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EVERSIBILITY WITH RESPECT TO AN IDEAL

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ABSTRACT. We define a right ideal I of an associative ring R to be eversible if $xRy \subseteq I$ for $x, y \in R$, then there exists $1 \neq z \in R$ such that $zRx \subseteq I$ and conversely. In case 0 is an eversible ideal, then R is called an eversible ring. We prove that the eversible condition is preserved by certain sorts of ring extensions and localizations. Numerous examples and relations between the other well known rings are provided throughout.

1. INTRODUCTION

A ring is reduced if it has no nonzero nilpotent elements. Cohn called a ring R reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, [2]. An element $a \in R$ is a left zero-divisor if there is $0 \neq r \in R$ with $ar = 0$. An element which is not a left zero-divisor is called a non-left zero-divisor. Right zero-divisors and non-right zero-divisors are defined analogously. Anderson and Camillo [1] investigate that the rings whose zero products commute, used the term ZC_2 for what is called reversible, but Krempa and Niewieczerzal [5] took the term C_0 for it. Therefore, in the class of reversible rings every left zero-divisor is a right zero-divisor and conversely.

In [3], Ghashghaei et. al. show that there exist rings in which every left zero-divisor in R is also a right zero-divisor while are not reversible rings. Such as rings are called eversible rings.

In [8], G. Mason introduced the reflexive property for ideals (a right ideal I of a ring R (possibly without identity) is called reflexive if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$) and J.-Y. Kim and J. U. Baik generalized this concept by defining idempotent reflexive right ideals and rings [4] (if $aRe \subseteq I$ implies $eRa \subseteq I$ for any a, $e^2 = e \in R$). In [6], the authors give some characterizations of reflexive rings and show that the concept of idempotent reflexive ring is not left-right symmetric, and that the reflexive condition is Morita invariant. Moreover, they prove that both the polynomial ring and the power series ring over a reflexive ring are idempotent reflexive. Reflexive rings are obviously one-sided idempotent reflexive, but not conversely by ([6], Example 2.3(1)). It is proved in ([6], Theorem 2.6) that the reflexive condition is Morita invariant. A (right idempotent) reflexive ring which is not semiprime (resp., reflexive) is also constructed from any semiprime (resp., reflexive) ring in ([6], Proposition 2.5 and Theorem 3.9).

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In this paper, we call a right ideal I of an associative ring R to be (idempotent) eversible if $(xRe \subseteq I)$ $xRy \subseteq I$ for $x, y \in R$ (and $e^2 = e \in R$), then there exists $1 \neq z \in R$ such that $(eRz \subseteq I\ )$ $zRx \subseteq I$ and conversely. In case 0 is an (idempotent) eversible ideal, then R is called an (idempotent) eversible ring. In particular, we define a right ideal I of an associative ring R to be completely eversible if $xy \in I$, then there exists $1 \neq z \in R$ such that $zx \in R$. Also, R is completely eversible if 0 has the corresponding property.

Throughout this paper, R will be an associative ring with identity, $U(R)$ its group of units, $J(R)$ its Jacobson radical, $Id(R)$ its set of idempotents of R. $Soc(R_R)$ is the right socle of R. We observed the followings:

(1) if R is an eversible ring and $e \in Id(R)$, then eR is eversible (equivalently, eR is two-sided) and e is central;
(2) if R is a subdirectly irreducible semiprime ring such that every right ideal of R is eversible, then R is a division ring, and in case R is a right primitive ring in which every maximal modular right ideal is eversible, then R is a field;
(3) for all $a \in aR$, every right ideal $a \in aR$ is completely eversible if and only if every aR is completely eversible.

Concerning idempotent eversible condition, we shall obtain that:

(4) for an idempotent eversible maximal ring R which contains an injective maximal right ideal, then R is right self-injective and;
(5) R is a regular right self-injective ring with $Soc(R_R) \neq 0$, if and only if R is an idempotent eversible right p.p.-ring containing an injective maximal right ideal.

2. EVERSIBILITY

We begin with the formal definitons of the central concept of the article.

Definition 2.1. *For a one-sided ideal I of a ring R, $x \in R$ is called a left pivot for I if $xa \in I$ then $a \in I$,* [8].

Definition 2.2. *A right ideal I is called eversible if $xRy \subseteq I$ for $x, y \in R$, then there exists $1 \neq z \in R$ such that $zRx \subseteq I$ and conversely. A ring R is called eversible when 0 is an eversible ideal.*

Lemma 2.1. *Let I be a right ideal of a ring R.*

(1) *If I is an eversible right ideal of a ring R, then $sRr \subset I$ for all $s \in I$, $r \in R$. Hence $R^2 I \subseteq I$.*
(2) *We have the following:*
    (a) *if $R = rR$ for some $r \in R$ then every eversible right ideal is two-sided.*
    (b) *if I is an eversible right ideal with a left pivot, then I is two-sided.*
    (c) *if e is an idempotent, every eversible right ideal $I \subseteq eR$ is two-sided.*

Proof. (1) Let $x \in sRr$ such that $x = sr_1r$ for all $r_1 \in R$ and so $x \in I$. Thus we have, $sRr \subseteq I$. But also I is eversible, so there exists $1 \neq z \in R$ such that $zRs \subseteq I$ which implies that $R^2 I \subseteq I$.
(2) (a) and (b) are straightforward.
(c) Assume that $I \subseteq eR$, for all $i \in I$ and $r \in R$ we have $i = er = e^2.er \in R^2 I$. Thus $I = R^2 I$ is an ideal. □

Theorem 2.1. *Let R be an eversible ring and $e \in Id(R)$. The followings are equivalent:*

a) *eR is eversible*

b) *e is central*
c) *eR is two-sided.*

Proof. $(a) \Rightarrow (b)$ Since eR is two-sided by the Lemma 2.1, for any $r \in R$, we get $re.e = ex$ for some x and so $ere = re$. Thus $sere = sre$ and $(se - s)Re = 0$ for all $r, s \in R$. Since R is eversible, there exists $1 \neq z \in R$ such that $zR(se - s) = 0$. In particular for $z = r = e$, we obtain $ese = es$ for all s. But we also have $ese = se$. With this part of the proof, $(c) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ can be easily seen.
$(b) \Rightarrow (a)$ If e is central and $xRy \subset eR$, then $xry = es$ for all $r \in R$ and for some s, so $exry = xry$. Thus $(ex - x)ry = 0$ for all $r \in R$ and since R is eversible, there exists $1 \neq z \in R$ such that $zr(ex - x) = 0$. Therefore $zrex = zrx$. But e is central so $zRx \subset eR$ as desired. □

By taking into consideration the Theorem 2.1, in a ring in which every principal right ideal is eversible, the idempotents are central. So we have the same consequence for the following.

Proposition 2.1. *For a ring R, if every non-zero principal right ideal is eversible, then idempotents are central.*

Proof. For all $x, y, z \in R$, $zRx \subset xR$ since $xRy \subset xR$, and so $z^2Rx \subset zxR$. By the hypothesis, $mRz^2 \subset zxR$ for $1 \neq m \in R$. In particular, if we take $m = x = e$ and $z = 1 - e$, then $(1 - e)Re = 0$ and $eR(1 - e) = 0$. Thus e is central. □

Recall that a ring R is said to be prime if the product of any two nonzero ideals of R is nonzero. Equivalently, $aRb = 0$ with $a, b \in R$ implies $a = 0$ or $b = 0$. A ring R is called semiprime if it has no nonzero nilpotent ideals. Equivalently, $aRa = 0$ with $a \in R$ implies $a = 0$.

Theorem 2.2. *If R is a subdirectly irreducible semiprime ring such that every right ideal of R is eversible, then R is a division ring.*

Proof. For a unique smallest non-zero ideal K of R which is also a minimal right ideal, if K contains any non-zero right ideal I of R, then $R^2I \subset I$. So R^2I is an ideal in K and $R^2I = 0$ or $R^2I = K$. If $R^2I = 0$ then $I^3 = 0$ which contradicts with R is semiprime. Thus K is a minimal right ideal so $K = eR$ and e is central by Theorem 2.1. Thus $I =< er-r >$ is an ideal and it must contain K, but it has known that $I \cap K = 0$. Hence $I = 0$ and $R = eR$ is a division ring. □

Theorem 2.2 still holds only if we say that every principal right ideal is eversible. If a non-zero right ideal L is contained in K, then K contains all the principal right ideals xR for $x \in L$ and also $I = xR$ is non-zero since R is semiprime ring.

Since reduced rings are semiprime and every right ideal is eversible in eversible rings, we have the following:

Proposition 2.2. *If every non-zero principal right ideal of R is eversible, then every minimal left ideal I with $I^2 \neq 0$ is a division ring.*

Proof. Let I be a minimal left ideal with $I^2 \neq 0$ which has the form Re for $e^2 = e \in R$. By Proposition 2.1, we have eR is eversible ideal and $xRe \subset eR$ for all $x \in R$. So $xee = es$ for some $s \in R$ and $exe = xe$ for all $x \in R$. Therefore $eRe = Re$ is a division ring. □

An ideal P of a ring R is called left primitive if it is the largest ideal of R contained in some maximal left ideal M of R. A ring is called primitive if 0 is a primitive ideal.

Theorem 2.3. *Let R be a right primitive ring whose every maximal modular right ideal is eversible, then R is a field.*

Proof. By the assumption, R is primitive ring so there exists a maximal modular right ideal M which has the form $(M : R) = 0$. By Lemma 2.1 (b), M is two-sided ideal and so that $M \subseteq (M : R) = \{r \in R : Rr \subset M\}$, i.e. $M = 0$. Thus R is a field. □

Proposition 2.3. *Let $1 \in R$, if M, N, P are right ideals with $MN \subseteq I$, then $PM \subseteq I$ if and only if I is an eversible right ideal .*

Proof. $(\Leftarrow)$ If $MN \subseteq I$, then $mRn \subseteq I$ for all $m \in M$, $n \in N$ so $pRm \subseteq I$ for all $p \in P$ which means $PM \subseteq I$.
$(\Rightarrow)$ If $mRn \subseteq I$ then $RmRn \subseteq I$ so $RpRm \subseteq I$ and $pRm \subseteq I$. □

Definition 2.3. *A right ideal I is called completely eversible if $xy \in I$ then there exists $1 \neq z \in R$ such that $zx \in I$. We say that R is completely eversible if 0 has the corresponding property.*

Clearly, completely eversible ideals are eversible and these are two-sided.

Definition 2.4. *I is called left (right) symmetric if $abc \in I$ implies $acb \in I$ (respectively, $abc \in I$ implies $bac \in I$) .*

Definiton 2.4 is a modification of one in [7] which deal with unital rings, in that case the left and right definitons are equivalent. Actually, it is true that I is (left) symmetric if and only if $\prod_{i=1}^{n} a_i \in I$ implies $\prod_{i=1}^{n} a_{\sigma(i)} \in I$ for all n and permutations $\sigma \in S_n$.

Example 1. *Every completely prime ideal is completely eversible and any prime completely eversible ideal is completely prime. The set of completely eversible ideals is closed under intersection. Hence every ideal of R is completely eversible, if every ideal of R is an intersection of completely prime ideals.*

Example 2. *If R is strongly regular, then every one-sided ideal is completely eversible.*

Example 3. *For a ring R, a maximal right ideal is completely eversible if and only if it is two-sided, in this case it is completely prime.*

Theorem 2.4. *Let I be a right ideal of a ring R.*

(1) *I is a right symmetric left ideal if and only if Ix^{-1} is completely eversible and hence Ix^{-1} is right symmetric*
(2) *If Ix^{-1} is right symmetric, then it is two-sided.*

Proof. (1) If $ab \in Ix^{-1}$ then $abx \in I$ so $bax \in I$ and $ax \in I$. For $z \in R$, $zax \in I$ and $za \in Ix^{-1}$. Furthermore, $abc \in Ix^{-1}$ implies $abcx \in I$ so $bacx \in I$ and hence $bac \in Ix^{-1}$.
(2) Let $r \in Ix^{-1}$ and $s \in R$. Clearly, $srx \in I$ and $rsx \in I$. Thus $rs \in Ix^{-1}$. □

Now consider a ring in which every aR is completely eversible. Since $ab \in aR$, there exists $z \in R$ such that $za \in aR$.

Theorem 2.5. *If $a \in aR$ for all $a \in R$, then every right ideal is completely eversible if and only if every aR is completely eversible.*

Proof. If $xy \in I$, then $xy \in xyR$ so $zx \in xyR \subset I$, as desired. □

3. Idempotent Eversible Ideals

A right ideal I is called idempotent eversible if $aRe \subseteq I$ if and only if there exists $1 \neq c \in R$ such that $eRc \subseteq I$. We say that R is an idempotent eversible ring when 0 is an idempotent eversible ideal.

Proposition 3.1. *If R is an idempotent eversible ring and e is an idempotent of R, then the following are equivalent:*

(1) *eR is an idempotent eversible right ideal.*
(2) *eR is a two-sided ideal.*
(3) *e is central.*

Proof. $(1) \Rightarrow (2)$ Let $i \in eR = I$. So we have $i = ex \in R^2I$ for some $x \in R$. Thus $I \subseteq R^2I$. By the hypothesis eR is an idempotent eversible right ideal and $eRa \subseteq eR$ for any $a \in R$, so we get $cRe \subseteq eR$ for some $1 \neq c \in R$. Hence $R^2e \subseteq eR$. Therefore $I = R^2I$. Also we have $bi = bex = beex \in R^2I$ for any $b \in R$ and $i \in I$. Hence $I = eR$ is a two-sided ideal.
$(2) \Rightarrow (3)$ For some $x \in R$, $xe = xee \in x(eR) \subseteq eR$. So we have $xe = er$ for some $r \in R$. Thus $exe = er = xe$. For any $s \in R$, we obtain $sexe = sxe$. Now $(se - s)xe = 0$ and so $(se - s)Re = 0$. Since R is idempotent eversible, there exists $1 \neq c \in R$ such that $eRc = 0$ and so we have $eR(ce - c) = 0$. Therefore $ece = ec$ which implies that $ex = xe$ for any $x \in R$.
$(3) \Rightarrow (1)$ Suppose $xRf \subseteq eR$ where $f = f^2 \in R$. For any $r \in R$, we have $xrf = ey$ for any $y \in R$. Therefore $exrf = ey = xrf$, so $(ex - x)rf = 0$. Since R is idempotent eversible and $(ex - x)Rf = 0$, we get $fRc = 0$ so $fR(ec - c) = 0$ for some $1 \neq c \in R$. Thus $frec = frc$. Since e is central, so $frc \in eR$. Hence $fRc \in eR$. □

Corollary 3.1. *If every principal right ideal of R is idempotent eversible, then R is abelian.*

In general, the existence of an injective maximal right ideal in a ring R may not guarantee the right self-injectivity of R. However, we obtain the following result.

Proposition 3.2. *If an idempotent eversible maximal ring R contains an injective maximal right ideal, then R is right self-injective.*

Proof. Let M be an injective maximal right ideal of R and N a maximal right ideal. So we have $R = M \oplus N$. Thus we obtain $M = eR$ and $N = (1 - e)R$ for some nonzero idempotent $e \in R$. If $NM = 0$, then we get $(1 - e)Re = 0$. Since R is idempotent eversible, $cR(1 - e) = 0$. Thus, in particular for $c = e$, e is central. Therefore we heve $R = M \oplus N = Re \oplus R(1 - e)$. So ${}_R(R/M)$ is projective. By [10], $(R/M)_R$ is injective. Thus N_R is injective. If $NM \neq 0$, then $NM = N$. So there is $b \in N$ such that $bM \neq 0$, whence $N = bM$. Let $f : M \to N$ be the map defined by $f(m) = bm$ for all $m \in M$. Then f is an epimorphism. Since the right module N_R is projective and $M/kerf \cong N$, we have $M \cong kerf \oplus M/kerf \cong kerf \oplus N$ as right R-modules. Thus N_R is injective. Therefore $R = M \oplus N$ is right self-injective. □

Corollary 3.2. *If a semiprime ring R contains an injective maximal right ideal, then R is right self-injective.*

A ring R is called a right p.p.-ring if every principal right ideal of R is projective. So, as an application of Proposition 3.2, we have the following

Corollary 3.3. *Let R be a ring, the following are equivalent:*

(1) *R is a regular right self-injective ring with $Soc(R_R) \neq 0$.*

(2) *R is an idempotent eversible right p.p.-ring containing an injective maximal right ideal.*

Proof. $(1) \Rightarrow (2)$ Firstly, regularity of R implies that R is an idempotent eversible right p.p.-ring. If every maximal right ideal of R is essential, then $Soc(R_R)$ is contained in $J(R)$, which is imposible. So there exists a maximal right ideal M of R which is not essential. Therefore M is a direct summand of R. Since R is right self-injective, M is an injective right ideal.
$(2) \Rightarrow (1)$ By Proposition 3.2, R is right self-injective. Since R is right p.p.-ring, R is regular . Moreover we have $Soc(R_R) \neq 0$ because there is an injective maximal right ideal. □

By [11], a ring R is called a right HI-ring if R is a right hereditary ring containing an injective maximal right ideal. Osofsky [9] proves that a right self-injective right hereditary ring is semisimple Artinian. The next corollary extends [11, Theorem 8].

Corollary 3.4. *The following statements are equivalent for a ring R:*

(1) *R is semisimple Artinian.*
(2) *R is an idempotent eversible right HI-ring.*

Proof. $(1) \Rightarrow (2)$ This is obvious.
$(2) \Rightarrow (1)$ This follows from Proposition 3.2 and Osofsky Theorem (see [9]). □

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