

**UNIFORMLY CONVERGENT DIFFERENCE SCHEME FOR A SEMILINEAR  
REACTION-DIFFUSION PROBLEM ON A SHISHKIN MESH**SAMIR KARASULJIĆ<sup>1</sup>, ENES DUVNJAKOVIĆ, AND ELVIR MEMIĆ

ABSTRACT. In this paper we consider two difference schemes for numerical solving of a one-dimensional singularly perturbed boundary value problem. We proved an  $\varepsilon$ -uniform convergence for both difference schemes on a Shishkin mesh. Finally, we present four numerical experiments to confirm the theoretical results.

**1. INTRODUCTION**

We consider the semilinear singularly perturbed problem

$$(1.1) \quad \varepsilon^2 y''(x) = f(x, y) \quad \text{on } (0, 1),$$

$$(1.2) \quad y(0) = 0, \quad y(1) = 0,$$

where  $\varepsilon$  is a small positive parameter. We assume that the nonlinear function  $f$  is continuously differentiable, i.e. for  $k \geq 2$ ,  $f \in C^k([0, 1] \times \mathbb{R})$ , and that it has a strictly positive derivative with respect to  $y$

$$(1.3) \quad \frac{\partial f}{\partial y} = f_y \geq m > 0 \quad \text{on } [0, 1] \times \mathbb{R} \quad (m = \text{const}).$$

The boundary value problem (1.1)–(1.2), under the condition (1.3), has a unique solution (see [14]). Numerical treatment of the problem (1.1), has been considered by many authors, under different condition on the function  $f$ , and made a significant contribution.

We are going to analyze two difference schemes for the problem (1.1)–(1.3). These difference schemes were constructed using the method first introduced by Boglaev [1], who constructed a difference scheme and showed convergence of order 1 on a modified Bakhvalov mesh. In our previous papers using the method [1], we constructed new difference schemes in [3, 4, 10, 6, 7, 8, 9, 13] and performed numerical tests, in [5, 11] we constructed new difference schemes and we proved the theorems on the uniqueness of the numerical solution and the  $\varepsilon$ -uniform convergence on the modified Shishkin mesh, and again performed the numerical test. In [12] we used the difference schemes from

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[11] and calculated the values of the approximate solutions of the problem (1.1)–(1.3) on the mesh points and then we constructed an approximate solution.

Since in the boundary layers, i.e. near  $x = 0$  and  $x = 1$ , the solution of the problem (1.1)–(1.3) changes rapidly, when parameter tends to zero, in order to get the  $\varepsilon$ -uniform convergence, we have to use a layer-adapted mesh. In the present paper we are going to use a Shishkin mesh [15], which is piecewise equidistant and consequently simpler than the modified Shishkin mesh we have already used in our mentioned papers.

## 2. DIFFERENCE SCHEMES

For a given positive integer  $N$ , let it be an arbitrary mesh

$$0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1,$$

with  $h_i = x_i - x_{i-1}$ , for  $i = 1, \dots, N$ .

Our first difference scheme has the following form

$$(2.1) \quad \begin{aligned} & \left( \frac{a_i + d_i}{2} \right) \bar{y}_{i-1} - \left( \frac{a_i + d_i}{2} + \frac{a_{i+1} + d_{i+1}}{2} \right) \bar{y}_i + \left( \frac{a_{i+1} + d_{i+1}}{2} \right) \bar{y}_{i+1} = \\ & = \frac{\Delta d_i}{\gamma} \bar{f}_{i-1/2} + \frac{\Delta d_{i+1}}{\gamma} \bar{f}_{i/2}, \end{aligned}$$

where  $d_i = \frac{\beta}{\tanh(\beta h_{i-1})}$ ,  $a_i = \frac{\beta}{\sinh(\beta h_{i-1})}$ ,  $\bar{f}_{i-1/2} = f\left(\frac{x_{i-1} + x_i}{2}, \frac{\bar{y}_{i-1} + \bar{y}_i}{2}\right)$  and  $\Delta d_i = d_i - a_i$ .

From (2.1), we obtain next discrete problem

$$F\bar{y} = (F\bar{y}_0, F\bar{y}_1, \dots, F\bar{y}_N)^T,$$

where

$$\begin{aligned} F\bar{y}_0 &= \bar{y}_0 = 0, \\ F\bar{y}_i &= \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[ \frac{a_i + d_i}{2} \bar{y}_{i-1} - \left( \frac{a_i + d_i}{2} + \frac{a_{i+1} + d_{i+1}}{2} \right) \bar{y}_i \right. \\ & \quad \left. + \frac{a_{i+1} + d_{i+1}}{2} \bar{y}_{i+1} - \frac{\Delta d_i}{\gamma} \bar{f}_{i-1/2} - \frac{\Delta d_{i+1}}{\gamma} \bar{f}_{i/2} \right] = 0, \\ & \quad i = 1, 2, \dots, N-1, \\ F\bar{y}_N &= \bar{y}_N = 0, \end{aligned}$$

and  $\bar{y} := (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_N)^T$  is the solution of the problem

$$(2.2) \quad F\bar{y} = 0.$$

Second difference scheme has the following form

$$(2.3) \quad \begin{aligned} & (3a_i + d_i + \Delta d_{i+1})(\tilde{y}_{i-1} - \tilde{y}_i) - (3a_{i+1} + d_{i+1} + \Delta d_i)(\tilde{y}_i - \tilde{y}_{i+1}) \\ & = \frac{\tilde{f}_{i-1} + 2\tilde{f}_i + \tilde{f}_{i+1}}{\gamma} (\Delta d_i + \Delta d_{i+1}), \end{aligned}$$

where  $d_i = \frac{\beta}{\tanh(\beta h_{i-1})}$ ,  $a_i = \frac{\beta}{\sinh(\beta h_{i-1})}$ ,  $\tilde{f}_i = f(x_i, \tilde{y}_i)$  and  $\Delta d_i = d_i - a_i$ .

From (2.3), we obtain second discrete problem

$$(2.4) \quad G\tilde{y} = (G\tilde{y}_0, G\tilde{y}_1, \dots, G\tilde{y}_N)^T,$$

where

$$\begin{aligned}
 G\tilde{y}_0 &= \tilde{y}_0 = 0 \\
 G\tilde{y}_i &= \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[ (3a_i + d_i + \Delta d_{i+1})(\tilde{y}_{i-1} - \tilde{y}_i) - (3a_{i+1} + d_{i+1} + \Delta d_i)(\tilde{y}_i - \tilde{y}_{i+1}) \right. \\
 (2.5) \quad &\quad \left. - \frac{\tilde{f}_{i-1} + 2\tilde{f}_i + \tilde{f}_{i+1}}{\gamma} (\Delta d_i + \Delta d_{i+1}) \right] = 0, \quad i = 1, \dots, N-1, \\
 G\tilde{y}_N &= \tilde{y}_N = 0,
 \end{aligned}$$

and  $\tilde{y} = (\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_N)^T$  is the solution of the problem

$$(2.6) \quad G\tilde{y} = 0.$$

### 3. THEORETICAL BACKGROUND

In this paper we use the maximum norm

$$\|u\|_\infty = \max_{0 \leq i \leq N} |u_i|,$$

for any vector  $u = (u_0, u_1, \dots, u_N)^T \in \mathbb{R}^{N+1}$  and the corresponding matrix norm.

The next two theorems hold:

**Theorem 3.1.** [11] *The discrete problem (2.2) for  $\gamma \geq f_y$ , has the unique solution  $\bar{y} = (\bar{y}_0, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{N-1}, \bar{y}_N)^T$ , with  $\bar{y}_0 = \bar{y}_N = 0$ . Moreover, the following stability inequality holds*

$$\|w - v\|_\infty \leq \frac{1}{m} \|Fw - Fv\|_\infty,$$

for any vectors  $v = (v_0, v_1, \dots, v_N)^T \in \mathbb{R}^{N+1}$ ,  $w = (w_0, w_1, \dots, w_N)^T \in \mathbb{R}^{N+1}$ .

**Theorem 3.2.** [5] *The discrete problem (2.6) has a unique solution  $\tilde{y}$  for  $\gamma \geq f_y$ . Also, for every  $u, v \in \mathbb{R}^{N+1}$  we have the following stabilizing inequality*

$$\|u - v\|_\infty \leq \frac{1}{m} \|Gu - Gv\|_\infty.$$

In the following analysis we need the decomposition of the solution  $y$  of the problem (1.1) – (1.2) to the layer component  $s$  and a regular component  $r$ , given in the following assertion.

**Theorem 3.3.** [18] *The solution  $y$  to problem (1.1) – (1.2) can be represented in the following way:*

$$y = r + s,$$

where for  $j = 0, 1, \dots, k+2$  and  $x \in [0, 1]$  we have that

$$|r^{(j)}(x)| \leq C,$$

and

$$|s^{(j)}(x)| \leq C\epsilon^{-j} \left( e^{-\frac{x}{\epsilon}\sqrt{m}} + e^{-\frac{1-x}{\epsilon}\sqrt{m}} \right).$$

#### 4. CONSTRUCTION OF THE MESH

The solution of the problem (1.1)–(1.3) changes fast near the ends of our domain  $[0, 1]$ . Therefore, the mesh has to be refined there. A Shishkin mesh is used to resolve the layers. This mesh is piecewise equidistant and it's quite simple. It is constructed as follows (see [16]). For given a positive integer  $N$ , where  $N$  is divisible by 4, we divide the interval  $[0, 1]$  into three subintervals

$$[0, \lambda], \quad [\lambda, 1 - \lambda], \quad [1 - \lambda, 1].$$

We use equidistant meshes on each of these subintervals, with  $1 + \frac{N}{4}$  points in each of  $[0, \lambda]$  and  $[1 - \lambda, 1]$ , and  $1 + \frac{N}{2}$  points in  $[\lambda, 1 - \lambda]$ . We define the parameter  $\lambda$  by

$$\lambda = \min \left\{ \frac{1}{4}, \frac{2\varepsilon \ln N}{\sqrt{m}} \right\},$$

which depends on  $N$  and  $\varepsilon$ . The basic idea here is to use a fine mesh to resolve the part of the boundary layers. More precisely, we have

$$0 = x_0 < x_1 < \dots < x_{i_0} < \dots < x_{N-i_0} < \dots < x_{N-1} < x_N = 1,$$

with  $i_0 = N/4$ ,  $x_{i_0} = \lambda$ ,  $x_{N-i_0} = 1 - \lambda$ , and

$$(4.1) \quad h_{i-1} = \frac{4\lambda}{N} \text{ for } i = 1, \dots, i_0, N - i_0, \dots, N,$$

$$(4.2) \quad h_{i-1} = \frac{2(1-2\lambda)}{N} \text{ for } i = i_0 + 1, \dots, N - i_0.$$

If  $\lambda = \frac{1}{4}$  i.e.  $\frac{1}{4} \leq \frac{2\varepsilon \ln N}{\sqrt{m}}$ , then  $\frac{1}{N}$  is very small relative to  $\varepsilon$ . This is unlike in practice, and in this case the method can be analyzed using standard techniques. Hence, we assume that

$$\lambda = \frac{2\varepsilon \ln N}{\sqrt{m}}.$$

From (4.1) and (4.2), we conclude that that the interval lengths satisfy

$$h_{i-1} = \frac{8\varepsilon \ln N}{\sqrt{m}} \text{ for } i = 1, \dots, i_0, N - i_0, \dots, N,$$

and

$$\frac{1}{N} \leq h_{i-1} \leq \frac{2}{N} \text{ for } i = i_0 + 1, \dots, N - i_0.$$

#### 5. UNIFORM CONVERGENCE

We will prove the theorem on uniform convergence of the difference schemes (2.1) and (2.3) on the part of the mesh which corresponds to  $[0, 1/2]$ , while the proof on  $[1/2, 1]$  can be analogously derived.

Namely, in the analysis of the value of the error the functions  $e^{-\frac{x}{\varepsilon}\sqrt{m}}$  and  $e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$  appear. For these functions we have that  $e^{-\frac{x}{\varepsilon}\sqrt{m}} \geq e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$ ,  $\forall x \in [0, 1/2]$  and  $e^{-\frac{x}{\varepsilon}\sqrt{m}} \leq e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$ ,  $\forall x \in [1/2, 1]$ . In the boundary layer in the neighbourhood of  $x = 0$ , we have that  $e^{-\frac{x}{\varepsilon}\sqrt{m}} \gg e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$ , while in the boundary layer in the neighbourhood of  $x = 1$  we have that  $e^{-\frac{x}{\varepsilon}\sqrt{m}} \ll e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$ . Based on the above, it is enough to prove the theorem on the part of the mesh which corresponds to  $[0, 1/2]$  with the exclusion of the function  $e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$ , or on  $[1/2, 1]$  but with the exclusion of the function  $e^{-\frac{x}{\varepsilon}\sqrt{m}}$ . Note that we need to take care of the fact that in the first case  $h_{i-1} \leq h_i$ , and in the second case  $h_{i-1} \geq h_i$ .

Let us start with the following two lemmas that will be further used in the proof of the first uniform convergence theorem on the part of the mesh from Section 3 which corresponds to  $[x_{N/4-1}, 1/2]$  and  $x_{N/4} = \lambda$ .

**Lemma 5.1.** *Assume that  $\varepsilon \leq \frac{C}{N}$ . In the part of the Shishkin mesh from Section 3, when  $x_i, x_{i\pm 1} \in [x_{N/4}, 1/2]$ , we have the following estimate*

$$|Fy_i| \leq \frac{C}{N^2}, \quad i = N/4, \dots, N/2 - 1.$$

*Proof.* On this part of the mesh holds  $h_{i-1} = h_i$ , so we have that

$$\begin{aligned} Fy_i &= \frac{\gamma}{2(\cosh(\beta h_i) - 1)} \\ &\quad \cdot \left[ (1 + \cosh(\beta h_i))(y_{i-1} - 2y_i + y_{i+1}) - \frac{\cosh(\beta h_i) - 1}{\gamma}(f_{i-1/2} + f_{i/2}) \right] \\ &= \frac{\gamma}{2} \left( y_{i-1} - 2y_i + y_{i+1} - \frac{f_{i-1/2} + f_{i/2}}{\gamma} \right) - \frac{\gamma}{\cosh(\beta h_i) - 1} (y_{i-1} - 2y_i + y_{i+1}). \end{aligned}$$

Because of Theorem 3.3, and the fact that  $\varepsilon^2 y'' = f(x, y)$ ,  $x \in (0, 1)$ , we obtain

$$\begin{aligned} |Fy_i| &\leq C_1 \left[ |r_{i-1} - 2r_i + r_{i+1}| + |s_{i-1} - 2s_i + s_{i+1}| + \varepsilon^2 |y''_{i-1}| \right. \\ &\quad \left. + \frac{1}{\cosh(\beta h_i) - 1} (|r_{i-1} - 2r_i + r_{i+1}| + |s_{i-1} - 2s_i + s_{i+1}|) \right]. \end{aligned}$$

Again, due to Theorem 3.3 and Taylor expansion, the following inequalities hold

$$\begin{aligned} |r_{i-1} - 2r_i + r_{i+1}| &= \left| \frac{r''(\xi_i^-)}{2} h_i^2 + \frac{r''(\xi_i^+)}{2} h_i^2 \right| \leq C_2 h_i^2, \\ |s_{i-1} - 2s_i + s_{i+1}| &\leq \frac{C_3}{N^2}, \\ \frac{1}{\cosh(\beta h_i) - 1} &\leq \frac{2}{(\beta h_i)^2} = \frac{2\varepsilon^2}{\gamma h_i^2} \leq C_4, \\ \varepsilon^2 |y''_{i-1}| &\leq C_5 \varepsilon^2 \left| \varepsilon^{-2} (e^{-\frac{x_{i-1}}{\varepsilon} \sqrt{m}} + e^{-\frac{1-x_{i-1}}{\varepsilon} \sqrt{m}}) + r''_{i-1} \right| \leq C_6 \left( \frac{1}{N^2} + \varepsilon^2 \right), \end{aligned}$$

where  $\xi_i^- \in (x_{i-1}, x_i)$  and  $\xi_i^+ \in (x_i, x_{i+1})$ . Finally, we have that

$$|Fy_i| \leq \frac{C}{N^2}.$$

□

**Lemma 5.2.** *Assume that  $\varepsilon \leq \frac{C}{N}$ . In the part of the Shishkin mesh from Section 3, when  $x_i = x_{N/4}$ , we have the following estimate*

$$|Fy_{N/4}| \leq \frac{C}{N}.$$

*Proof.* Let us estimate  $\|Fy_{N/4}\|_\infty$ , consider  $Fy_i$  in the following form

$$\begin{aligned}
 (5.1) \quad Fy_i = & \frac{\gamma}{\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \\
 & \cdot \left[ \frac{1 + \cosh(\beta h_{i-1})}{2 \sinh(\beta h_{i-1})} y_{i-1} - \left( \frac{1 + \cosh(\beta h_{i-1})}{2 \sinh(\beta h_{i-1})} + \frac{1 + \cosh(\beta h_i)}{2 \sinh(\beta h_i)} \right) y_i \right. \\
 & \left. + \frac{1 + \cosh(\beta h_i)}{2 \sinh(\beta h_i)} y_{i+1} - \frac{\cosh(\beta h_{i-1}) - 1}{\gamma \sinh(\beta h_{i-1})} f_{i-1/2} - \frac{\cosh(\beta h_i) - 1}{\gamma \sinh(\beta h_i)} f_{i/2} \right], \quad i = N/4
 \end{aligned}$$

Let us first estimate the expressions from (5.1) using the nonlinear terms. Due to Theorem 3.3, and the fact that  $\varepsilon^2 y'' = f(x, y)$ ,  $x \in (0, 1)$ , we have that

$$\begin{aligned}
 (5.2) \quad & \frac{\gamma}{\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \left| - \frac{\cosh(\beta h_{i-1}) - 1}{\gamma \sinh(\beta h_{i-1})} f_{i-1/2} - \frac{\cosh(\beta h_i) - 1}{\gamma \sinh(\beta h_i)} f_{i/2} \right| \\
 & \leq C_3 \varepsilon^2 y''(x_{N/4}) \leq \frac{C_4}{N^2}.
 \end{aligned}$$

For the linear terms from (5.1), we have that

$$\begin{aligned}
 & \frac{\gamma}{\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \cdot \left[ \frac{1 + \cosh(\beta h_{i-1})}{2 \sinh(\beta h_{i-1})} y_{i-1} \right. \\
 & \quad \left. - \left( \frac{1 + \cosh(\beta h_{i-1})}{2 \sinh(\beta h_{i-1})} + \frac{1 + \cosh(\beta h_i)}{2 \sinh(\beta h_i)} \right) y_i + \frac{1 + \cosh(\beta h_i)}{2 \sinh(\beta h_i)} y_{i+1} \right] \\
 & = \frac{\gamma}{\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \cdot \left[ \frac{1 + \cosh(\beta h_{i-1})}{2 \sinh(\beta h_{i-1})} (y_{i-1} - y_i) \right. \\
 & \quad \left. - \frac{1 + \cosh(\beta h_i)}{2 \sinh(\beta h_i)} (y_i - y_{i+1}) \right].
 \end{aligned}$$

According Theorem 3.3, for the layer component  $s$ , we have that

$$\begin{aligned}
 (5.3) \quad & \frac{\gamma}{\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \cdot \left| \frac{1 + \cosh(\beta h_{i-1})}{2 \sinh(\beta h_{i-1})} (s_{i-1} - s_i) \right. \\
 & \quad \left. - \frac{1 + \cosh(\beta h_i)}{2 \sinh(\beta h_i)} (s_i - s_{i+1}) \right| \leq C_5 (|s_{i-1} - s_i| + |s_i - s_{i+1}|) \leq \frac{C_6}{N^2}.
 \end{aligned}$$

For the regular component  $r$ , due to  $\frac{\cosh x - 1}{\sinh x} = \tanh \frac{x}{2}$  and our assumption  $\varepsilon \leq 1/N$ , we get that

$$\begin{aligned}
& \frac{\gamma}{\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \left| \frac{1 + \cosh(\beta h_{i-1})}{2 \sinh(\beta h_{i-1})} (r_{i-1} - r_i) - \frac{1 + \cosh(\beta h_i)}{2 \sinh(\beta h_i)} (r_i - r_{i+1}) \right| \\
&= \frac{\gamma}{\tanh \frac{\beta h_{i-1}}{2} + \tanh \frac{\beta h_i}{2}} \\
&\quad \cdot \left| \frac{\tanh \frac{\beta h_{i-1}}{2}}{2} (r_{i-1} - r_i) + \frac{\tanh \frac{\beta h_i}{2}}{2} (r_i - r_{i+1}) + 2(r_{i-1} - r_i) - 2(r_i - r_{i+1}) \right| \\
&\leq C_7 \left( |r_{i-1} - r_i| + |r_i - r_{i+1}| + \frac{|r_{i-1} - r_i| + |r_i - r_{i+1}|}{\tanh(\beta h_i)} \right) \\
(5.4) \quad &\leq C_8 \left( \frac{\varepsilon \ln N}{N} + \frac{1}{N} + \frac{\frac{\varepsilon \ln N}{N} + \frac{1}{N}}{\tanh(\beta h_i)} \right) \leq \frac{C}{N}.
\end{aligned}$$

Now, collecting (5.2), (5.3) and (5.4), the statement of the lemma is therefore proven.  $\square$

**Theorem 5.1.** *The discrete problem (2.2) on the mesh from Section 3 is uniformly convergent with respect to  $\varepsilon$  and*

$$\max_i |y_i - \bar{y}_i| \leq C \begin{cases} \frac{\ln^2 N}{N^2}, & i \in \{0, 1, \dots, N/4 - 1\} \\ \frac{1}{N^2}, & i \in \{N/4 + 1, \dots, 3N/4 - 1\} \\ \frac{1}{N}, & i \in \{N/4, 3N/4\} \\ \frac{\ln^2 N}{N^2}, & i \in \{3N/4 + 1, \dots, N\}, \end{cases}$$

where  $y$  is the solution of the problem (1.1)–(1.3),  $\bar{y}$  is the corresponding solution of (2.2) and  $C > 0$  is a constant independent of  $N$  and  $\varepsilon$ .

*Proof.* We are going to divide the proof of this theorem in four parts.

Suppose first that  $x_i, x_{i\pm 1} \in [0, \lambda]$ ,  $i = 1, \dots, N/4$ . The proof for this part of the mesh has already been done in [11, Theorem 4.2]. It is hold that

$$(5.5) \quad |Fy_i| \leq C \frac{\ln^2 N}{N^2}, \quad i = 0, 1, \dots, N/4 - 1.$$

Now, suppose that  $x_i, x_{i\pm i} \in [x_{N/4+1}, x_{N/2-1}]$ . Based on Lemma 5.1, we have that

$$(5.6) \quad |Fy_i| \leq \frac{C}{N^2}.$$

In the case  $i = N/4$ , now based on Lemma 5.2, we have that

$$(5.7) \quad |Fy_{N/4}| \leq \frac{C}{N}.$$

Finally, the proof in the case  $i = N/2$  is trivial, because the mesh on this part is equidistant and the influence of the layer component is negligible. Therefore

$$(5.8) \quad |Fy_{N/2}| \leq \frac{C}{N^2}.$$

Using inequalities (5.5), (5.6), (5.7) and (5.8), we complete the proof of the theorem.  $\square$

Let us show the  $\varepsilon$ -uniform convergence of second difference scheme, i.e (2.3).

**Lemma 5.3.** *Assume that  $\varepsilon \leq \frac{C}{N}$ . In the part of the Shishkin mesh from Section 3, when  $x_i, x_{i\pm 1} \in [x_{N/4}, 1/2]$ , we have the following estimate*

$$(5.9) \quad |Gy_i| \leq \frac{C}{N^2}, \quad i = N/4, \dots, N/2 - 1.$$

*Proof.* Let us rewrite  $G\tilde{y}_i$  in the following form

$$\begin{aligned} G\tilde{y}_i &= \frac{\gamma}{2 \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \left[ \frac{2(\cosh(\beta h_i) + 1)}{\sinh(\beta h_i)} (y_{i-1} - y_i) - \frac{2(\cosh(\beta h_i) + 1)}{\sinh(\beta h_i)} (y_i - y_{i+1}) \right. \\ &\quad \left. - \varepsilon^2 \frac{y''_{i-1} - y''_i + y''_{i+1}}{\gamma} \cdot \frac{2(\cosh(\beta h_i) - 1)}{\sinh(\beta h_i)} \right] \\ &= \frac{\gamma}{\cosh(\beta h_i) - 1} \left[ (\cosh(\beta h_i) - 1)(y_{i-1} - 2y_i + y_{i+1}) - 2(y_{i-1} - 2y_i + y_{i+1}) \right. \\ &\quad \left. - \varepsilon^2 (y''_{i-1} - 2y''_i + y''_{i+1}) \cdot \frac{\cosh(\beta h_i) - 1}{\gamma} \right] \\ (5.10) \quad &= \gamma(y_{i-1} - 2y_i + y_{i+1}) - \frac{2\gamma(y_{i-1} - 2y_i + y_{i+1})}{\cosh(\beta h_i) - 1} - \varepsilon^2 (y''_{i-1} - y''_i + y''_{i+1}). \end{aligned}$$

Using Theorem 3.3, Taylor expansion, assumption  $\varepsilon \leq \frac{1}{N}$  and the properties of the mesh from Section 3, let us estimate the expressions from (5.10). We get that

$$\begin{aligned} |y_{i-1} - 2y_i + y_{i+1}| &\leq C_1 (|r_{i-1} - r_i + r_{i+1}| + |s_{i-1} - 2s_i + s_{i+1}|) \\ (5.11) \quad &\leq C_2 \left( \frac{|r''(\xi_i^+) + r''(\xi_i^-)|}{2} h_i^2 + e^{-\frac{x_{i-1}}{\varepsilon} \sqrt{m}} \right) \leq \frac{C_3}{N^2}, \end{aligned}$$

$$(5.12) \quad \frac{1}{\cosh(\beta h_i) - 1} \leq \frac{2}{(\beta h_i)^2} = \frac{2\varepsilon^2}{\gamma h_i^2} \leq C_3,$$

$$\begin{aligned} \varepsilon^2 |y''_{i-1} - y''_i + y''_{i+1}| &\leq \varepsilon^2 |r''_{i-1} - r''_i + r''_{i+1}| + \varepsilon^2 |s''_{i-1} - s''_i + s''_{i+1}| \\ (5.13) \quad &\leq C_4 \varepsilon^2 \left( 1 + \frac{e^{-\frac{x_{i-1}}{\varepsilon} \sqrt{m}}}{\varepsilon^2} \right) \leq \frac{C_5}{N^2}, \end{aligned}$$

where  $\xi_i^- \in (x_{i-1}, x_i)$ ,  $\xi_i^+ \in (x_i, x_{i+1})$ .

Now using (5.10), (5.11), (5.12) and (5.13), we obtain (5.9).  $\square$

**Lemma 5.4.** *Assume that  $\varepsilon \leq \frac{C}{N}$ . In the part of the Shishkin mesh from Section 3, when  $x_i = x_{N/4}$ , we have the following estimate*

$$(5.14) \quad |Gy_{N/4}| \leq \frac{C}{N}.$$

*Proof.* Using (2.5), let us write  $Gy_i$  in the following form

$$\begin{aligned}
 (5.15) \quad Gy_i &= \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} [(4a_i + \Delta d_i + \Delta d_{i+1})(y_{i-1} - y_i) - (4a_{i+1} + \Delta d_i + \Delta d_{i+1})(y_i - y_{i+1})] \\
 &\quad - (f_{i-1} + 2f_i + f_{i+1}) \\
 &= \frac{4\gamma}{\Delta d_i + \Delta d_{i+1}} [a_i(y_{i-1} - y_i) - a_{i+1}(y_i - y_{i+1})] \\
 &\quad + \gamma(y_{i-1} - 2y_i + y_{i+1}) - (f_{i-1} + 2f_i + f_{i+1}).
 \end{aligned}$$

In a similar way, as in the previously lemmas, we can get

$$\begin{aligned}
 (5.16) \quad |y_{i-1} - 2y_i + y_{i+1}| &\leq |s_{i-1} - 2s_i + s_{i+1}| + |r_{i-1} - 2r_i + r_{i+1}| \leq C_1 \left( \frac{1}{N^2} + \frac{1}{N} \right), \\
 (5.17) \quad |f_{i-1} + 2f_i + f_{i+1}| &\leq \frac{C_2}{N^2}.
 \end{aligned}$$

Using the identity  $\frac{\cosh x - 1}{\sinh x} = \tanh \frac{x}{2}$  and Theorem 3.3, we have that

$$\begin{aligned}
 (5.18) \quad &\frac{4\gamma}{\Delta d_i + \Delta d_{i+1}} [a_i(y_{i-1} - y_i) - a_{i+1}(y_i - y_{i+1})] \\
 &= \frac{4\gamma}{\tanh \frac{\beta h_{i-1}}{2} + \tanh \frac{\beta h_i}{2}} \left[ \frac{1}{\sinh(\beta h_{i-1})} |s_{i-1} - s_i| - \frac{1}{\sinh(\beta h_i)} |s_i - s_{i+1}| \right. \\
 &\quad \left. + \frac{1}{\sinh(\beta h_{i-1})} |r_{i-1} - r_i| - \frac{1}{\sinh(\beta h_i)} |r_i - r_{i+1}| \right].
 \end{aligned}$$

Due to Theorem 3.3 and assumption  $\varepsilon \leq \frac{C}{N}$ , hold the next inequalities

$$(5.19) \quad \frac{\gamma}{\tanh \frac{\beta h_{i-1}}{2} + \tanh \frac{\beta h_i}{2}} \leq \frac{4\gamma}{\tanh \frac{\beta h_i}{2}} \leq C_1,$$

$$(5.20) \quad \frac{1}{\sinh(\beta h_{i-1})} |s_{i-1} - s_i| \leq \frac{1}{\beta h_{i-1}} |s_{i-1} - s_i| \leq C_2 \cdot \frac{1}{\frac{\ln N}{N}} \cdot \frac{1}{N^2} = \frac{C_2}{N \ln N},$$

$$(5.21) \quad \frac{1}{\sinh(\beta h_i)} |s_i - s_{i+1}| \leq \frac{1}{\beta h_i} |s_i - s_{i+1}| \leq \frac{C_3}{N^2},$$

$$(5.22) \quad \frac{1}{\sinh(\beta h_{i-1})} |r_{i-1} - r_i| \leq \frac{1}{\beta h_{i-1}} |r_{i-1} - r_i| \leq \frac{1}{\frac{\ln N}{N}} \cdot C_4 \frac{\varepsilon \ln N}{N} = C_4 \varepsilon,$$

$$(5.23) \quad \frac{1}{\sinh(\beta h_i)} |r_i - r_{i+1}| \leq \frac{1}{\beta h_i} |r_i - r_{i+1}| \leq \frac{C_5}{N}.$$

Now, using (5.15), (5.16), (5.17), (5.18), (5.19), (5.20), (5.21), (5.22) and (5.23), we obtain (5.14).  $\square$

**Theorem 5.2.** *The discrete problem (2.4) on the mesh from Section 3 is uniformly convergent with respect to  $\varepsilon$  and*

$$\max_i |y_i - \tilde{y}_i| \leq C \begin{cases} \frac{\ln^2 N}{N^2}, & i \in \{0, 1, \dots, N/4 - 1\} \\ \frac{1}{N^2}, & i \in \{N/4 + 1, \dots, 3N/4 - 1\} \\ \frac{1}{N}, & i \in \{N/4, 3N/4\} \\ \frac{\ln^2 N}{N^2}, & i \in \{3N/4 + 1, \dots, N\}, \end{cases}$$

where  $y$  is the solution of the problem (1.1)–(1.3),  $\tilde{y}$  is the corresponding solution of (2.6) and  $C > 0$  is a constant independent of  $N$  and  $\varepsilon$ .

*Proof.* Again, let us divide the proof on four parts.

Suppose first that  $x_i, x_{i\pm 1} \in [0, \lambda]$ ,  $i = 1, \dots, N/4$ . The proof for this part of the mesh has already been done in [5, Theorem 4.4]. It is proved that

$$(5.24) \quad |Gy_i| \leq C \frac{\ln^2 N}{N^2}, \quad i = 0, 1, \dots, N/4 - 1.$$

Secondly, suppose that  $x_i, x_{i\pm 1} \in [x_{N/4+1}, x_{N/2} - 1]$ . Due to Lemma 5.3, we have that

$$(5.25) \quad |Gy_i| \leq \frac{C}{N^2}.$$

In the case  $i = N/4$ , based on Lemma 5.4, we have the following estimate

$$(5.26) \quad |Gy_{N/4}| \leq \frac{C}{N}.$$

At the end, in the case  $i = N/2$ , the proof is trivial, because of the properties of the mesh and the layer component. Hence, it is true that

$$(5.27) \quad |Gy_i| \leq \frac{C}{N^2}.$$

Using (5.24), (5.25), (5.26) and (5.27), we complete the statement of the theorem.  $\square$

## 6. NUMERICAL EXPERIMENTS

In this section we present numerical results to confirm the uniform accuracy of the discrete problems (2.2) and (2.6). Both discrete problems will be checked on two different examples. First one is the linear boundary value problem, whose exact solution is known. Second example is the nonlinear boundary value problem whose exact solution is unknown.

For the problems from our examples whose exact solution is known, we calculate  $E_N$  as

$$E_N = \max_{0 \leq i \leq N} |y(x_i) - \bar{y}^N(x_i)| \quad \text{or} \quad E_N = \max_{0 \leq i \leq N} |y(x_i) - \tilde{y}^N(x_i)|,$$

for the problems, whose exact solution is unknown, we calculate  $E_N$ , as

$$(6.1) \quad E_N = \max_{0 \leq i \leq N} |\bar{y}_S^{2N}(x_i) - \bar{y}^N(x_i)| \quad \text{or} \quad E_N = \max_{0 \leq i \leq N} |\tilde{y}_S^{2N}(x_i) - \tilde{y}^N(x_i)|,$$

the rate of convergence Ord we calculate in the usual way

$$\text{Ord} = \frac{\ln E_N - \ln E_{2N}}{\ln \frac{2k}{k+1}}$$

where  $N = 2^k$ ,  $k = 6, 7, \dots, 11$ ,  $\bar{y}^N(x_i)$ ,  $\tilde{y}^N(x_i)$  are the values of the numerical solutions on a mesh with  $N + 1$  mesh points, and  $\bar{y}_S^{2N}(x_i)$ ,  $\tilde{y}_S^{2N}(x_i)$  are the values of the numerical solutions on a mesh with  $2N + 1$  mesh points and the transition points altered slightly to  $\lambda_S = \min \left\{ \frac{2}{4}, \frac{2\varepsilon}{\sqrt{m}} \ln \frac{N}{2} \right\}$ .

**Remark 6.1.** In a case when the exact solution is unknown we use the double mesh method, see [2, 16, 17] for details.

**Example 1.** Consider the following problem

$$\varepsilon^2 y'' = y + 1 - 2\varepsilon^2 + x(x - 1) \quad \text{for } x \in (0, 1), \quad y(0) = y(1) = 0.$$

The exact solution of this problem is given by  $y(x) = \frac{e^{-\frac{x}{\varepsilon}} + e^{-\frac{1-x}{\varepsilon}}}{1 + e^{-\frac{1}{\varepsilon}}} - x(x - 1) - 1$ . The nonlinear system was solved using the initial condition  $y_0 = -0.5$  and the value of the constant  $\gamma = 1$ .

| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
|---------------|--------------|------|--------------|------|--------------|------|
| $2^6$         | $8.1585e-04$ | 2.00 | $2.8932e-03$ | 2.02 | $2.5827e-02$ | 2.05 |
| $2^7$         | $2.7762e-04$ | 2.00 | $9.7397e-04$ | 2.01 | $8.5547e-03$ | 1.96 |
| $2^8$         | $9.0650e-05$ | 2.00 | $3.1625e-04$ | 2.00 | $2.8566e-03$ | 1.99 |
| $2^9$         | $3.5410e-05$ | 2.00 | $1.2353e-04$ | 2.00 | $1.2111e-03$ | 2.00 |
| $2^{10}$      | $1.5738e-05$ | 2.00 | $5.4904e-05$ | 2.00 | $4.9827e-04$ | 2.00 |
| $2^{11}$      | $7.7116e-06$ | —    | $2.6903e-05$ | —    | $2.4415e-04$ | —    |
| $\varepsilon$ | $2^{-3}$     |      | $2^{-5}$     |      | $2^{-10}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $3.9901e-02$ | 2.04 | $3.9901e-02$ | 2.04 | $3.9901e-02$ | 2.04 |
| $2^7$         | $1.3288e-02$ | 1.93 | $1.3288e-02$ | 1.93 | $1.3288e-02$ | 1.93 |
| $2^8$         | $4.5122e-03$ | 1.99 | $4.5122e-03$ | 1.99 | $4.5122e-03$ | 1.99 |
| $2^9$         | $1.7709e-03$ | 1.98 | $1.7709e-03$ | 1.98 | $1.7709e-03$ | 1.98 |
| $2^{10}$      | $7.9347e-04$ | 1.98 | $7.9347e-04$ | 1.98 | $7.9347e-04$ | 1.98 |
| $2^{11}$      | $3.9158e-04$ | —    | $3.9158e-04$ | —    | $3.9158e-04$ | —    |
| $\varepsilon$ | $2^{-15}$    |      | $2^{-25}$    |      | $2^{-30}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $3.9901e-02$ | 2.04 | $4.0243e-02$ | 2.02 | $4.0248e-02$ | 2.02 |
| $2^7$         | $1.3288e-02$ | 1.93 | $1.3581e-02$ | 1.92 | $1.3582e-02$ | 1.92 |
| $2^8$         | $4.5122e-03$ | 1.99 | $4.6375e-03$ | 1.97 | $4.6381e-03$ | 1.97 |
| $2^9$         | $1.7709e-03$ | 1.98 | $1.8372e-03$ | 1.98 | $1.8375e-03$ | 1.98 |
| $2^{10}$      | $1.7709e-03$ | 1.98 | $8.2321e-04$ | 1.98 | $8.2331e-04$ | 1.98 |
| $2^{11}$      | $3.9158e-04$ | —    | $4.0626e-04$ | —    | $4.0631e-04$ | —    |
| $\varepsilon$ | $2^{-35}$    |      | $2^{-40}$    |      | $2^{-45}$    |      |

TABLE 1. Errors  $E_N$  and convergence rates Ord for approximate solutions from Example 1.

**Example 2.** Consider the following problem

$$(6.2) \quad \varepsilon^2 y'' = y^3 + y - 2 \quad \text{for } (0, 1), \quad y(0) = y(1) = 0,$$

whose exact solution is unknown. The nonlinear system was solved using the initial condition  $y_0 = 1$ , that represents the reduced solution. The value of the constant  $\gamma = 4$  has been chosen so that the condition  $\gamma \geq f_y(x, y)$ ,  $\forall (x, y) \in [0, 1] \times [y_L, y_U] \subset [0, 1] \times \mathbb{R}$  is fulfilled, where  $y_L$  and  $y_U$  are lower and upper solutions, respectively, of the problem

(6.2). Because of the fact that the exact solution is unknown, we are going to calculate  $E_n$  using (6.1).

| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
|---------------|--------------|------|--------------|------|--------------|------|
| $2^6$         | $7.1345e-04$ | 2.02 | $3.7134e-03$ | 2.01 | $1.5182e-02$ | 2.09 |
| $2^7$         | $2.4017e-04$ | 2.01 | $1.2564e-04$ | 2.01 | $4.9236e-03$ | 1.96 |
| $2^8$         | $7.7985e-05$ | 2.00 | $3.1655e-04$ | 2.00 | $1.6403e-03$ | 2.09 |
| $2^9$         | $3.0463e-05$ | 2.00 | $1.2959e-04$ | 2.00 | $5.1903e-04$ | 2.00 |
| $2^{10}$      | $1.3539e-05$ | 2.00 | $3.9986e-05$ | 2.00 | $1.6001e-04$ | 2.00 |
| $2^{11}$      | $6.6341e-06$ | —    | $1.2096e-05$ | —    | $4.8389e-05$ | —    |
| $\varepsilon$ | $2^{-3}$     |      | $2^{-5}$     |      | $2^{-10}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $1.5181e-02$ | 2.09 | $1.5181e-02$ | 2.09 | $1.5181e-02$ | 2.09 |
| $2^7$         | $4.9236e-03$ | 1.96 | $4.9236e-03$ | 1.96 | $4.9236e-03$ | 1.96 |
| $2^8$         | $1.6403e-03$ | 2.00 | $1.6403e-03$ | 2.00 | $1.6403e-03$ | 2.00 |
| $2^9$         | $5.1903e-04$ | 2.00 | $5.1903e-04$ | 2.00 | $5.1903e-04$ | 2.00 |
| $2^{10}$      | $1.6001e-04$ | 2.00 | $1.6001e-04$ | 2.00 | $1.6001e-04$ | 2.00 |
| $2^{11}$      | $4.8389e-05$ | —    | $4.8389e-05$ | —    | $4.8389e-05$ | —    |
| $\varepsilon$ | $2^{-15}$    |      | $2^{-25}$    |      | $2^{-30}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $1.5181e-02$ | 2.09 | $1.5184e-02$ | 2.09 | $1.5795e-02$ | 2.09 |
| $2^7$         | $4.9236e-03$ | 1.96 | $4.9221e-03$ | 1.96 | $5.1202e-03$ | 1.96 |
| $2^8$         | $1.6403e-03$ | 2.00 | $1.6436e-03$ | 1.99 | $1.7097e-03$ | 1.99 |
| $2^9$         | $5.1903e-04$ | 2.00 | $6.4509e-04$ | 2.00 | $6.7102e-04$ | 2.00 |
| $2^{10}$      | $1.6002e-04$ | 2.00 | $2.8669e-04$ | 2.00 | $2.9823e-04$ | 2.00 |
| $2^{11}$      | $4.8390e-05$ | —    | $1.4048e-04$ | —    | $1.4613e-04$ | —    |
| $\varepsilon$ | $2^{-35}$    |      | $2^{-40}$    |      | $2^{-45}$    |      |

TABLE 2. Errors  $E_N$  and convergence rates Ord for approximate solutions from Example 2.

**Example 3.** Consider the following problem

$$\epsilon^2 y'' = y + 1 - 2\epsilon^2 + x(x-1) \quad \text{for } x \in (0, 1), \quad y(0) = y(1) = 0.$$

The exact solution of this problem is given by  $y(x) = \frac{e^{-\frac{x}{\epsilon}} + e^{-\frac{1-x}{\epsilon}}}{1 + e^{-\frac{1}{\epsilon}}} - x(x-1) - 1$ . The nonlinear system was solved using the initial condition  $y_0 = -0.5$  and the value of the constant  $\gamma = 1$ .

**Example 4.** Consider the following problem

$$\epsilon^2 y'' = y^3 + y - 2 \quad \text{for } (0, 1), \quad y(0) = y(1) = 0,$$

whose exact solution is unknown. The nonlinear system was solved using the initial condition  $y_0 = 1$ , that represents the reduced solution. The value of the constant  $\gamma = 4$  has been chosen so that the condition  $\gamma \geq f_y(x, y)$ ,  $\forall (x, y) \in [0, 1] \times [y_L, y_U] \subset [0, 1] \times \mathbb{R}$  is fulfilled, where  $y_L$  and  $y_U$  are lower and upper solutions, respectively, of the problem (6.2). Because of the fact that the exact solution is unknown, we are going to calculate  $E_n$  using (6.1).

| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
|---------------|--------------|------|--------------|------|--------------|------|
| $2^6$         | $9.0262e-04$ | 2.05 | $4.4799e-03$ | 2.03 | $3.9479e-02$ | 2.01 |
| $2^7$         | $2.8729e-04$ | 1.91 | $1.4999e-03$ | 1.92 | $1.3362e-03$ | 1.93 |
| $2^8$         | $9.8102e-05$ | 1.95 | $5.1221e-04$ | 1.95 | $4.5373e-03$ | 1.96 |
| $2^9$         | $3.9049e-05$ | 1.99 | $2.0484e-04$ | 1.99 | $1.8060e-03$ | 1.97 |
| $2^{10}$      | $1.7496e-05$ | 1.99 | $9.1409e-05$ | 1.99 | $8.1249e-04$ | 1.97 |
| $2^{11}$      | $8.6345e-06$ | —    | $4.4951e-05$ | —    | $4.0241e-04$ | —    |
| $\varepsilon$ | $2^{-3}$     |      | $2^{-5}$     |      | $2^{-10}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $3.9479e-02$ | 2.01 | $3.9479e-02$ | 2.01 | $3.9479e-02$ | 2.01 |
| $2^7$         | $1.3362e-03$ | 1.93 | $1.3362e-03$ | 1.93 | $1.3362e-03$ | 1.93 |
| $2^8$         | $4.5373e-03$ | 1.96 | $4.5373e-03$ | 1.96 | $4.5373e-03$ | 1.96 |
| $2^9$         | $1.8060e-03$ | 1.97 | $1.8060e-03$ | 1.97 | $1.8060e-03$ | 1.97 |
| $2^{10}$      | $8.1249e-04$ | 1.97 | $8.1249e-04$ | 1.97 | $8.1249e-04$ | 1.97 |
| $2^{11}$      | $4.0241e-04$ | —    | $4.0241e-04$ | —    | $4.0241e-04$ | —    |
| $\varepsilon$ | $2^{-15}$    |      | $2^{-25}$    |      | $2^{-30}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $3.9479e-02$ | 2.01 | $3.9483e-02$ | 2.01 | $3.9485e-02$ | 2.01 |
| $2^7$         | $1.3362e-03$ | 1.93 | $1.3363e-03$ | 1.93 | $1.3364e-03$ | 1.93 |
| $2^8$         | $4.5373e-03$ | 1.96 | $4.5377e-03$ | 1.95 | $4.5378e-03$ | 1.95 |
| $2^9$         | $1.8060e-03$ | 1.97 | $1.8147e-03$ | 1.97 | $1.8180e-03$ | 1.97 |
| $2^{10}$      | $8.1249e-04$ | 1.97 | $8.1641e-04$ | 1.97 | $8.1645e-04$ | 1.97 |
| $2^{11}$      | $4.0241e-04$ | —    | $4.0434e-04$ | —    | $4.0436e-04$ | —    |
| $\varepsilon$ | $2^{-35}$    |      | $2^{-40}$    |      | $2^{-45}$    |      |

TABLE 3. Errors  $E_N$  and convergence rates Ord for approximate solutions from Example 3.

| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
|---------------|--------------|------|--------------|------|--------------|------|
| $2^6$         | $8.8623e-04$ | 2.09 | $3.4567e-03$ | 2.11 | $1.1656e-02$ | 2.10 |
| $2^7$         | $2.8728e-05$ | 1.92 | $1.1085e-03$ | 1.93 | $3.7537e-03$ | 1.91 |
| $2^8$         | $9.8102e-05$ | 1.96 | $3.7643e-04$ | 1.95 | $1.2923e-03$ | 1.98 |
| $2^9$         | $3.9049e-05$ | 1.98 | $1.5054e-04$ | 1.98 | $4.1404e-04$ | 1.99 |
| $2^{10}$      | $1.7496e-05$ | 1.98 | $6.7451e-05$ | 1.99 | $1.2855e-04$ | 2.00 |
| $2^{11}$      | $8.6345e-06$ | —    | $3.3169e-05$ | —    | $3.8914e-05$ | —    |
| $\varepsilon$ | $2^{-3}$     |      | $2^{-5}$     |      | $2^{-10}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $1.1656e-02$ | 2.10 | $1.1656e-02$ | 2.10 | $1.1656e-02$ | 2.10 |
| $2^7$         | $3.7537e-03$ | 1.91 | $3.7537e-03$ | 1.91 | $3.7537e-03$ | 1.91 |
| $2^8$         | $1.2923e-03$ | 1.98 | $1.2923e-03$ | 1.98 | $1.2923e-03$ | 1.98 |
| $2^9$         | $4.1404e-04$ | 1.99 | $4.1404e-04$ | 1.99 | $4.1404e-04$ | 1.99 |
| $2^{10}$      | $1.2855e-04$ | 2.00 | $1.2855e-04$ | 2.00 | $1.2855e-04$ | 2.00 |
| $2^{11}$      | $3.8914e-05$ | —    | $3.8914e-05$ | —    | $3.8914e-05$ | —    |
| $\varepsilon$ | $2^{-15}$    |      | $2^{-25}$    |      | $2^{-30}$    |      |
| $N$           | $E_n$        | Ord  | $E_n$        | Ord  | $E_n$        | Ord  |
| $2^6$         | $1.1656e-02$ | 2.10 | $1.1656e-02$ | 2.10 | $1.1656e-02$ | 2.10 |
| $2^7$         | $3.7537e-03$ | 1.91 | $3.7537e-03$ | 1.91 | $3.7537e-03$ | 1.91 |
| $2^8$         | $1.2923e-03$ | 1.98 | $1.2923e-03$ | 1.98 | $1.2923e-03$ | 1.98 |
| $2^9$         | $4.1404e-04$ | 1.99 | $4.1404e-04$ | 1.99 | $4.1404e-04$ | 1.99 |
| $2^{10}$      | $1.2855e-04$ | 2.00 | $1.2855e-04$ | 2.00 | $1.2855e-04$ | 2.00 |
| $2^{11}$      | $3.8914e-05$ | —    | $3.8914e-05$ | —    | $3.8914e-05$ | —    |
| $\varepsilon$ | $2^{-35}$    |      | $2^{-40}$    |      | $2^{-45}$    |      |

TABLE 4. Errors  $E_N$  and convergence rates Ord for approximate solutions from Example 4.

## 7. CONCLUSION

In this paper we proved  $\varepsilon$ -uniform convergence on two differential schemes. In the parts of the domain where lies boundary layers we showed  $\varepsilon$ -convergence of the order  $\mathcal{O}(\ln^2 N^{-2})$ , while out of the layers the order of the  $\varepsilon$ -convergence are of order  $\mathcal{O}(N^{-2})$  and in the transition points  $\lambda$  and  $1 - \lambda$  are of order  $\mathcal{O}(N^{-1})$ . At the end of the paper we perform numerical experiments which agrees with theoretical results.

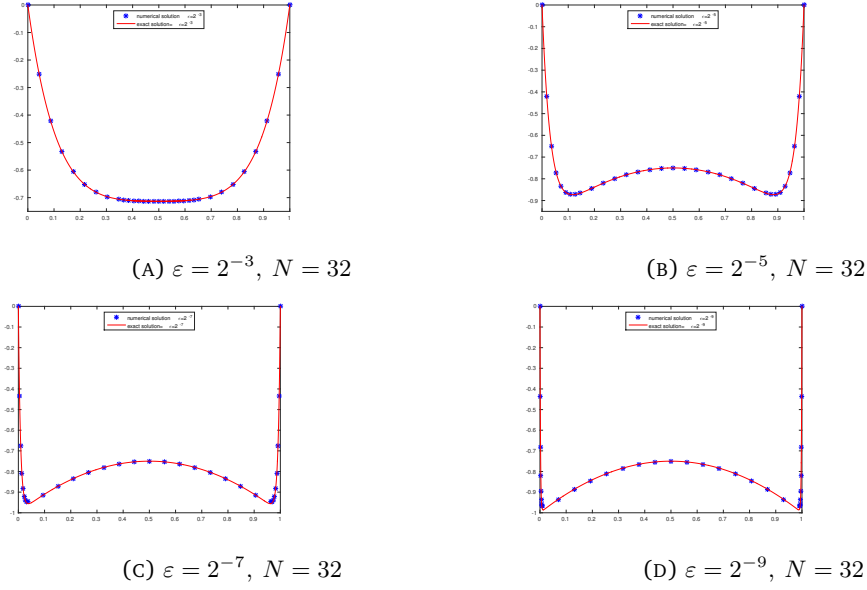


FIGURE 1. Graphics of the numerical and exact solutions for  $N = 32$  and  $\varepsilon = 2^{-3}, 2^{-5}, 2^{-7}, 2^{-9}$  for Example 1 and Example 3

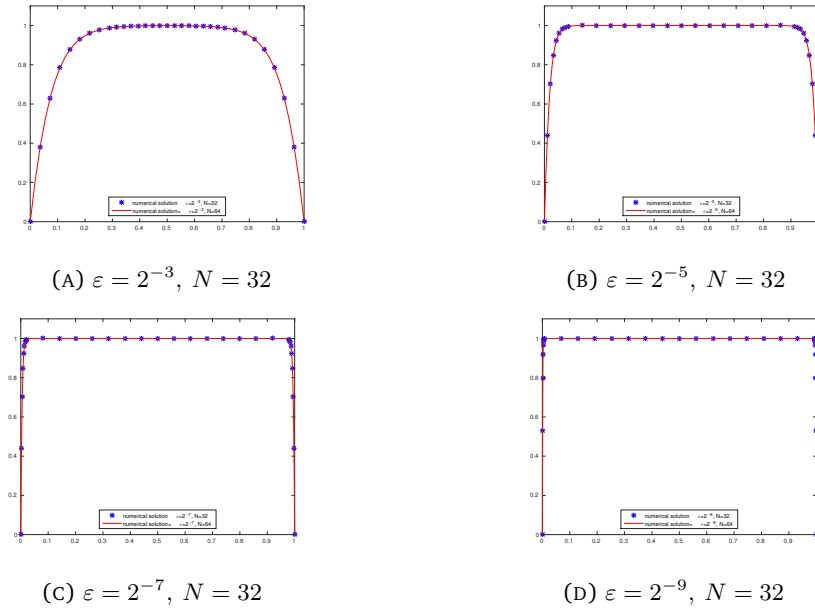


FIGURE 2. Graphics of the numerical and solutions for  $N = 32, 64$  and  $\varepsilon = 2^{-3}, 2^{-5}, 2^{-7}, 2^{-9}$  for Example 2 and Example 4

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## REFERENCES

- [1] I. BOGLAEV: *Approximate solution of a non-linear boundary value problem with a small parameter for the highest-order differential*, Zh. Vychisl. Mat. i Mat. Fiz., **24**(11), (1984), 1649–1656.
- [2] E.P. DOOLAN, J.J.H. MILLER, W.H.A. SCHILDERS: *Uniform numerical methods for problems with initial and boundary layers*, Boole Press, Dublin, 1980.
- [3] E. DUVNJAKOVIĆ, S. KARASULJIĆ, N. OKIČIĆ : *Difference Scheme for Semilinear Reaction-Diffusion Problem*, In: 14th International Research/Expert Conference Trends in the Development of Machinery and Associated Technology TMT 2010, 7. Mediterranean Cruise, (2010), 793–796.
- [4] E. DUVNJAKOVIĆ, S. KARASULJIĆ : *Difference Scheme for Semilinear Reaction-Diffusion Problem on a Mesh of Bakhvalov Type*, Mathematica Balkanica, **25**(5), (2011), 499–504.
- [5] E. DUVNJAKOVIĆ, S. KARASULJIĆ, V. PAŠIĆ, H. ZARIN: *A uniformly convergent difference scheme on a modified Shishkin mesh for the singularly perturbed reaction-diffusion boundary value problem*, Journal of Modern Methods in Numerical Mathematics, **6**(1), (2015), 28-43.
- [6] E. DUVNJAKOVIĆ, S. KARASULJIĆ: *Uniformly Convergent Difference Scheme for Semilinear Reaction-Diffusion Problem*, SEE Doctoral Year Evaluation Workshop, Skopje, Macedonia, 2011.
- [7] E. DUVNJAKOVIĆ, S. KARASULJIĆ: *Difference Scheme for Semilinear Reaction-Diffusion Problem*, The Seventh Bosnian-Herzegovinian Mathematical Conference, Sarajevo, BiH, 2012.
- [8] E. DUVNJAKOVIĆ, S. KARASULJIĆ: *Class of Difference Scheme for Semilinear Reaction-Diffusion Problem on Shishkin Mesh*, MASSEE International Congress on Mathematics - MICOM 2012, Sarajevo, Bosnia and Herzegovina, 2012.
- [9] E. DUVNJAKOVIĆ, S. KARASULJIĆ: *Collocation Spline Methods for Semilinear Reaction-Diffusion Problem on Shishkin Mesh*, IECMSA-2013, Second International Eurasian Conference on Mathematical Sciences and Applications, Sarajevo, Bosnia and Herzegovina, 2013.
- [10] E. DUVNJAKOVIĆ, S. KARASULJIĆ: *Uniformly Convergent Difference Scheme for Semilinear Reaction-Diffusion Problem*, Conference on Applied and Scientific Computing, Trogir, Croatia, pp. 25, 2011.
- [11] S. KARASULJIĆ, E. DUVNJAKOVIĆ, H. ZARIN: *Uniformly convergent difference scheme for a semilinear reaction-diffusion problem*, Advances in Mathematics: Scientific Journal, **4**(2), (2015), 139–159.
- [12] S. KARASULJIĆ, E. DUVNJAKOVIĆ, V. PAŠIĆ, E. BARAKOVIĆ: *Construction of a global solution for the one dimensional singularly-perturbed boundary value problem*, Journal of Modern Methods in Numerical Mathematics, **8**(1–2), (2017),. 52–65.
- [13] S. KARASULJIĆ, E. DUVNJAKOVIĆ : *Construction of the Difference Scheme for Semilinear Reaction-Diffusion Problem on a Bakhvalov Type Mesh*, The Ninth Bosnian-Herzegovinian Mathematical Conference, Sarajevo, BiH, 2015.
- [14] J. LORENZ: *Stability and monotonicity properties of stiff quasilinear boundary problems*, Zb.rad. Prirod. Mat. Fak. Univ. Novom Sadu, Ser. Mat., **12**, (1982), 151–176.
- [15] G.I. SHISHKIN: *Grid approximation of singularly perturbed parabolic equations with internal layers*, Sov. J. Numer. Anal. M. Russian Journal of Numerical Analysis and Mathematical Modelling, **3**(5), (1988), 393–408.
- [16] G. SUN, M. STYNES: *A uniformly convergent method for a singularly perturbed semilinear reaction-diffusion problem with multiple solutions*, Math. Comput., **215**(65), (1996), 1085–1109.
- [17] Z. UZELAC, K. SURLA: *A uniformly accurate collocation method for a singularly perturbed problem*, Novi Sad J. Math., **33**(1), (2003), 133–143.
- [18] R. VULANOVIĆ: *On a Numerical Solution of a Type of Singularly Perturbed Boundary Value Problem by Using a Special Discretization Mesh*, Novi Sad J. Math., **13** 1983, 187–201.

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