ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **7** (2018), no.2, 45–49
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version UDC: 517.546.1

A NOTE ON UNIVALENCY FOR ANALYTIC FUNCTION

NESLIHAN UYANIK AND SHIGEYOSHI OWA[1]

ABSTRACT. Let $\mathcal{N}$ be the class of functions $p(z)$ which are analytic in the open unit disk $\mathbb{U}$ with $p(0) = 1$. Further, $\mathcal{A}$ denotes the class of analytic functions in $\mathbb{U}$ with $f(0) = 0$ and $f'(0) = 1$. In 1989, Nunokawa, Obradović and Owa gave a condition for $f(z) \in \mathcal{A}$ to be univalent in $\mathbb{U}$. The object of the present paper is to consider some conditions for $f(z) \in \mathcal{A}$ to be in the class $S(\alpha)$ $(0 \leq \alpha < 1)$.

1. INTRODUCTION

Let $\mathcal{N}$ be the class of functions $p(z)$ which are analytic in the open unit disk $\mathbb{U} = \{ z \in \mathbb{C}: |z| < 1\}$ with $p(0) = 1$.

Let $\mathcal{A}$ denote the class of functions $f(z)$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\mathbb{U}$. For analytic functions $g(z)$ and $G(z)$ in $\mathbb{U}$, we say that $g(z)$ is subordinat to $G(z)$ in $\mathbb{U}$, written $g(z) \prec G(z)$, if there exists an analytic function $w(z)$ in $\mathcal{N}$ such that $g(z) = G(w(z))$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$. If $G(z)$ is univalent in $\mathcal{N}$, then $g(z) \prec G(z)$ is equivalent to $g(0) = G(0)$ and $g(\mathbb{U}) \subset G(\mathbb{U})$ (see Pommerenke [6], Duren [1]).

Ozaki and Nunokawa [5] gave the following result for univalency of $f(z) \in \mathcal{A}$.

Theorem 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\text{Re}\left\{ \frac{f(z)^2}{z^2 f'(z)} \right\} \geq \frac{1}{2} \qquad (z \in \mathbb{U}), \tag{1.1}$$

then $f(z)$ is univalent in $\mathbb{U}$.

The condition (1.1) for $f(z) \in \mathcal{A}$ is equivalent to

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| \leq 1 \qquad (z \in \mathbb{U}). \tag{1.2}$$

---

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic function, univalent function, starlike function, convex function, subordination.

Using (1.2), Nunokawa, Obradović and Owa [4] gave

Theorem 1.2. *If* $f(z) \in \mathcal{A}$ *satisfies* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *and*

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1 \qquad (z \in \mathbb{U}),$$

then $f(z)$ *is univalent in* $\mathbb{U}$*.*

Let $S(\alpha)$ be the subclass of $\mathcal{A}$ consisting of $f(z) \in \mathcal{A}$ which satisfy

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| \leq 1 - \alpha \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real α $(0 \leq \alpha < 1)$. Considering (1.2) and (1.3), we know that $S(\alpha)$ is the subclass of S which is the subclass of $\mathcal{A}$ consisting of univalent functions $f(z)$ in $\mathbb{U}$, that is, that $S(\alpha) \subset S$. To discuss for $f(z) \in S(\alpha)$, we have to recall here the following result due to Suffridge [7] (or Miller and Mocanu [3]).

Lemma 1.1. *Let* $h(z)$ *be starlike in* $\mathbb{U}$ *with* $h(0) = 0$*. If* $p(z) \in \mathcal{N}$ *satisfies*

$$zp'(z) \prec h(z) \qquad (z \in \mathbb{U}),$$

then

$$p(z) \prec g(z) = 1 + \int_0^z \frac{h(t)}{t} dt \qquad (z \in \mathbb{U}).$$

The function $g(z)$ *is convex in* $\mathbb{U}$*.*

2. Univalency for functions

Applying Lemma 1.1, we derive

Theorem 2.1. *If* $p(z) \in \mathcal{N}$ *satisfies*

$$z\, p'(z) \prec (1-\alpha) z \qquad (z \in \mathbb{U})$$

for $0 \leq \alpha < 1$*, then*

$$p(z) \prec 1 + (1-\alpha) z \qquad (z \in \mathbb{U}).$$

Proof. It is clear that $(1-\alpha) z$ is starlike in $\mathbb{U}$. Therefore, Lemma 1.1 gives us that

$$p(z) \prec 1 + \int_0^z \frac{(1-\alpha) t}{t} dt = 1 + (1-\alpha) z \qquad (z \in \mathbb{U}).$$

□

Remark 2.1. *If* $\alpha = 0$ *in Theorem 2.1, then we have the result by Miller and Mocanu* [2].

Theorem 2.2. *If* $f(z) \in \mathcal{A}$ *satisfies* $\dfrac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *and*

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1 - \alpha \qquad (z \in \mathbb{U}) \tag{2.1}$$

for $0 \leq \alpha < 1$*, then* $f(z) \in S(\alpha)$*.*

Proof. Define a function $p(z)$ by

$$p(z) = \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' = \frac{z^2 f'(z)}{f(z)^2} \qquad (z \in \mathbb{U}). \tag{2.2}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Thus we see that $p(z) \in \mathcal{N}$. It follows from (2.2) that

$$p'(z) = -z\left(\frac{z}{f(z)}\right)''.$$

Therefore, our condition (2.1) gives us that

$$\left|-z^2\left(\frac{z}{f(z)}\right)''\right| \leq (1-\alpha)|z|^2 < 1-\alpha \qquad (z \in \mathbb{U}).$$

This means that

$$-z^2\left(\frac{z}{f(z)}\right)'' \prec (1-\alpha)z \qquad (z \in \mathbb{U}).$$

Applying Theorem 2.1, we have that if

$$-z^2\left(\frac{z}{f(z)}\right)'' \prec (1-\alpha)z \qquad (z \in \mathbb{U}),$$

then

$$\frac{z^2 f'(z)}{f(z)^2} \prec 1 + (1-\alpha)z \qquad (z \in \mathbb{U}). \tag{2.3}$$

The subordination (2.3) shows us that

$$\left|\frac{z^2 f''(z)}{f(z)^2} - 1\right| \leq (1-\alpha)|z| < 1-\alpha \qquad (z \in \mathbb{U}),$$

that is, that $f(z) \in S(\alpha)$. This completes the proof of the theorem. □

Remark 2.2. *Letting $\alpha = 0$ in Theorem 2.2, we obtain Theorem 1.2 by Nunokawa, Obradović, and Owa* [4].

Remark 2.3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathbb{U})$$

for some real β $(0 \leq \beta < 1)$, then $f(z)$ is starlike of order β in $\mathbb{U}$. We denote by $S^*(\beta)$ the class of starlike functions of order β in $\mathbb{U}$. It is well known that $S^*(\beta) \subset S^*(0) \subset \mathcal{A}$.

Let us now consider a function $f(z)$ given by

$$f(z) = \frac{2z}{2 + (1-\alpha)z^2} \qquad (z \in \mathbb{U}). \tag{2.4}$$

Since $f(0) = 0$ and $f'(0) = 1$, $f(z)$ is in the class $\mathcal{A}$ with $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$. It follows from (2.4) that

$$\left(\frac{z}{f(z)}\right)'' = 1-\alpha \qquad (z \in \mathbb{U}).$$

Therefore, $f(z)$ satisfies the condition (2.1). Thus Theorem 2.2 shows us that $f(z) \in S(\alpha)$. For such a function $f(z)$, we have that

$$\frac{zf'(z)}{f(z)} = 1 - \frac{2(1-\alpha)z^2}{2+(1-\alpha)z^2}.$$

Letting $z = e^{-i\theta}$ $(0 \leq \theta < 2\pi)$, we see that

$$\begin{aligned}
\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \mathrm{Re}\left(1 - \frac{2(1-\alpha)}{2e^{-i2\theta}+(1-\alpha)}\right) \\
&= 1 - 2(1-\alpha)\,\mathrm{Re}\left(\frac{1}{((1-\alpha)+2\cos 2\theta) - i2\sin 2\theta}\right) \\
&= 1 - 2(1-\alpha)\frac{(1-\alpha)+2\cos 2\theta}{((1-\alpha)+2\cos 2\theta)^2 + 4\sin^2 2\theta} \\
&= 1 - 2(1-\alpha)\frac{(1-\alpha)+2\cos 2\theta}{(1-\alpha)^2+4+4(1-\alpha)\cos 2\theta}.
\end{aligned}$$

Consider a function $h(t)$ by

$$h(t) = \frac{(1-\alpha)+2t}{(1-\alpha)^2+4+4(1-\alpha)t} \qquad (t = \cos 2\theta).$$

Since $h'(t) > 0$, we have that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 1 - \frac{1}{3-\alpha} = \frac{2-\alpha}{3-\alpha} \qquad (z \in \mathbb{U}).$$

This means that $f(z) \in S^*\left(\frac{2-\alpha}{3-\alpha}\right)$.

From the above, we have

Problem 1. *Find some real* β $(0 \leq \beta < 1)$ *such that* $S(\alpha) \subset S^*(\beta)$. *The function* $f(z)$ *given by* (2.4) *gives us that* $f(z) \in S(\alpha)$ *and* $f(z) \in S^*\left(\frac{2-\alpha}{3-\alpha}\right)$.

References

[1] P.L.Duren: *Univalent Functions*, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[2] S. S. Miller, P. T. Mocanu: *On some classes of first-order differential subordinations*, Michigan Math.J. **32** (1985), 185-195.

[3] S. S. Miller, P. T. Mocanu: *Differential Subordinations*, Theory and Applications, Marcel Dekker, New York, Basel, 2000.

[4] M. Nunokawa, M. Obradović, S. Owa: *One criterion for univalency*, Proc. Amer. Math. Soc., **106** (1989), 1035-1037.

[5] S. Ozaki, M. Nunokawa: *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc.,**33** (1972), 392-394.

[6] Ch. Pommerenke: *Univalent Functions*, Vanderhoeck and Ruprecht, Göttingen, 1975.

[7] T.J.Suffridge: *Some remarks on convex maps of the unit disc*, Duke Math.J., **37** (1970), 775-777.

DEPARTMENT OF MATHEMATICS
KAZIM KARABEKIR FACULTY OF EDUCATION
ATATÜRK UNIVERSITY
ERZURUM T-25240, TURKEY
Email address: nesuyan@yahoo.com

HONORARY PROFESSOR
1 DECEMBRIE 1918, UNIVERSITY OF ALBA LULIA
510009 ALBA LULIA, ROMANIA
Email address: shige21@ican.zag.ne.jp