge}{\delta} \text{Nrv}_2\mathcal{H}_1$. Then cyc$A \in \text{Nrv}_1\mathcal{H}_1 \underset{\Phi}{\cap} \text{Nrv}_2\mathcal{H}_1$. Hence, from Prop. 2.1, $\text{Nrv}_1\mathcal{H}_1\ \delta_\Phi\ \text{Nrv}_2\mathcal{H}_1$.
3^o: Immediate from Lemma 2.3. □

Corollary 2.3. *Let* $\left(\mathcal{H}_1, \left\{\delta_\Phi, \overset{\wedge\!\wedge}{\delta}\right\}\right)$ *be a collection of homology groups endowed with proximal relator, homology nerves* $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1 \in 2^{2^{\mathcal{H}_1}}$ *with* $Nrv_2\mathcal{H}_1$ *on shape* shB. *Then*

1^o $Nrv_1\mathcal{H}_1 \overset{\wedge\!\wedge}{\delta} Nrv_2\mathcal{H}_1$ *implies* $Nrv_1\mathcal{H}_1\ \delta_\Phi\ shB$.
2^o $Nrv_1\mathcal{H}_1 \cap Nrv_2\mathcal{H}_1 \neq \emptyset$ *implies* $Nrv_1\mathcal{H}_1 \underset{\Phi}{\cap} Nrv_2\mathcal{H}_1$.

2.5. **Descriptive Homology Nerves and Shape Signature.** This section introduces descriptive homology nerves and the components of a shape signature.

Definition 2.2. *Let* K *be a simplicial complex covering a shape* shA *and let* $\mathcal{H}_1$ *be the collection of 1-cycles in homology groups* H_1 *on* K. *A* descriptive homology nerve *on* $\mathcal{H}_1(K)$ *(denoted by* $Nrv_\Phi\mathcal{H}_1$*) is defined by*

$$Nrv_\Phi\mathcal{H}_1 = \left\{cycA \in \mathcal{H}_1 : \bigcap_{\Phi} cycA \neq \emptyset\right\} \ (\textbf{Descriptive Homology Nerve}).$$

The nucleus *of a descriptive homology nerve is any member* $cycA \in Nrv_\Phi\mathcal{H}_1$ *that serves as a representative of the nerve inasmuch as* $cycA$ *defines a cluster* X *that contains all 1-cycles* $cycB$ *such that* $cycA\ \delta_\Phi\ cycB$.

Theorem 2.5. *Let* K *be a simplicial complex covering a finite, bounded planar shape,* $\mathcal{H}_1$ *a collection of homology groups on* K*, and* $\Phi(\mathcal{H}_1)$ *a set of descriptions of the 1-cycles in* $\mathcal{H}_1$. *Every member of* $\Phi(\mathcal{H}_1)$ *is the nucleus of a descriptive homology nerve* $Nrv_\Phi\mathcal{H}_1$.

Proof.
By definition, $\Phi(\mathcal{H}_1) = \{\Phi(\text{cyc}A) : \text{cyc}A \in \mathcal{H}_1\}$. Let $\Phi(\text{cyc}A) \in \Phi(\mathcal{H}_1)$. Since cyc$A\ \delta_\Phi$ cyc$A$, then, from Def. 2.2, cyc$A$ is the nucleus of a descriptive nerve $\text{Nrv}_\Phi\mathcal{H}_1$ containing one cycle, namely, cyc$A$. Let

$$X = \{\text{cyc}B \in \mathcal{H}_1 : \text{cyc}A\ \delta_\Phi\ \text{cyc}B \neq \emptyset\}.$$

Hence, by Def. 2.2, X is a descriptive homology nerve and cyc$A$ is the nucleus of the nerve X, *i.e.*, cyc$A\ \delta_\Phi$ cyc$B$ for every member cyc$B \in X$. □

Example 6. *Let the collection of homology groups* $\mathcal{H}_1$ *be represented the 1-cycles* $cycA$, $cycB$, $cycC$, $cycD$ *in Fig. 4. Let uniform iso-curvature be used to describe a 1-cycle in* $\mathcal{H}_1$. *Notice that curved edge* $\widehat{v_3v_6} \in cycB$ *has the same uniform iso-curvature as* $\widehat{v_{12}v_{13}} \in cycC$ *and* $\widehat{v_{12}v_{13}} \in cycD$. *Hence,*

$$Nrv_\Phi\mathcal{H}_1 = \{\text{cyc}B, cycC, cycD \in \mathcal{H}_1\} \ (\textbf{Descriptive Homology Nerve}),$$

since

$$\bigcap_{\substack{cycX\in\\ \{cycB, cycC, cycD\}}} cycX \neq \emptyset.$$

From Theorem 2.5, $cycB$ *is the nucleus of* $Nrv_\Phi\mathcal{H}_1$.

Conjecture 2.3. *Every finite, bounded, planar shape with a decomposition and with at least one hole contains a descriptive homology nerve that intersects with the boundary of a hole.*

Conjecture 2.4. *Every finite, bounded, planar shape with a decomposition and with at least one hole contains a descriptive homology nerve that does not intersect with the boundary of any hole.*

Consider next a basis for a shape signature.

Definition 2.3. Shape Signature.
Let $\mathcal{H}_1$ be a collection of homology groups on a simplicial complex covering a shape shA, a finite bounded planar region with nonempty interior and let $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1 \in 2^{\mathcal{H}_1}$. Assume that $\mathcal{H}_1$ is equipped with a proximal relator $\left\{ \overset{\wedge\!\!\wedge}{\delta}, \delta_\Phi \right\}$. A signature of shape shA (denoted by sig(shA)) is a feature vector that includes at least one of the following components.

1^o **Geometry**: *One or more features of the curvature of each 1-cycle cyc$A \in \mathcal{H}_1$ are included in sig(shA) that describes shape shA.*

2^o **Homology**: *rank of the homology group H_1 (denoted by rH_1),* i.e., *number of 1-cycle generators of H_1 is defined in terms of the rank of the cycles group Z_1 (denoted by rZ_1) and the rank of the boundaries group B_1 (denoted by rB_1) . Recall that*

$rH_1 = r(Z_1/B_1) = rZ_1 - rB_1$ (**Rank of a homology group**) [43, p. 63].

The rank rH_1 (a Betti number) can change over time and provides a useful in indicator of planar shape persistence. Hence, its inclusion in a shape shA signature sig(shA) (barcode) is important in considering the persistent topology of data such as that found in R. Ghrist [16].

3^o **Homology Nerve**: *Since every cyc$A \in \mathcal{H}_1$ is the nucleus of a descriptive homology nerve Nrv$_\Phi\mathcal{H}_1$ (from Theorem 2.5), select a component of Φ(cyc$A$) (call it x) with a description that matches the description of the same component in the other members of Nrv$_\Phi\mathcal{H}_1$. Include $\Phi(x)$ in the signature of shA,* i.e.,

sig(shA) $= (\dots, \Phi(x)\dots)$ ($\Phi(x)$ in feature vector that describes shA).

4^o **Closure Finiteness**: *Let $\widehat{vv'}$ be an arc in a 1-cycle cyc$A \in \mathcal{H}_1$ and cl($\widehat{vv'}$) intersects only a finite number of other arcs in $\mathcal{H}_1$. cl($\widehat{vv'}$) is the closure of an arc in cyc$A \underset{\Phi}{\cap}$ cyc$B$ for a finite number of 1-cycles. For cyc$A$, cyc$B \in \mathcal{H}_1$, choose Φ(cl($\widehat{vv'}$)) $\in$ sig(shA) or Φ(cyc$A$) $\in$ sig(shA) for a selected number of 1-cycles in $\mathcal{H}_1$.*

5^o **descriptive CW**: *(*i.e., **descriptive Weak Topology***) Assume that Closure Finiteness holds for the collection of homology groups $\mathcal{H}_1$ equipped with the descriptive proximity δ_Φ. Let $\widehat{vv'}$ be an arc in $H_1 \in \mathcal{H}_1$ and let 1-cycle cyc$A \in \mathcal{H}_1$. Then cyc$A$ is closed in $\mathcal{H}_1$, provided cyc$A \cap \widehat{vv'} \neq \emptyset$ is also closed in H_1 . Then cyc$A \ \overset{\wedge\!\!\wedge}{\delta} \ \widehat{vv'}$. Hence, from Lemma 2.3, cyc$A \ \delta_\Phi \ \widehat{vv'}$. For example, 1-cycles cyc$A$, cyc$B$ in Fig. 4 overlap, since arc $\widehat{v_3v_6}$ is common to both 1-cycles. Such arcs provide an incisive feature for a shape signature. Then, for a shape shA, include the description of such arcs in the shape signature sig(shA).*

Remark 2.2. *The original idea of a CW topology (***C***losure finite* **W***eak topology) was to shift from structures in simplicial complexes K that are the focus in P. Alexandroff* [2] *and in P. Alexandroff, H. Hopf* [4] *to homological structures called cells and cell complexes (*e.g., *0-cells (vertices) and 1-cells (open arcs) attached to a shape skeleton via maps to obtain a* *cell complex) in a homology on K* [45, p. 214]. *A* cell complex *is a finite collection of cells* [19]. *With a descriptive CW, we shift from a description of structures (*e.g., *simplicial nerves* [34, p. 2] *and nerve spokes* [34, §2.2, p. 4] [1, Def. 9, p. 8]*) in simplicial complexes to a description of structures such as homology nerves, collections of 1-cycles and overlapping arcs in a collection of homology groups $\mathcal{H}_1$ in cell complexes on finite bounded planar shapes. Basically, with a descriptive CW on $\mathcal{H}_1$, we include those features of arcs, 1-cycles and homology nerves in $\mathcal{H}_1$ that provide a complete signature sig(sh$A$) for a shape sh$A$. The motivation for doing this is an interest in measuring the persistence of the feature values of arcs, 1-cycles and homology nerves in homology groups over time. This descriptive CW is based on the Closure finiteness and Weak topology axioms for a traditional CW complex given by K. Jänich* [21, §VII.3, p. 95] *founded on its original introduction by J.H.C. Whitehead* [45].

3. Main result

Theorem 3.1. *Every finite, bounded planar shape sh$A$ covered by a simplicial complex has a signature derived from the homology group on the complex.*

Proof. From Def. 2.3, it is enough to include the rank of H_1 in sig(sh$A$) for a shape sh$A$ to have a signature. □

Lemma 3.1. *Let $\mathcal{H}_1$ be a collection of homology groups equipped with the proximal relator $\mathcal{R}_\Phi = \left\{ \overset{\wedge\!\wedge}{\delta}, \delta_\Phi \right\}$ on a simplicial complex covering a finite, bounded shape. Every collection of 1-dimensional homology groups $H_1 \in \mathcal{H}_1$ endowed with the proximal relator $\mathcal{R}_\Phi$ defines a descriptive uniform Leader topology on $\mathcal{H}_1$.*

Proof. The basic approach in this proof is to use the steps for constructing a uniform topology introduced by S. Leader [23] in constructing a descriptive uniform topology.

$\underset{\Phi}{\cap}$: For each $\mathrm{Nrv}_1\mathcal{H}_1 \in \mathcal{H}_1$, select all $\mathrm{Nrv}_2\mathcal{H}_1 \in \mathcal{H}_1$ such that $\mathrm{Nrv}_1\mathcal{H}_1 \ \overset{\wedge\!\wedge}{\delta}\ \mathrm{Nrv}_2\mathcal{H}_1$, *i.e.*, the pair of homology nerves $\mathrm{Nrv}_1\mathcal{H}_1 \in \mathcal{H}_1$ overlap (have strong proximity). From Lemma 2.3, $\mathrm{Nrv}_1\mathcal{H}_1 \underset{\Phi}{\cap} \mathrm{Nrv}_2\mathcal{H}_1 \neq \emptyset$. Hence, $\mathrm{Nrv}_1\mathcal{H}_1 \underset{\Phi}{\cap} \mathrm{Nrv}_2\mathcal{H}_1 \in \Phi(\mathcal{H}_1)$.

$\underset{\Phi}{\cup}$: By definition,

$$\begin{aligned}\mathrm{Nrv}_1\mathcal{H}_1 \underset{\Phi}{\cup} \mathrm{Nrv}_2\mathcal{H}_1 = \{&\mathrm{cyc}A \in \mathcal{H}_1 : \mathrm{cyc}A \in \mathrm{Nrv}_1\mathcal{H}_1 \underset{\Phi}{\cap} \mathrm{Nrv}_2\mathcal{H}_1 \\ &\text{or } \Phi(\mathrm{cyc}A) \in \Phi(\mathrm{Nrv}_1\mathcal{H}_1) \text{ or } \Phi(\mathrm{cyc}A) \in \Phi(\mathrm{Nrv}_2\mathcal{H}_1)\}.\end{aligned}$$

Hence, $\mathrm{Nrv}_1\mathcal{H}_1 \underset{\Phi}{\cup} \mathrm{Nrv}_2\mathcal{H}_1 \in \Phi(\mathcal{H}_1)$. □

Remark 3.1. *Lemma 3.1 is a stronger result than we need to derive a descriptive CW, which is a convenient setting for the study of finite, bounded planar shapes signatures. Theorem 1.1 is a direct result of Lemma 3.1.*

Theorem 3.2. [14, §III.2, p. 59] *Let $\mathcal{F}$ be a finite collection of closed, convex sets in Euclidean space. Then the nerve of $\mathcal{F}$ and the union of the sets in $\mathcal{F}$ have the same homotopy type.*

Lemma 3.2. *Let $\mathcal{H}_1$ be a collection of homology groups on a simplicial complex covering a finite, bounded shape. Then a homology nerve Nrv$\mathcal{H}_1 \in 2^{\mathcal{H}_1}$ and $\bigcup\limits_{\mathrm{cyc}A \in \mathcal{H}_1}$ cyc$A$ have same homotopy type.*

Proof. $\mathcal{H}_1$ is a collection of 1-cycles, which are closed, convex sets in Euclidean space. Then from Theorem 3.2, Nrv$\mathcal{H}_1$ and $\bigcup\limits_{\mathrm{cyc}A\in\mathcal{H}_1}$ cyc$A$ have same homotopy type. □

Theorem 3.3. *Let* $\left(\mathcal{H}_1, \left\{\overset{\wedge\wedge}{\delta}, \delta_\Phi\right\}\right)$ *be a collection of homology groups* H_1 *equipped with a proximal relator on a simplicial complex covering a finite, bounded shape. Then* $\Phi(Nrv\mathcal{H}_1)) \in 2^{\mathbb{R}^n}$ *(a description of a homology nerve) and* $\bigcup\limits_{\Phi(cycA)\in\Phi(Nrv\mathcal{H}_1)} \Phi(cycA)$ *(union of the descriptions) have same homotopy type.*

Proof. Each member of $\Phi(\mathcal{H}_1)$ is feature vector in $\mathbb{R}^n$ and each point in $\mathbb{R}^n$ is a closed, convex singleton set. Then from Lemma 3.2, $\Phi(\mathrm{Nrv}\mathcal{H}_1)$ and $\bigcup\limits_{\Phi(\mathrm{cyc}A)\in\Phi(\mathrm{Nrv}\mathcal{H}_1)} \Phi(\mathrm{cyc}A)$ have same homotopy type. □

Remark 3.2. Open Problems.
Let sh$A$ be a finite, bounded planar shape covered with a simplicial complex K and let $H_1(K)$ be a homology group on K.
An open problem in shape theory is selecting each 1-cycle that is the contour of a subshape containing a hole in sh$A$.

A second open problem in shape theory is the construction of a collection of homology nerves that overlap a subshape of interest in a shape shA.

Let $\mathcal{H}_1(K)$ be a collection of homology groups on a simplicial complex K. A third open problem in shape theory is detecting space curves (also called twisted curves by D. Hilbert and S. Cohn-Vossen [20, §27]*) overlapping with 1-cycles in $\mathcal{H}_1(K)$.*

A fourth open problem in shape theory is to use homology nerves as a basis for measuring the persistence over time of object shapes in digital images.

A fifth open problem in shape theory is to measure the persistence of a finite, bounded shape over time using a shape signature that includes the uniform iso-curvature of the 1-cycles and the Betti number of a homology group on the shape.

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