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MAJORIZATION OF BEREZIN TRANSFORM

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ABSTRACT. In this paper, we majorize the Berezin transform of positive invertible operators defined from the Bergman space $L_a^2(\mathbb{D})$ into itself. We also present sufficient conditions on bounded operators $S, T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\rho(|S|) = \rho(T)$ in terms of the Schatten norm of these operators. Here $\rho(T)$ is the Berezin transform of T. Further, given $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, we find conditions on the existence of a projection operator $E \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\rho(TE) = 0$.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $dA(z) = \frac{1}{\pi}dxdy$ denote the normalized Lebesgue area measure on $\mathbb{D}$ in the complex plane $\mathbb{C}$. For $1 \le p < \infty$ and $f : \mathbb{D} \longrightarrow \mathbb{C}$ Lebesgue measurable let $\|f\|_p = \left(\int_{\mathbb{D}} |f|^p dA(z)\right)^{1/p}$. The Bergman space $L_a^p(\mathbb{D})$ is the Banach space of analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ such that $\|f\|_p < \infty$. The Bergman space $L_a^2(\mathbb{D})$ is a Hilbert space; it is a closed subspace [3] of the Hilbert space $L^2(\mathbb{D}, dA)$ with the inner product given by $\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z),\ f, g \in L^2(\mathbb{D}, dA)$. Let P denote the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. Let $K(z, \bar{w})$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the reproducing kernel of $L_a^2(\mathbb{D})$. For any $n \ge 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$, then $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$. Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. Let $L^\infty(\mathbb{D}, dA)$ be the Banach space of all essentially bounded measurable functions f on $\mathbb{D}$ with $\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\}$ and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $\mathcal{L}(H)$ be the space of all bounded linear operators from the separable Hilbert space H into itself and $\mathcal{LC}(H)$ be the space of all compact operators in $\mathcal{L}(H)$. An operator $A \in \mathcal{L}(H)$ is called positive if $\langle Ax, x\rangle \ge 0$ holds for every $x \in H$ in which case we write $A \ge 0$. The absolute value of an operator A is the positive operator $|A|$ defined

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as $|A| = (A^*A)^{\frac{1}{2}}$. If H is infinite-dimensional, the map $|\cdot|$ on $\mathcal{L}(H)$ is not Lipschitz continuous. We define $\rho : \mathcal{L}(L^2_a(\mathbb{D})) \longrightarrow L^\infty(\mathbb{D})$ by $\rho(T)(z) = \widetilde{T}(z) = \langle Tk_z, k_z\rangle,\ z \in \mathbb{D}$. A function $g(x, \bar{y})$ on $\mathbb{D} \times \mathbb{D}$ is called of positive type (or positive definite), written $g \gg 0$, if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} g(x_j, \overline{x_k}) \geq 0$$

for any n-tuple of complex numbers $c_1, \ldots, c_n$ and points $x_1, \ldots, x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$. We shall say $\Upsilon \in \mathcal{A}$ if $\Upsilon \in L^\infty(\mathbb{D})$ and is such that

$$\Upsilon(z) = \Theta(z, \bar{z}) \tag{1.1}$$

where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y and if there exists a constant $c > 0$ such that

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0 \text{ for all } x, y \in \mathbb{D}.$$

It is a fact that (see [7], [8]) Θ as in (1.1), if it exists, is uniquely determined by Υ. In this paper, we majorize the Berezin transform of positive invertible operators belonging to $\mathcal{L}(L^2_a(\mathbb{D}))$. The organization of this paper is as follows: In Section 2, we find conditions on positive invertible operators $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(XB^{-1}X) \leq \rho(A)$ where $X \in \mathcal{L}(L^2_a(\mathbb{D}))$ is self-adjoint. In Section 3, we establish that if f is an operator monotone function on $[0, \infty)$ and $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ is positive then $\Theta_{f(EAE)}(x, \bar{y})K(x, \bar{y}) \gg \Theta_{Ef(A)E}(x, \bar{y})K(x, \bar{y})$ for all $x, y \in \mathbb{D}$ and $\rho(f(EAE)) = \rho(Ef(A)E)$ if and only if E and A commute $f(0) = 0$ and f is not a linear function. Section 4 is devoted to Schatten norm and contractions. In this section, we obtain sufficient conditions on Schatten norm of $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(|S|) = \rho(T)$ and $\rho(S) \leq \rho(T)$. Further, we also find conditions on the existence of projection operator $E \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(TE) = 0$.

2. On invertible positive operators

In this section, we find conditions on positive invertible operators $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(XB^{-1}X) \leq \rho(A)$ where $X \in \mathcal{L}(L^2_a(\mathbb{D}))$ is self-adjoint. If $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ and S is positive, then let $\Theta_S(x, \bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle}$ for all $x, y \in \mathbb{D}$.

Theorem 2.1. *Let $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ are positive and invertible and $X \in \mathcal{L}(L^2_a(\mathbb{D}))$ is self-adjoint. Then*

$$\Theta_A(x, \bar{y})K(x, \bar{y}) \gg \Theta_{XB^{-1}X}(x, \bar{y})K(x, \bar{y}) \tag{2.1}$$

if and only if

$$|\langle XK_y, K_x\rangle|^2 \leq \langle AK_x, K_x\rangle\langle BK_y, K_y\rangle \tag{2.2}$$

for all $x, y \in \mathbb{D}$. In this case $\rho(XB^{-1}X) \leq \rho(A)$.

Proof. Suppose (2.1) holds. Then

$$\langle AK_y, K_x\rangle \geq \langle XB^{-1}XK_y, K_x\rangle$$

for all $x, y \in \mathbb{D}$. The last inequality is valid if and only if

$$\sum_{i,j=1}^{n} c_j \bar{c}_i \langle AK_{x_j}, K_{x_i}\rangle \geq \sum_{i,j=1}^{n} c_j \bar{c}_i \langle XB^{-1}XK_{x_j}, K_{x_i}\rangle$$

where $x_1, x_2, \cdots, x_n \in \mathbb{D}$ and $c_j, j = 1, 2, \cdots, n$ are constants. Thus (2.1) holds if and only if

$$\left\langle A\left(\sum_{j=1}^{n} c_j K_{x_j}\right), \left(\sum_{i=1}^{n} c_i K_{x_i}\right)\right\rangle \geq \left\langle XB^{-1}X\left(\sum_{j=1}^{n} c_j K_{x_j}\right), \left(\sum_{i=1}^{n} c_i K_{x_i}\right)\right\rangle.$$

Since $\left\{\sum_{j=1}^{n} c_j K_{x_j}; x_j \in \mathbb{D}, j = 1, \cdots, n\right\}$ is dense in $L_a^2(\mathbb{D})$, hence (2.1) holds if and only if $\langle Ag, g\rangle \geq \langle XB^{-1}Xg, g\rangle$ for all $g \in L_a^2(\mathbb{D})$. That is, if and only if $A \geq XB^{-1}X$. Now considering the congruence

$$\begin{aligned}\begin{pmatrix} A & X \\ X & B \end{pmatrix} &\sim \begin{pmatrix} I & -XB^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} I & 0 \\ -B^{-1}X & I \end{pmatrix} \\ &= \begin{pmatrix} A - XB^{-1}X & 0 \\ 0 & B \end{pmatrix}\end{aligned}$$

we obtain $A \geq XB^{-1}X$ if and only if $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive. Thus (2.1) holds if and only if $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive. Suppose $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0$ in $\mathcal{L}(L_a^2 \oplus L_a^2)$. Then it follows from [2], that $\left|\left\langle \begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} K_x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ K_y \end{pmatrix}\right\rangle\right|^2$
$\leq \left\langle \begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} K_x \\ 0 \end{pmatrix}, \begin{pmatrix} K_x \\ 0 \end{pmatrix}\right\rangle \left\langle \begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} 0 \\ K_y \end{pmatrix}, \begin{pmatrix} 0 \\ K_y \end{pmatrix}\right\rangle$ for all $x, y \in \mathbb{D}$. A simplification of these inner products yields

$$|\langle XK_x, K_y\rangle|^2 \leq \langle AK_x, K_x\rangle\langle BK_y, K_y\rangle$$

for all $x, y \in \mathbb{D}$. That is,

$$|\langle XK_y, K_x\rangle|^2 \leq \langle AK_x, K_x\rangle\langle BK_y, K_y\rangle$$

for all $x, y \in \mathbb{D}$. That is, (2.2) holds. Suppose (2.2) holds for all $x, y \in \mathbb{D}$. Let $f = \sum_{j=1}^{n} c_j K_{y_j}$ and $g = \sum_{i=1}^{m} d_i K_{x_i}$ where c_j are constants for $j = 1, 2, \cdots, n$ and d_i are constants, $x_i \in \mathbb{D}$, for $i = 1, 2, \cdots, m$. Then using Heinz inequality [5] we obtain

$$|\langle Xf, g\rangle|^2 \leq \langle |X|f, f\rangle\langle |X|g, g\rangle \tag{2.3}$$

for all $f, g \in L_a^2(\mathbb{D})$. Now it follows from (2.3), that

$$\begin{aligned}\left\langle \begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle &= \langle Af, f\rangle + \langle Xg, f\rangle + \langle Xf, g\rangle + \langle Bg, g\rangle \\ &= \langle Af, f\rangle + \langle Bg, g\rangle + 2\mathrm{Re}\langle Xf, g\rangle \\ &\geq 2\langle Af, f\rangle^{1/2}\langle Bg, g\rangle^{1/2} + 2\mathrm{Re}\langle Xf, g\rangle \\ &\geq 2|\langle Xf, g\rangle| + 2\mathrm{Re}\langle Xf, g\rangle \\ &\geq 2|\langle Xf, g\rangle| - 2|\langle Xf, g\rangle| = 0\end{aligned}$$

for all $f, g \in L^2_a(\mathbb{D})$. Hence $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive. From the first part it follows that $A \geq XB^{-1}X$ and

$$\Theta_A(x,\bar{y})K(x,\bar{y}) \gg \Theta_{XB^{-1}X}(x,\bar{y})K(x,\bar{y})$$

for all $x, y \in \mathbb{D}$. The result follows. $\square$

Corollary 2.1. *Let $0 < m \leq A \leq M$ and E is a projection operator from $L^2_a(\mathbb{D})$ onto a closed subspace $\mathcal{M}$. Let $A^{-1} |_{\mathcal{M}} = A_1$ and $A |_{\mathcal{M}} = A_2$. Then*

$$\Theta_{EA_1}(x,\bar{y})K(x,\bar{y}) \gg \Theta_{(EA_2)^{-1}}(x,\bar{y})K(x,\bar{y}) \tag{2.4}$$

for all $x, y \in \mathbb{D}$. Further $\rho(EA_1) = \rho((EA_2)^{-1})$ if and only if E and A commute.

Proof. The inequality in (2.4) follows from Theorem 2.1. Notice that EA_1 and $(I-E)A^{-1} |_{\mathcal{M}^\perp}$ are invertible and

$$(EA_2)^{-1} = EA_1 - EA^{-1}((I-E)A^{-1} |_{\mathcal{M}^\perp})^{-1}(I-E)A_1.$$

Now let $(EA_2)^{-1} = EA_1$. Then $(I-E)A_1 = 0$ and this implies $EA^{-1} = A^{-1}E$. Thus $EA = AE$. $\square$

3. Operator monotone function

In this section, we establish that if f is an operator monotone function on $[0,\infty)$ and $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ is positive then

$$\Theta_{f(EAE)}(x,\bar{y})K(x,\bar{y}) \gg \Theta_{Ef(A)E}(x,\bar{y})K(x,\bar{y})$$

for all $x, y \in \mathbb{D}$ and $\rho(f(EAE)) = \rho(Ef(A)E)$ if and only if E and A commute $f(0) = 0$ and f is not a linear function.

Theorem 3.1. *Let f be an operator monotone function on $[0,\infty)$ and assume $f(0) \geq 0$. Let $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $A \geq 0$ and E is the projection operator from $L^2_a(\mathbb{D})$ onto a closed subspace $\mathcal{M}$ of $L^2_a(\mathbb{D})$. Then*

$$\Theta_{f(EAE)}(x,\bar{y})K(x,\bar{y}) \gg \Theta_{Ef(A)E}(x,\bar{y})K(x,\bar{y})$$

for all $x, y \in \mathbb{D}$. Further, $\rho(f(EAE)) = \rho(Ef(A)E)$ if and only if E and A commute $f(0) = 0$ and f is not a linear function.

Proof. Since f is operator monotone on $[0,\infty)$, hence f can be represented as

$$f(s) = a + bs + \int_0^\infty \left(\frac{1}{t} - \frac{1}{t+s}\right) d\mu(t)$$

where $a = f(0), b \geq 0$ and μ is a positive Borel measure such that

$$\int_0^\infty \frac{1}{1+t^2} d\mu(t) < \infty.$$

Let E be the projection operator from $L^2_a(\mathbb{D})$ onto the closed subspace $\mathcal{M}$ of $L^2_a(\mathbb{D})$. Then

$$\langle Ef(A)Eg, g\rangle = \langle (a+bA)Eg, Eg\rangle + \int_0^\infty \left(\frac{1}{t}I - (tI+A)^{-1}Eg, Eg\right) d\mu(t)$$

and

$$\begin{aligned}\langle f(EAE)g,g\rangle &= \langle (a+bEAE)g,g\rangle + \int_0^\infty \left\langle \left(\frac{1}{t}I-(tI+EAE)^{-1}\right)g,g\right\rangle d\mu(t)\\ &= \langle (a+bEAE)g,g\rangle + \int_0^\infty \left\langle \left(\frac{1}{t}E-(E(tI+A)\mid_{M})^{-1}\right)Eg,Eg\right\rangle d\mu(t).\end{aligned}$$

By Corollary 2.1,

$$\frac{1}{t}E-(E(tI+A)\mid_{\mathcal{M}})^{-1}\geq \frac{1}{t}E-E(tI+A)^{-1}E$$

for $t>0$ implies that $f(EAE)\geq Ef(A)E$. Thus

$$\Theta_{f(EAE)}(x,\bar{y})K(x,\bar{y})\gg \Theta_{Ef(A)E}(x,\bar{y})K(x,\bar{y})$$

for all $x,y\in\mathbb{D}$. Now if, $f(EAE)=Ef(A)E$, then for every $g\in L^2_a(\mathbb{D})$, $\langle ag,g\rangle = \langle aEg,Eg\rangle$ and

$$\langle (E(tI+A)\mid_{\mathcal{M}})^{-1}Eh,Eh\rangle = \langle (tI+A)^{-1}Eh,Eh\rangle$$

for almost every $t>0$ with respect to μ. Since $L^2_a(\mathbb{D})$ is separable, we obtain

$$(E(tI+A)\mid_{\mathcal{M}})^{-1}=E(tI+A)^{-1}E$$

for almost every $t>0$. Thus by Corollary 2.1, $E(tI+A)=(tI+A)E$ and hence $EA=AE$ and $f(0)=a=0$. $\square$

4. Schatten norm and contractions

In this section, we obtain sufficient conditions on Schatten norm of $S,T\in\mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(|S|)=\rho(T)$ and $\rho(S)\leq\rho(T)$. Further, we also find conditions on the existence of $E\in L^2_a(\mathbb{D})$ such that $\rho(TE)=0$. From [5], it follows that if $T\in\mathcal{L}(H)$, then $|\langle Tx,y\rangle|^2\leq\langle |T|^{2\alpha}x,x\rangle\langle |T^*|^{2(1-\alpha)}y,y\rangle$ for all $x,y\in H$ and for $0\leq\alpha\leq 1$. An operator $T\in\mathcal{LC}(H)$ is said to be in the Schatten p-class $S_p(H)$ $(1\leq p<\infty)$, if trace$(|T|^p)<\infty$. Let S_∞ be the set of all bounded operators from $L^2_a(\mathbb{D})$ into itself. The Schatten p-norm of T is defined by $\|T\|_p=(\text{trace}|T|^p)^{1/p}$. It is well known that if $T\in S_p(H)$ then, $\|T\|_p=\|T^*\|_p=\||T|\|_p$. The class $S_1(H)$ is also called the trace class of H.

$$\|T\|_1=\text{trace}|T|=\|T\|_{tr}=\sum_{k=1}^{\infty}|\langle T\phi_k,\phi_k\rangle|$$

where $\{\phi_k\}$ is an orthonormal basis for H. Let x and y be two nonzero vectors in H. Suppose $\langle x,y\rangle=0$. Let $T=x\otimes y+y\otimes x$. Then T is self-adjoint on H. Further, $\|T^2\|_p=2^{\frac{1}{p}}\|x\|^2\|y\|^2$ and $\|T\|_p=2^{\frac{1}{p}}\|x\|\|y\|$, where $\|\cdot\|_p$ is the Schatten p-class norm for $p\geq 1$. Thus $\|T^2\|_p\neq\|T\|_p^2$. Notice that $T^2=\|y\|^2x\otimes x+\|x\|^2y\otimes y$, so the square root $|T|$ of the positive operator T^2 is

$$|T|=\|x\|\|y\|\frac{x}{\|x\|}\otimes\frac{x}{\|x\|}+\|x\|\|y\|\frac{y}{\|y\|}\otimes\frac{y}{\|y\|}.$$

Proposition 4.1. *Let T be a rank k normal operator on H with $\{\lambda\}_{j=1}^k$ the k eigenvalues of T repeated according to multiplicity. Then*

$$\textit{trace}|T^2|\leq(\textit{trace}|T|)^2\leq k\,\textit{trace}|T^2|.$$

Proof. Notice that $\|T\|_{\text{tr}} = \sum_{j=1}^{k} |\lambda_j|$. Since T^2 is also of rank k and normal with the eigenvalues $\{\lambda_j^2\}_{j=1}^k$, by functional calculus, $\|T^2\|_{\text{tr}} = \sum_{j=1}^{k} |\lambda_j|^2$. So the first inequality is trivial. The second inequality follows from the Cauchy-Schwarz inequality. □

Proposition 4.2. *Let $S, T \in \mathcal{L}(L_a^2(\mathbb{D}))$. If tr$(ESE)$ = tr(ETE) for every rank-one projection $E \in \mathcal{L}(L_a^2(\mathbb{D}))$, then $\rho(S) = \rho(T)$.*

Proof. For $z \in \mathbb{D}$, let $E = k_z \otimes k_z$ where $k_z \in L_a^2(\mathbb{D})$ is the normalized reproducing kernel. Then E is a rank-one projection and every rank-one projection takes this form. By the assumption, we have

$$\begin{aligned}\langle Sk_z, k_z\rangle &= \text{tr}(Sk_z \otimes k_z)\\ &= \text{tr}(ESE) = \text{tr}(ETE)\\ &= \text{tr}(Tk_z \otimes k_z) = \langle Tk_z, k_z\rangle.\end{aligned}$$

Thus for all $z \in \mathbb{D}$, $\langle Sk_z, k_z\rangle = \langle Tk_z, k_z\rangle$ and $\rho(S) = \rho(T)$. □

Lemma 4.1. *Let $S, T \in S_p$ for some $p \in [1, \infty)$. If $0 \le S \le T$ and $\|S\|_p = \|T\|_p$, then $S = T$.*

Proof. For proof see [6]. □

Let $\mathcal{F}_1(H)$ be the set of all rank-one projections on the Hilbert space H.

Theorem 4.1. *Let $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ be a positive operator. The following hold:*

(i) $\lim_{b\to\infty} (\|S + bE\| - b) =$ *tr(SE) for all $E \in \mathcal{F}_1(L_a^2(\mathbb{D}))$, $b > 0$.*

(ii) *If $S \in S_p$, $1 < p < \infty$, then $\lim_{b\to\infty} (\|S + bE\|_p - b) =$ tr(SE) holds for all $E \in \mathcal{F}_1(L_a^2(\mathbb{D}))$, $b > 0$.*

Proof. To prove (i), Suppose $f \in (\text{Range}E) \cap L_a^2(\mathbb{D})$ with $\|f\| = 1$ and $\epsilon > 0$. Assume $T = (\langle Sf, f\rangle + \epsilon)E + bE^\perp$ where $E^\perp = I - E$. Then

$$T^{-1/2}ST^{-1/2} = \frac{1}{\langle Sf, f\rangle + \epsilon} ESE + \frac{1}{\sqrt{t}\sqrt{\langle Sf, f\rangle + \epsilon}} ESE^\perp +$$

$$\frac{1}{\sqrt{t}\sqrt{\langle Sf, f\rangle + \epsilon}} E^\perp SE + \frac{1}{t} E^\perp SE^\perp = \frac{1}{\langle Sf, f\rangle + \epsilon} ESE + V_b$$

where V_b is the sum of the last three terms. Notice that

$$ESEf = ES(\langle f, f\rangle f) = \langle f, f\rangle ESf = ESf = \langle Sf, f\rangle f.$$

Thus $\|ESEf\| = \langle Sf, f\rangle$. Hence

$$\|T^{-1/2}ST^{-1/2}\| \le \frac{\langle Sf, f\rangle}{\langle Sf, f\rangle + \epsilon} + \|V_b\|.$$

Letting $b \longrightarrow \infty$, we obtain

$$\|T^{-1/2}ST^{-1/2}\| \le \frac{\langle Sf, f\rangle}{\langle Sf, f\rangle + \epsilon} \le 1$$

since $\|V_b\| \longrightarrow 0$ as $b \longrightarrow \infty$. Thus $T^{-1/2}ST^{-1/2} \le I$ and therefore

$$S \le (\langle Sf, f\rangle + \epsilon)E + bE^\perp.$$

Hence we obtain the inequality $\|(S+bE\| \leq \|\langle Sf,f\rangle+\epsilon+b)E+bE^{\perp}\|$ which holds for sufficiently large $b \geq 0$. Further, $\langle Sf,f\rangle+b \leq \|S+bE\|$ and

$$\|(\langle Sf,f\rangle+\epsilon+b)E+bE^{\perp}\| = \max\{\langle Sf,f\rangle+\epsilon+b, b\} = \langle Sf,f\rangle+\epsilon+b.$$

Thus for sufficiently large $b \geq 0$, we obtain

$$0 \leq \|S+bE\| - \langle Sf,f\rangle - b \leq \langle Sf,f\rangle+\epsilon+b-\langle Sf,f\rangle-b = \epsilon.$$

Hence we get,

$$\lim_{b\to\infty}(\|S+bE\|-b) = \langle Sf,f\rangle = \mathrm{tr}(SE).$$

To prove (ii), first notice that $\|bE\|_p = b$ and

$$\|S+bE\|_p - b = \frac{\|\frac{1}{b}S+E\|_p - \|E\|_p}{\frac{1}{b}}.$$

From [1], it follows that the Schatten-norm $\|\cdot\|_p$ is Fréchet differentiable at any point of $S_p(L^2_a(\mathbb{D}))$ and computing the derivative at the point E in the direction of S, we obtain

$$\begin{aligned}\lim_{b\to\infty}(\|S+bE\|_p-b) &= \lim_{b\to\infty}\frac{\|\frac{1}{b}S+E\|_p - \|E\|_p}{\frac{1}{b}}\\ &= \mathrm{tr}\left(\frac{|E|^{p-1}U^*S}{\|E\|_p^{p-1}}\right)\end{aligned}$$

where U is the partial isometry in the polar decomposition of E. Clearly, $U=E$, $|E|^{p-1}=E$ and $\|E\|_p = 1$ and hence we obtain that

$$\lim_{b\to\infty}(\|S+bE\|_p-b) = \mathrm{tr}(SE)$$

for $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$. $\square$

Theorem 4.2. *Suppose $S,T \in \mathcal{L}(L^2_a(\mathbb{D}))$. The following hold:*

(i) *Suppose S is self-adjoint, $T \geq 0$ and $\pm S \leq T$. If further $S,T \in S_p$ for some p with $1 \leq p < \infty$ and $\|S\|_p = \|T\|_p$ then $\rho(|S|) = \rho(T)$.*

(ii) *If $S \geq 0$, $T \geq 0$, $S,T \in S_p$ for $1 < p \leq \infty$, then $\|S+bE\|_p \leq \|T+bE\|_p$ holds for all $b \geq 0$ and $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$ if and only if $S \leq T$. In this case, $\rho(S) \leq \rho(T)$.*

Proof. (i) Since $S = S^*$, the space $L^2_a(\mathbb{D})$ can be written as $L^2_a(\mathbb{D}) = X_+ \oplus X_-$ so that $S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}$, where S_+ and S_- are positive operators on X_+ and X_- respectively. Let $T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{pmatrix}$ relative to the decomposition $X = X_+ \oplus X_-$. Since $T \geq \pm S$, it follows that

$$\begin{pmatrix} T_1 - S_+ & T_2 \\ T_2^* & T_3 + S_- \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} T + S_+ & T_2 \\ T_2^* & T_3 - S_- \end{pmatrix} \geq 0. \tag{4.1}$$

Hence

$$T_1 \geq S_+ \text{ and } T_3 \geq S_-. \tag{4.2}$$

By [6], (4.2) and the min-max principle, we obtain

$$\|T\|_p \geq \left\|\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}\right\|_p = \left(\|T_1\|_p^p + \|T_3\|_p^p\right)^{1/p} \geq \left(\|S_+\|_p^p + \|S_-\|_p^p\right)^{1/p} = \|S\|_p. \tag{4.3}$$

Now suppose that $\|S\|_p = \|T\|_p$. Then it follows from (4.2) and (4.3) that

$$\|T_1\|_p = \|S_+\|_p \text{ and } \|T_3\|_p = \|S_-\|_p. \tag{4.4}$$

From Lemma 4.1, it follows from (4.2) and (4.4) that $T_1 = S_+$ and $T_3 = S_-$. From (4.1), it follows that $T_2 = 0$ and so $\rho(T) = \rho(|S|)$.
(ii) Suppose $1 < p < \infty$ and assume $S \leq T$. It follows from the monotonicity of Schatten-p-norms [9] that $\|S + bE\|_p \leq \|T + bE\|_p$ for all $b \geq 0$ and for all $E \in \mathcal{F}_1(L_a^2(\mathbb{D}))$. Now assume that

$$\|S + bE\|_p \leq \|T + bE\|_p \tag{4.5}$$

holds for all $b \geq 0$ and for all $E \in \mathcal{F}_1(L_a^2(\mathbb{D}))$. From Theorem 4.1, it follows that $\lim\limits_{b\to\infty} (\|S + bE\|_p - b) = \text{tr}(SE)$. Thus we obtain from (4.5) that $\text{tr}(SE) \leq \text{tr}(TE)$ for all $E \in \mathcal{F}_1(L_a^2(\mathbb{D}))$. Thus it follows that

$$\langle Sf, f\rangle = \text{tr}(S(f \otimes f)) \leq \text{tr}(T(f \otimes f)) = \langle Tf, f\rangle$$

for all $f \in L_a^2(\mathbb{D})$ and $\rho(S) \leq \rho(T)$. For $p = \infty$, $S \leq T$ implies $\|S + bE\| \leq \|T + bE\|$ for all $b \geq 0$ and for all $E \in \mathcal{F}_1(L_a^2(\mathbb{D}))$. It follows from the monotonicity of the operator norm. Now suppose $\|S + bE\| \leq \|T + bE\|$ for all $t \geq 0$ and for all $E \in \mathcal{F}_1(L_a^2(\mathbb{D}))$. From Theorem 4.1, it follows that $\lim\limits_{b\to\infty} (\|S + bE\| - b) = \text{tr}(SE)$. Thus $\text{tr}(SE) \leq \text{tr}(TE)$. Hence

$$\begin{aligned}\langle Sf, f\rangle &= \lim_{b\to\infty} (\|S + b(f \otimes f)\| - b)\\ &\leq \lim_{b\to\infty} (\|T + b(f \otimes f)\| - b) = \langle Tf, f\rangle\end{aligned}$$

for all $f \in L_a^2(\mathbb{D})$. Therefore $S \leq T$ and $\rho(S) \leq \rho(T)$. The theorem follows. □

Definition 4.1. *An operator $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ is a contraction if $\|A\| \leq 1$.*

Theorem 4.3. *Suppose $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is a contraction and $|T|^2 \leq |T^2|$. Then $\rho(K^{n+1}) \leq \rho(K^n)$ for all $n \in \mathbb{N}$ where $K = |T^2| - |T|^2$. Further $\{K^n\}$ converges strongly to a projection operator E and $\rho(K^n) \to \rho(E)$ and $\rho(TE) = 0$.*

Proof. Since $|T|^2 \leq |T^2|$, hence $K > 0$. Further, since $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ it follows from [4] that $\||T|^{1/2}\|^2 = \|T\|$ and $\||T|f\| = \|Tf\|$ for all $f \in L_a^2(\mathbb{D})$. Let $S = K^{1/2}$ be the unique [4] non-negative square root of K. Now because T is a contraction, we obtain

$$\||T^2|^{1/2}\|^2 = \|T^2\| \leq 1.$$

Thus

$$\begin{aligned}\langle K^{n+1}f, f\rangle &= \|S^{n+1}f\|^2\\ &= \langle KS^n f, S^n f\rangle\\ &= \||T^2|^{1/2}S^n f\|^2 - \||T|S^n f\|^2\\ &\leq \|S^n f\|^2 - \|TS^n f\|^2 \leq \|S^n f\|^2 = \langle K^n f, f\rangle.\end{aligned}$$

Therefore $\langle K^{n+1}k_z, k_z\rangle \leq \langle K^n k_z, k_z\rangle$, for all $z \in \mathbb{D}$. That is $\rho(K^{n+1}) \leq \rho(K^n)$ for all $n \in \mathbb{N}$ and $\{K^n\}$ is a monotonically decreasing sequence of bounded positive operators. Now since $K \geq 0$, it follows from [2] that $\{K^n\}$ converges strongly to a projection E. Moreover,

$$\sum_{n=0}^{m} \|TS^n f\|^2 \leq \sum_{n=0}^{m} \left(\|S^n f\|^2 - \|S^{n+1}f\|^2\right) = \|f\|^2 - \|S^{m+1}f\|^2 \leq \|f\|^2$$

for all non-negative integers m and $f \in L^2_a(\mathbb{D})$. Therefore, $\|TS^n f\| \longrightarrow 0$ as $n \longrightarrow \infty$, and hence

$$TEf = T(\lim_{n\to\infty} K^n f) = \lim_{n\to\infty} TS^{2n} f = 0,$$

for every $f \in L^2_a(\mathbb{D})$. Thus $\rho(TE) = 0$. $\square$

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