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THE POINT AND RHODIUS SPECTRA OF CERTAIN NONLINEAR SUPERPOSITION OPERATORS

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ABSTRACT. In this paper we consider the nonlinear superposition operator F in l_p spaces of sequences $(1 \leq p \leq \infty)$, generated by the function $f(s,u) = a(s) + \frac{u}{u^2+1}$. We find out the Rhodius spectrum $\sigma_R(F)$ and the point spectra $\sigma_p(F)$ of these operators and the spectral radius. We make comparison and give some conclusions about these spectra.

1. INTRODUCTION AND PRELIMINARIES

The nonlinear superposition operators arise in a large field of mathematics problems and have various applications in mathematical physics, mathematical economics, mathematical biology, discrete and continuous dynamical systems and so on. Hence, the eigenproblem of such operators deserves a substantial attention. The spectral theories for nonlinear operators on Banach spaces is now quite well-established research topic and it is still in developing process ([5], [11]). The term spectrum for nonlinear operators, in the beginning, was used just for the set of eigenvalues i.e. the point spectrum. Later it became clear that the notion spectrum need to have wider meaning and more complete description. The several nonlinear spectra and spectral theories have been introduced in the literature by now (see [1], [2], [4], [7], [10], [13]). For the class $\mathfrak{C}(X)$ of all continuous operators F on Banach space X, the following definition has been introduced by Rhodius in 1984. ([13]).

Definition 1.1. *For the continuous operator $F : X \to X$ the set*

$$\rho_R(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}(X)\}$$

is called the Rhodius resolvent set and

$$\sigma_R(F) = \mathbb{K} \setminus \rho_R(F).$$

is the Rhodius spectrum.

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$\mathbb{K}$ is the field of real or complex numbers ($\mathbb{R}$ or $\mathbb{C}$). We may notice that a point $\lambda \in \mathbb{K}$ belongs to $\rho_R(F)$ if and only if $\lambda I - F$ is a homeomorphism on X. The Rhodius spectral radius of $F \in \mathfrak{C}(X)$ is the number

$$r_R(F) = \sup\{|\lambda| : \lambda \in \sigma_R(F)\}.$$

The Rhodius spectrum of some nonlinear superposition operators may be found in [2], [8] and [9].

Definition 1.2. *The set of all eigenvalues of the operator F*

$$\sigma_p(F) = \{\lambda \in \mathbb{K} : Fx = \lambda x \text{ for some } x \neq 0\}$$

is called the point spectrum of F.

It is known that if F is a nonlinear operator F with $F(0) = 0$ then

$$\sigma_p(F) \subseteq \sigma_R(F).$$

Let Ω denotes an arbitrary set and $f = f(s,u)$ be a function defined on $\Omega \times \mathbb{R}$ and taking values in $\mathbb{R}$. For a given function $x = x(s)$ on Ω one can define another function $y(s) = f(s, x(s))$ for $s \in \Omega$. In this way, the function f generates an operator

$$Fx(s) = f(s, x(s)), \tag{1.1}$$

This operator F is usually called a nonautonomous superposition operator, Nemytskij operator or composition operator ([3], [6]). Superposition operators on sequence spaces are not studied so intensively as on spaces of functions (see [3]). We are going to observe the operator of superposition, defined in the Banach spaces of sequences l_p $(1 \leq p \leq \infty)$, so we have $s \in \mathbb{N}$ in (1.1).

Dedagić and Zabreiko in [6] have investigated the conditions for acting and continuity of superposition operators on the sequence spaces l_∞, c_o and l_p for $1 \leq p < \infty$ (see also [12]) and those conditions are given in the next two theorems.

Theorem 1.1. *Let $1 \leq p, q < \infty$. Then the following properties are equivalent:*

(i) *the operator F acts from l_p to l_q;*
(ii) *there are functions $a(s) \in l_q$ and constants $\delta > 0, n \in \mathbb{N}, b \geq 0$, for which $|f(s,u)| \leq a(s) + b|u|^{\frac{p}{q}}$ $(s \geq n, |u| < \delta)$;*
(iii) *for any $\varepsilon > 0$ there exists a function $a_\varepsilon \in l_q$ and constants $\delta_\varepsilon > 0, n_\varepsilon \in \mathbb{N}, b_\varepsilon \geq 0$, for which $\|a_\varepsilon(s)\|_q < \varepsilon$ and*

$$|f(s,u)| \leq a_\varepsilon(s) + b_\varepsilon |u|^{\frac{p}{q}} \ (s \geq n_\varepsilon, \ |u| \leq \delta_\varepsilon).$$

Theorem 1.2. *Let $1 \leq p, q < \infty$ and let the superposition operator* (1.1)*, generated by the function $f(s,u)$, acts from l_p to l_q. Then this operator is continuous if and only if each of the functions is continuous for every $s \in \mathbb{N}$.*

2. Main Results

We consider the superposition operator F, generated by the function $f(s,u) = a(s) + \frac{u}{u^2+1}$, where $a = (a(s))_{s\in\mathbb{N}}$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. We are going to show that this operator acts from the space l_p to the

space l_p.
If $1 \le p < \infty$ we have

$$|f(s,u)| = |a(s) + \frac{u}{u^2+1}| \le |a(s)| + \left|\frac{u}{u^2+1}\right|.$$

For $|u| < 1$ we certainly have $|\frac{u}{u^2+1}| \le |u|$, so

$$|f(s,u)| \le |a(s)| + |\frac{u}{u^2+1}| \le d(s) + 1 \cdot |u|, \tag{2.1}$$

where $d(s) = |a(s)|$. Since $a \in l_q$ and $l_q \subseteq l_p$ we conclude $d \in l_p$. Now we can see there exists constants $\delta = 1, n = 1, b = 1$ such that $\forall s \ge n$, $|u| < \delta$, inequality (2.1) holds. From the Theorem 1.1 it follows that $F : l_p \to l_p$.
In case $p = l_\infty$, we have the following considerations.

$$a \in l_q \subseteq l_\infty \Rightarrow \exists \sup_{s \in \mathbb{N}} a(s) = A < \infty.$$

For arbitrary $x = (x_1, x_2, \cdots) \in l_\infty$ we have

$$\sup_{s\in\mathbb{N}} |Fx(s)| = \sup_{s\in\mathbb{N}} |a(s) + \frac{x_s}{x_s^2+1}| \le \sup_{s\in\mathbb{N}} |a(s)| + \sup_{s\in\mathbb{N}} |\frac{x_s}{x_s^2+1}| \le A + \frac{1}{2} < \infty.$$

Here we have used the fact that $\frac{1}{2}$ is the global maximum of the function

$$f_1(x) = \frac{x}{x^2+1}, \ (x \in \mathbb{R}). \tag{2.2}$$

We see that for every $x \in l_\infty$ it holds $Fx \in l_\infty$, thus F acts from l_∞ to l_∞. For every $s \in \mathbb{N}$ the function $f(s,u) = a(s) + \frac{u}{u^2+1}$ is continuous, so according to the Theorem 1.2, the operator F is a continuous one.

Theorem 2.1. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = a(s) + \frac{u}{u^2+1}$, where $(a(s))_s$ is a sequence from the space l_q $(1 \le q \le p \le \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = (-\frac{1}{8}, 1)$.*

Proof. Denote $x = (x_1, x_2, ...) \in l_p$ and $a = (a_1, a_2, ...) \in l_q$.

$$F(x_1, x_2, ...) = (a_1 + \frac{x_1}{x_1^2+1}, a_2 + \frac{x_2}{x_2^2+1}, ...).$$

The operator $\lambda I - F$ for $\lambda = 0$ becomes

$$-Fx = (-a_1 - \frac{x_1}{x_1^2+1}, -a_2 - \frac{x_2}{x_2^2+1}, ...).$$

From $-Fx = -Fy \quad (x, y \in l_p)$, we have

$$(-a_1 - \frac{x_1}{x_1^2+1}, -a_2 - \frac{x_2}{x_2^2+1}, ...) = (-a_1 - \frac{y_1}{y_1^2+1}, -a_2 - \frac{y_2}{y_2^2+1}, ...)$$
$$-a_s - \frac{x_s}{x_s^2+1} = -a_s - \frac{y_s}{y_s^2+1}, \ \forall s \in \mathbb{N}$$
$$\frac{x_s}{x_s^2+1} = \frac{y_s}{y_s^2+1}, \ \forall s \in \mathbb{N}.$$

The function (2.2) is not injective, so from the last equation does not follow $x_s = y_s$, $\forall s \in \mathbb{N}$. That is why this operator $-F$ is not injective. This is neither surjective operator since $-a_s - \frac{x_s}{x_s^2+1} \in [-\frac{1}{2} - a_s, \frac{1}{2} - a_s] (s \in \mathbb{N})$. Hence, the operator $-F$ is not bijection and it follows that

$$0 \in \sigma_R(F).$$

If $\lambda \neq 0$ then $(\lambda I - F)x = (\lambda x_1 - a_1 - \frac{x_1}{x_1^2+1}, \lambda x_2 - a_2 - \frac{x_2}{x_2^2+1}, ...)$.

$$(\lambda I - F)x = (\lambda I - F)y \qquad (x, y \in l_p) \Rightarrow$$

$$(\lambda x_1 - a_1 - \tfrac{x_1}{x_1^2+1}, \lambda x_2 - a_2 - \tfrac{x_2}{x_2^2+1}, ...) = (\lambda y_1 - a_1 - \tfrac{y_1}{y_1^2+1}, \lambda y_2 - a_2 - \tfrac{y_2}{y_2^2+1}, ...)$$

$$\lambda x_s - a_s - \frac{x_s}{x_s^2+1} = \lambda y_s - a_s - \frac{y_s}{y_s^2+1}, \, \forall s \in \mathbb{N} \Rightarrow$$

$$\lambda x_s - \frac{x_s}{x_s^2+1} = \lambda y_s - \frac{y_s}{y_s^2+1}, \, \forall s \in \mathbb{N}.$$

For the function

$$f_2(x) = \lambda x - \frac{x}{x^2+1}$$

it holds $f_2(0) = 0$ and we are going to find if the equation $f_2(x) = 0$ has any nontrivial solutions.

$$f_2(x) = \lambda x - \frac{x}{x^2+1} = 0$$

$$x(\lambda - \frac{1}{x^2+1}) = 0 \Rightarrow x = 0 \quad \vee \quad \lambda - \frac{1}{x^2+1} = 0.$$

It is easy to see that we have nontrivial solutions $x = \pm\sqrt{\frac{1-\lambda}{\lambda}}$ for $\lambda \in (0,1)$. Thus, the function $f_2(x)$ is not injective and $\lambda I - F$ is not an injective operator for $\lambda \in (0,1)$. So, we have

$$(0,1) \subseteq \sigma_R(F).$$

The operator $\lambda I - F$ is generated by the function

$$f_3(s,u) = \lambda u - a(s) - \frac{u}{u^2+1}.$$

For arbitrary fixed s it may be considered as a function of one variable u and when we add $-a(s)$ to the function $f_2(u)$ we get the function $f_3(s,u)$. Therefore, the function $f_3(s,u)$ is bijective for arbitrary s, if and only if the function f_2 is bijective. The function f_2 is bijective for $\lambda = 1$ and for $\lambda > 1$ we have

$$f_2'(x) = \frac{\lambda(x^2+1)^2 - 1 + x^2}{(x^2+1)^2} = \frac{x^2(\lambda x^2 + 2\lambda + 1) + \lambda - 1}{(x^2+1)^2} > 0, \, \forall x \in \mathbb{R}.$$

The function f_2 is continuous, strictly increasing for $\lambda > 1$ and it also holds

$\lim\limits_{x\to+\infty} f_2(x) = \lim\limits_{x\to+\infty} (\lambda x - \dfrac{x}{x^2+1}) : \dfrac{x^2}{x^2} = \lim\limits_{x\to+\infty} \lambda x = +\infty$ and

$\lim\limits_{x\to-\infty} f_2(x) = \lim\limits_{x\to-\infty} (\lambda x - \dfrac{x}{x^2+1}) : \dfrac{x^2}{x^2} = \lim\limits_{x\to-\infty} \lambda x = -\infty$, so f_2 is a bijective function for $\lambda > 1$. We find that the function $f_3(s,u)$ is bijective for $\lambda \geq 1$, for every $s \in \mathbb{N}$ and it implies that operator $\lambda I - F$ is bijective for $\lambda \geq 1$.

If $\lambda < 0$ then $\lim\limits_{x\to+\infty} f_2(x) = \lim\limits_{x\to+\infty} (\lambda x - \dfrac{x}{x^2+1}) : \dfrac{x^2}{x^2} = \lim\limits_{x\to+\infty} \lambda x = -\infty$ and

$\lim\limits_{x\to-\infty} f_2(x) = \lim\limits_{x\to-\infty} (\lambda x - \dfrac{x}{x^2+1}) : \dfrac{x^2}{x^2} = \lim\limits_{x\to-\infty} \lambda x = +\infty$.

Consider the rational function that we get

$$f_2'(x) = \frac{\lambda(x^2+1)^2 - 1 + x^2}{(x^2+1)^2} = \frac{\lambda x^4 + (2\lambda+1)x^2 + \lambda - 1}{(x^2+1)^2}.$$

If we introduce the substitution $t = x^2$ then the numerator of this fraction becomes $\lambda t^2 + (2\lambda+1)t^2 + \lambda - 1$ and its discriminant is $8\lambda + 1$, so for $\lambda < -\frac{1}{8}$ this biquadratic function in numerator is always negative and since denominator is always positive, we

get $f_2'(x) < 0$, $\forall x \in \mathbb{R}$. We have that f_2 is a continuous, strictly decreasing function from $+\infty$ to $-\infty$ when $\lambda < -\frac{1}{8}$, thus f_2 and $f_3(s,u)$ are bijective functions (for all $s \in \mathbb{N}$) and it implies that operator $\lambda I - F$ is bijective for $\lambda < -\frac{1}{8}$.
For $\lambda = -\frac{1}{8}$ we have $f_2'(x) = -\frac{1}{8}\frac{(x^2-3)^2}{(x^2+1)^2}$ and $x = \pm\sqrt{3}$ are the points of inflection since $f_2''(x) = \frac{-2x(x^2-3)}{(x^2+1)^3}$, $f_2'(\pm\sqrt{3}) = 0$, $f'''(x) = \frac{-6(x^6-x^4+5x^2-1)}{(x^2+1)^5}$, $f_2'''(\pm\sqrt{3}) \neq 0$. Hence, the function $f_2(x)$ is continuous and always decreasing from $+\infty$ to $-\infty$ and we conclude that f_2 and $f_3(s,u)$ are bijective functions for $\lambda = -\frac{1}{8}$, so the operator $-\frac{1}{8}I - F$ is bijective.
For $\lambda \in (-\frac{1}{8}, 0)$ we will show that operator $\lambda I - F$ is not injective.

$$(\lambda I - F)x = (\lambda I - F)y \quad (x, y \in l_p) \Rightarrow$$

$$(\lambda x_1 - a_1 - \frac{x_1}{x_1^2+1}, \lambda x_2 - a_2 - \frac{x_2}{x_2^2+1}, ...) = (\lambda y_1 - a_1 - \frac{y_1}{y_1^2+1}, \lambda y_2 - a_2 - \frac{y_2}{y_2^2+1}, ...)$$

$$\lambda x_s - a_s - \frac{x_s}{x_s^2+1} = \lambda y_s - a_2 - \frac{y_s}{y_s^2+1}, \ \forall s \in \mathbb{N} \Rightarrow$$

$$\lambda x_s - \frac{x_s}{x_s^2+1} = \lambda y_s - \frac{y_s}{y_s^2+1}, \ \forall s \in \mathbb{N}.$$

$$\lambda x_s - \frac{x_s}{x_s^2+1} = \lambda y_s - \frac{y_s}{y_s^2+1}$$

$$\lambda x_s - \lambda y_s = \frac{x_s}{x_s^2+1} - \frac{y_s}{y_s^2+1}$$

$$\lambda(x_s - y_s) = \frac{x_s(y_s^2+1) - y_s(x_s^2+1)}{(x_s^2+1)(y_s^2+1)}$$

$$\lambda(x_s - y_s) = \frac{(x_s - y_s)(1 - x_s y_s)}{(x_s^2+1)(y_s^2+1)}.$$

If $x_s \neq y_s$ then it follows $\lambda = \frac{1-x_s y_s}{(x_s^2+1)(y_s^2+1)}$

$$(\lambda x_s^2 + \lambda)y_s^2 + x_s y_s + \lambda x_s^2 + \lambda - 1 = 0$$

$$y_s = \frac{-x_s \pm \sqrt{D}}{2\lambda(x_s^2+1)}, \tag{2.3}$$

where $D = x_s^2 - 4\lambda(x_s^2+1)(\lambda x_s^2 + \lambda - 1) = -4\lambda^2 x_s^4 + (1 + 4\lambda - 8\lambda^2)x_s^2 + 4\lambda - 4\lambda^2$. If we take $x_s = 3$ we get $D = -400\lambda^2 + 40\lambda + 9$ and then $D \geq 0$ for $\lambda \in (\frac{1-\sqrt{10}}{20}, \frac{1+\sqrt{10}}{20})$. Since $(-\frac{1}{8}, 0) \subseteq (\frac{1-\sqrt{10}}{20}, \frac{1+\sqrt{10}}{20})$, we get that for $\lambda \in (-\frac{1}{8}, 0)$ and $x_s = 3$, the equation (2.3) always has real solutions y_s. For example, for $\lambda = -\frac{1}{10}$ and $x_s = 3$ from (2.3) we get $y_s = 1$ and $y_s = 2$. It means that for $\lambda \in (-\frac{1}{8}, 0)$ the operator $\lambda I - F$ is not injective and we have shown:

$$\left(-\frac{1}{8}, 0\right) \subseteq \sigma_R(F).$$

For $\lambda \geq 1$ and $\lambda \leq -\frac{1}{8}$ operator $\lambda I - F$ is bijective and we need to research whether $(\lambda I - F)^{-1}$ is a continuous operator. If $\lambda \geq 1$, for arbitrary $s \in \mathbb{N}$ function $f(s,u) = \lambda u - a(s) - \frac{u}{u^2+1}$ is bijective, increasing and continuous, so there exists its inverse $f^{-1}(s,u)$ which is also bijective, increasing and continuous function ([14]). From Theorem 1.2 follows that operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(s,u)$, is a continuous operator. That is why $[1, +\infty) \subseteq \rho_R(F)$. For $\lambda \leq -\frac{1}{8}$, for arbitrary $s \in \mathbb{N}$, function $f(s,u) = \lambda u - a(s) - \frac{u}{u^2+1}$ is bijective, decreasing and continuous, so there exists its inverse $f^{-1}(s,u)$ which is also bijective, decreasing and continuous function ([14]). From

Theorem 1.2 follows that operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(s,u)$, is a continuous operator. Therefore $(-\infty, -\frac{1}{8}] \subseteq \rho_R(F)$.
After summerizing all above, we get that the Rhodius resolvent set is

$$\rho_R(F) = \left(-\infty, -\frac{1}{8}\right] \cup [1, +\infty)$$

and the Rhodius spectrum of F :

$$\sigma_R(F) = \left(-\frac{1}{8}, 1\right). \tag{2.4}$$

□

We get that the Rhodius spectrum (2.4) of this considering operator F is nonempty, bounded, but not closed set and the spectral radius is

$$r_R(F) = \sup\{|\lambda| : \lambda \in \sigma_R(F)\} = \sup\{|\lambda| : \lambda \in (-\frac{1}{8}, 1)\} = 1.$$

Theorem 2.2. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = a(s) + \frac{u}{u^2+1}$, where $(a(s))_s$ is a sequence from the space l_q $(1 \le q \le p \le \infty)$.*
If $a(s) = 0$, $\forall s \in \mathbb{N}$, then the point spectrum of F is $\sigma_p(F) = (0,1)$.
If $(\exists s \in \mathbb{N})\, a(s) \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$, then $\sigma_p(F) = \mathbb{R}$. If $(\forall s \in \mathbb{N})\, a(s) = (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, +\infty)$, then $\sigma_p(F) = \mathbb{R} \setminus \{0\}$.

Proof. We need to find out does the equation $(\lambda I - F)x = 0$ have any nontrivial solution.
For $\lambda = 0$ we have

$$-Fx = (-a_1 - \frac{x_1}{x_1^2+1}, -a_2 - \frac{x_2}{x_2^2+1}, ...) = (0, 0, ...) \Rightarrow$$

$$-a_s - \frac{x_s}{x_s^2+1} = 0, \ \forall s \in \mathbb{N}. \tag{2.5}$$

If $a(s) = 0$, $\forall s \in \mathbb{N}$, then equation (2.5) becomes $-\frac{x_s}{x_s^2+1} = 0$ and it has only trivial solution $x_s = 0$, $\forall s \in \mathbb{N}$ i.e. $x = (0, 0, ...)$, so $0 \notin \sigma_p(F)$. If there exists s such that $a_s \neq 0$ then from (2.5) it follows $a_s x_s^2 + x_s + a_s = 0$. The discriminant is $D = 1 - 4a_s^2$ and $D \ge 0$ for $a_s \in [-\frac{1}{2}, \frac{1}{2}]$. Thus $0 \in \sigma_p(F)$ if $(\exists s \in \mathbb{N})\, a(s) \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$ and if $(\forall s \in \mathbb{N})\, a_s \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$ then $0 \notin \sigma_p(F)$.
For $\lambda \neq 0$ we consider the equation

$$(\lambda x_1 - a_1 - \frac{x_1}{x_1^2+1}, \lambda x_2 - a_2 - \frac{x_2}{x_2^2+1}, ...) = (0, 0, ...).$$

$$\lambda x_s - a_s - \frac{x_s}{x_s^2+1} = 0, \ \forall s \in \mathbb{N}$$

$$\lambda x_s^3 - a_s x_s^2 + (\lambda - 1)x_s - a_s = 0, \quad \forall s \in \mathbb{N}. \tag{2.6}$$

These are cubic equations and all cubic equations have either one real root or three real roots; so every equation in (2.6) has at least one real solution.
Denote $K(x) = \lambda x^3 - a_s x^2 + (\lambda - 1)x - a_s$. If $(\exists s \in \mathbb{N}), a_s \neq 0$, then for such s is $K(0) \neq 0$ so there is (at least one) real solution $x_s \neq 0$ of the equation $K(x) = 0$. It follows that

equation $(\lambda I - F)x = 0$ has nontrivial solution $x = (x_1, x_2, ...)$ and $\mathbb{R} \setminus \{0\} \subseteq \sigma_p(F)$. If $a(s) = 0$, $\forall s \in \mathbb{N}$, then

$$(\lambda x_1 - \frac{x_1}{x_1^2+1}, \lambda x_2 - \frac{x_2}{x_2^2+1}, ...) = (0, 0, ...).$$

$$\lambda x_s - \frac{x_s}{x_s^2+1} = 0, \ \forall s \in \mathbb{N}.$$

The equation $\lambda x_s - \frac{x_s}{x_s^2+1} = 0$ has one trivial solution $x_s = 0$ and for $\lambda \in (0,1)$ it has also nontrivial solutions $x_s = \pm\sqrt{\frac{1-\lambda}{\lambda}}$. Hence, $(0,1) \subseteq \sigma_p(F)$. □

From the Theorem 2.1 and Theorem 2.2 we see that if $a(s) = 0$, $\forall s \in \mathbb{N}$, $(F0 = 0)$ we have

$$\sigma_p(F) = (0,1) \subseteq \sigma_R(F) = \left(-\frac{1}{8}, 1\right).$$

In other cases, when $F0 \neq 0$, we do not have this inclusion, i.e. $\sigma_p(F) \nsubseteq \sigma_R(F)$. In fact, if $(\exists s \in \mathbb{N})\, a(s) \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$, then we have even an opposite inclusion

$$\sigma_p(F) = \mathbb{R} \supseteq \sigma_R(F) = \left(-\frac{1}{8}, 1\right).$$

These results of the point and Rhodius spectra for considering nonlinear superposition operators may be used in solving some nonlinear operator equations and eigenvalue problems.

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