Advances in Mathematics: Scientific Journal 7 (2018), no.1, 17–22
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version UDC: 517.546

A NEW CLASS OF ANALYTIC FUNCTIONS CONCERNING WITH SUBORDINATIONS

TOSHIO HAYAMI AND SHIGEYOSHI OWA[1]

ABSTRACT. Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk. Also, let $\mathcal{S}^*(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of starlike functions $f(z)$ of order α $(0 \leq \alpha < 1)$. Considering of the extremal function for the class $\mathcal{S}^*(\alpha)$, a new class $\mathcal{S}_k(\alpha)$ of $f(z)$ concerned with subordinations is defined. The object of the present paper is to get some properties of $f(z)$ for $\mathcal{S}_k(\alpha)$.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $f(z) \in \mathcal{A}$ satisfies $f(z_1) \neq f(z_2)$ for any $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ with $z_1 \neq z_2$, then $f(z)$ is said to be univalent in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{S}$. If a function $f(z) \in \mathcal{A}$ maps $\mathbb{U}$ onto a starlike domain with respect to the origin, then $f(z)$ is said to be starlike in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{S}^*$. We say that $f(z)$ is starlike of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some real α $(0 \leqq \alpha < 1)$. We also denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions $f(z)$ of order α in $\mathbb{U}$. Furthermore, we call that $f(z)$ is convex of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies $zf'(z) \in \mathcal{S}^*(\alpha)$ for some real α $(0 \leqq \alpha < 1)$ and denote by $\mathcal{K}(\alpha)$. From the definitions for classes, we know that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$ and that $f(z) \in \mathcal{S}^*(\alpha)$ if and only if $\displaystyle\int_0^z \frac{f(t)}{t}\,dt \in \mathcal{K}(\alpha)$. The function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!} z^n$$

is the extremal function for the class $\mathcal{S}^*(\alpha)$, and the function $f(z)$ given by

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic function, extremal function, starlike function, subordination.

$$f(z)=\begin{cases}\dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}=z+\sum\limits_{n=2}^{\infty}\dfrac{\prod\limits_{j=2}^{n}(j-2\alpha)}{n!}z^n & \left(\alpha\neq\dfrac{1}{2}\right)\\[2ex] -\log(1-z)=z+\sum\limits_{n=2}^{\infty}\dfrac{1}{n}z^n & \left(\alpha=\dfrac{1}{2}\right)\end{cases}$$

is the extremal function for the class $\mathcal{K}(\alpha)$ (see [1] or [4]).

Taking the principal value for $\sqrt[k]{z}$, we consider a function $f(z)$ defined by

$$f(z)=\frac{z}{(1-\sqrt[k]{z})^{2(1-\alpha)}}=z+\sum_{n=2}^{\infty}\frac{\prod\limits_{j=2}^{n}(j-2\alpha)}{(n-1)!}z^{\frac{n-1+k}{k}}\quad(k=1,2,3,\cdots). \tag{1.1}$$

Then, $f(z)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)=\mathrm{Re}\left(\frac{k+(2-2\alpha-k)\sqrt[k]{z}}{k\,(1-\sqrt[k]{z})}\right)>\frac{k+\alpha-1}{k}\quad(z\in\mathbb{U}).$$

This means that $f(z)$ is starlike of order $\dfrac{k+\alpha-1}{k}$ in $\mathbb{U}$ and therefore, $f(z)$ is also starlike of order α in $\mathbb{U}$. With $f(z)$ given by (1.1), we introduce a new class of $\mathcal{A}$ applying the subordinations.

2. A NEW CLASS $\boldsymbol{\mathcal{S}_k(\alpha)}$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0)=0$, $|w(z)|<1$ $(z\in\mathbb{U})$ and $f(z)=g(w(z))$. We denote this subordination by, (see [4]):

$$f(z)\prec g(z)\,.$$

Let $\mathcal{A}_k$ be the class of functions $f(z)$ given by

$$f(z)=z+\sum_{n=2}^{\infty}a_{\frac{n-1+k}{k}}z^{\frac{n-1+k}{k}}\qquad(k=1,2,3,\dots)$$

which are analytic in $\mathbb{U}$. For such a function $f(z)$, we introduce the class $\mathcal{S}_k(\alpha)$ consisting of functions $f(z)$ which satisfy

$$f(z)\prec\frac{z}{(1-\sqrt[k]{z})^{2(1-\alpha)}}\qquad(z\in\mathbb{U}),$$

where $0\leqq\alpha<1$ and $k=1,2,3,\dots$.

Recently, Owa et. al. [7] have studied some problems for new classes $\mathcal{S}_k^*(\alpha)$ and $\mathcal{K}_k(\alpha)$ of $f(z)$ given by

$$f(z)=z+\sum_{n=1}^{\infty}a_{1+\frac{n}{k}}z^{1+\frac{n}{k}}\qquad(k=1,2,3,\dots)$$

satisfying

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\quad(z\in\mathbb{U})$$

and

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \quad (z\in\mathbb{U})$$

for some real α $(0 \leqq \alpha < 1)$, respectively. Also, Owa ([5] and [6]) and Srivastava and Owa [8]have discussed some properties of generalized Carathéodory functions.

For considering our problems for functions $f(z)$, we have to recall here the folowing lemma due to Miller and Mocanu ([3] and [4]) or due to Jack [2].

Lemma 2.1. *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0)=0$. Then if $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathbb{U}$, then we have $z_0w'(z_0)=mw(z_0)$, where m is real and $m\geqq 1$.*

Now, we derive

Theorem 2.1. *If $f(z)\in\mathcal{S}_k(\alpha)$ $(0\leqq\alpha<1)$, then*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\frac{k+\alpha-1}{k} \qquad (z\in\mathbb{U}).$$

Proof. For $f(z)\in\mathcal{S}_k(\alpha)$, there exists a function $w(z)$ which is analytic in $\mathbb{U}$, $w(0)=0$ and $|w(z)|<1$ $(z\in\mathbb{U})$ such that

$$f(z)=\frac{w(z)}{\left(1-\sqrt[k]{w(z)}\right)^{2(1-\alpha)}}\,.$$

This gives us that

$$\frac{zf'(z)}{f(z)}=\frac{zw'(z)}{w(z)}\left(1+\frac{2(1-\alpha)}{k}\,\frac{\sqrt[k]{w(z)}}{1-\sqrt[k]{w(z)}}\right). \tag{2.1}$$

We suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=\rho<1.$$

Then, applying Lemma 2.1, we write that $w(z_0)=\rho e^{i\theta}$ and $z_0w(z_0)=mw(z_0)$ $(m\geqq 1)$. It follows from (2.1) that

$$\begin{aligned}\operatorname{Re}\left(\frac{z_0f'(z_0)}{f(z_0)}\right) &= \operatorname{Re}\left\{\frac{z_0w'(z_0)}{w(z_0)}\left(1+\frac{2(1-\alpha)\sqrt[k]{w(z_0)}}{k\left(1-\sqrt[k]{w(z_0)}\right)}\right)\right\}\\ &= \operatorname{Re}\left\{m\left(1+\frac{2(1-\alpha)\rho^{\frac{1}{k}}e^{i\frac{\theta}{k}}}{k\left(1-\rho^{\frac{1}{k}}e^{i\frac{\theta}{k}}\right)}\right)\right\}.\end{aligned}$$

Letting $t=\cos\dfrac{\theta}{k}$, we see that

$$\operatorname{Re}\left(\frac{e^{i\frac{\theta}{k}}}{1-\rho^{\frac{1}{k}}e^{i\frac{\theta}{k}}}\right)=\frac{t-\rho^{\frac{1}{k}}}{1+\rho^{\frac{2}{k}}-2\rho^{\frac{1}{k}}t}\,.$$

If we write that

$$g(t)=\frac{t-\rho^{\frac{1}{k}}}{1+\rho^{\frac{2}{k}}-2\rho^{\frac{1}{k}}t} \qquad (-1\leqq t\leqq 1)\,,$$

then

$$g'(t) = \frac{1-\rho^{\frac{2}{k}}}{\left(1+\rho^{\frac{2}{k}}-2\rho^{\frac{1}{k}}t\right)^2} > 0\,.$$

This means that

$$g(t) \geqq g(-1) = -\frac{1}{1+\rho^{\frac{1}{k}}}\,,$$

that is, that

$$\begin{aligned}\operatorname{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) &\geqq m\left(1-\frac{2(1-\alpha)\rho^{\frac{1}{k}}}{k\left(1+\rho^{\frac{1}{k}}\right)}\right)\\ &> \frac{k+\alpha-1}{k}.\end{aligned}$$

This completes the proof of the theorem. □

Letting $\alpha = 0$ in Theorem 2.1, we have

Corollary 2.1. *If $f(z) \in \mathcal{S}_k(0)$, then*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{k-1}{k} \qquad (z \in \mathbb{U}).$$

Remark 2.1. *If we take $k = 1$ in Theorem 2.1, then $f(z)$ is starlike of order α in $\mathbb{U}$.*

Next, we derive

Theorem 2.2. *If $f(z) \in \mathcal{S}_k(\alpha)$ $(0 \leqq \alpha < 1)$, then*

$$\frac{|z|}{\left(1+|z|^{\frac{1}{k}}\right)^{2(1-\alpha)}} \leqq |f(z)| \leqq \frac{|z|}{\left(1-|z|^{\frac{1}{k}}\right)^{2(1-\alpha)}} \tag{2.2}$$

for $z \in \mathbb{U}$. The equalities in (2.2) holds true for

$$f(z) = \frac{z}{\left(1-z^{\frac{1}{k}}\right)^{2(1-\alpha)}}. \tag{2.3}$$

Proof. Note that there exists an analytic function $w(z)$ which is called the Schwarz function $w(z)$ such that

$$f(z) = \frac{w(z)}{\left(1-\sqrt[k]{w(z)}\right)^{2(1-\alpha)}}.$$

Letting $w(z) = |w(z)|e^{i\theta}$, we have that

$$\begin{aligned}|f(z)| &= \frac{|w(z)|}{\left(1-|w(z)|^{\frac{1}{k}}e^{i\frac{\theta}{k}}\right)^{2(1-\alpha)}}\\ &= \frac{|w(z)|}{\left\{\left(1-|w(z)|^{\frac{1}{k}}\cos\frac{\theta}{k}\right)^2+|w(z)|^{\frac{2}{k}}\sin^2\frac{\theta}{k}\right\}^{1-\alpha}}\\ &= \frac{|w(z)|}{\left(1+|w(z)|^{\frac{2}{k}}-2|w(z)|^{\frac{1}{k}}\cos\frac{\theta}{k}\right)^{1-\alpha}}\,.\end{aligned}$$

In view of the Schwarz lemma for $w(z)$, we know that $|w(z)| \leqq |z|$ $(z \in \mathbb{U})$. Therefore, we obtain that

$$\frac{|z|}{\left(1+|z|^{\frac{1}{k}}\right)^{2(1-\alpha)}} \leqq |f(z)| \leqq \frac{|z|}{\left(1-|z|^{\frac{1}{k}}\right)^{2(1-\alpha)}}$$

for $z \in \mathbb{U}$. Further, if $f(z)$ is given by (2.3), then $f(z) \in \mathcal{S}_k(\alpha)$ and $f(z)$ satisfies (2.2). □

Making $\alpha = 0$ in Theorem 2.2, we have

Corollary 2.2. *If $f(z) \in \mathcal{S}_k(0)$, then*

$$\frac{|z|}{\left(1+|z|^{\frac{1}{k}}\right)^{2}} \leqq |f(z)| \leqq \frac{|z|}{\left(1-|z|^{\frac{1}{k}}\right)^{2}} \qquad (z \in \mathbb{U}). \tag{2.4}$$

The equality in (2.4) holds true for

$$f(z) = \frac{z}{\left(1-z^{\frac{1}{k}}\right)^{2}}.$$

If we let $|z| \to 1$ in Theorem 2.2, then we have

Corollary 2.3. *If $f(z) \in \mathcal{S}_k(\alpha)$, then*

$$|f(z)| \geqq \left(\frac{1}{4}\right)^{1-\alpha}.$$

The equality is attained for $f(z)$ given by (2.3) with $z = e^{ik\pi}$.

Further, we consider

Theorem 2.3. *If $f(z) \in \mathcal{S}_k(\alpha)$, then*

$$|f'(z)| \geqq \frac{1}{\left(1+|z|^{\frac{1}{k}}\right)^{2(1-\alpha)}}\left(1-\frac{2(1-\alpha)}{k}-\frac{|z|^{\frac{1}{k}}}{1-|z|^{\frac{1}{k}}}\right) \qquad (z \in \mathbb{U}).$$

Proof. For $f(z) \in \mathcal{S}_k(\alpha)$, we have (2.1). With Lemma 2.1, we say that $|w(z)| \leqq |z|$ and

$$\frac{zw'(z)}{w(z)} = m \geqq 1$$

for $z \in \mathbb{U}$. This shows that

$$\begin{aligned} |f'(z)| &= \left|\frac{f(z)}{z}\right| \left|\frac{zw'(z)}{w(z)}\left(1+\frac{2(1-\alpha)}{k}\frac{\sqrt[k]{w(z)}}{1-\sqrt[k]{w(z)}}\right)\right| \\ &\geqq \left|\frac{f(z)}{z}\right|\left(1-\frac{2(1-\alpha)}{k}\left|\frac{\sqrt[k]{w(z)}}{1-\sqrt[k]{w(z)}}\right|\right) \\ &\geqq \frac{1}{\left(1+|z|^{\frac{1}{k}}\right)^{2(1-\alpha)}}\left(1-\frac{2(1-\alpha)}{k}\frac{|z|^{\frac{1}{k}}}{1-|z|^{\frac{1}{k}}}\right) \end{aligned}$$

for $z \in \mathbb{U}$. □

REFERENCES

[1] P. L. DUREN: *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
[2] I. S. JACK: *Functions starlike and convex of order α*, J. London Math. Soc. **3**(1971), 469 – 474.
[3] S. S. MILLER, P. T. MOCANU: *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65**(1978), 289 – 305.
[4] S. S. MILLER, P. T. MOCANU: *Differential Subordination, Theory and Applications*, Marcel Dkker, New York and Basel, 2000.
[5] S. OWA: *A new class of functions concerning with Carathéodory functions*, PanAmerican Math. J. **26**(2016), 96 – 100.
[6] S. OWA: *An application of Carathéodory functions*, Konuralp J. Math., **5**(2017), 19 – 23.
[7] S. OWA, H. M. SRIVASTAVA, T. HAYAMI, K. KUROKI: *A new general idea for starlike and convex functions*, Tamkang J. Math., **47**(2016), 445 – 454.
[8] H. M. SRIVASTAVA, S. OWA: *Some new results associated with Carathéodory functions of order α*, J. Comput. Sci. Appl. Math. **2**(2016), 11 – 13.

DEPARTMENT OF MATHEMATICS AND PHYSICS
SETSUNAN UNIVERSITY
IKEDANAKA 17-8, NEYAGAWA, OSAKA 572-8508
JAPAN
Email address: ha_ya_to112@hotmail.com

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION, YAMATO UNIVERSITY
KATAYAMA 2-5-1, SUITA, OSAKA 564-0082
JAPAN
Email address: owa.shigeyoshi@yamato-u.ac.jp