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SOME PROPERTIES OF THE CONJUGATE FOURIER-JACOBI AND FOURIER-CHEBYSHEV SERIES

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ABSTRACT. In the present paper we prove a new result on determination of jump discontinuities by the n-th order tails of the conjugate Fourier-Jacobi series. Also, we prove the equiconvergence related to "harmonic" function (Poisson integral) and conjugate "harmonic" function (conjugate Poisson integral) of the Fourier-Chebyshev series.

1. INTRODUCTION AND PRELIMINARIES

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n and order $(\alpha,\beta), \alpha,\beta > -1$, normalized so that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. They are orthogonal on the interval $(-1,1)$ with respect to the measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$.

Define $R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$, and denote by $L_p(\alpha,\beta), (1 \le p < \infty)$ the space of functions $f(x)$ for which $\|f\|_{p(\alpha,\beta)} = \{\int_{-1}^1 |f(x)|^p d\mu_{\alpha,\beta}(x)\}^{\frac{1}{p}}$ is finite.

For functions $f \in L_1(\alpha,\beta)$, its Fourier-Jacobi expansion is

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x), \tag{1.1}$$

where

$$\hat{f}(n) = \int_{-1}^{1} f(y)R_n^{(\alpha,\beta)}(y)d\mu_{\alpha,\beta}(y)$$

are the Fourier coefficients and

$$\omega_n^{(\alpha,\beta)} = \{\int_{-1}^{1}[R_n^{(\alpha,\beta)}(y)]^2 d\mu_{\alpha,\beta}(y)\}^{-1} \sim n^{2\alpha+1}.$$

An alternative way is to define Fourier-Jacobi expansion of a function f on $(0,\pi)$ by (1.2).

$$f(\theta) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta), \tag{1.2}$$

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where

$$\hat{f}(n) = \int_0^{\pi} f(\varphi) R_n^{(\alpha,\beta)}(\cos\varphi) d\mu_{\alpha,\beta}(\varphi),$$

$$\omega_n^{(\alpha,\beta)} = \{\int_0^{\pi} [R_n^{(\alpha,\beta)}(\cos\varphi)]^2 d\mu_{\alpha,\beta}(\varphi)\}^{-1} \sim n^{2\alpha+1},$$

and correspondingly $d\mu_{\alpha,\beta}(\theta) = 2^{\alpha+\beta+1} sin^{2\alpha+1}\frac{\theta}{2}\cos^{2\beta+1}\frac{\theta}{2}d\theta$.

To the Fourier-Jacobi series of the form (1.2), its conjugate series is defined by

$$\tilde{f}(\theta) \sim \frac{1}{2\alpha+2}\sum_{n=1}^{\infty} n\hat{f}(n)\omega_n^{(\alpha,\beta)} R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta, \tag{1.3}$$

(see [7]). If we start with (1.1), and $x = \cos\theta$, this would correspond to

$$\tilde{f}(x) \sim \frac{-1}{2\alpha+2}\sum_{n=1}^{\infty} n\hat{f}(n)\omega_n^{(\alpha,\beta)} R_{n-1}^{(\alpha+1,\beta+1)}(x)\sqrt{1-x^2}.$$

Conjugate Fourier-Jacobi series was introduced by B. Muckenhoupt and E. M. Stein [7] when $\alpha = \beta$, and by Zh.-K. Li [6] for general α and β. "Conjugacy" is an important concept in classical Fourier analysis which links the study of the more fundamental properties of harmonic functions to that of analytic functions and is used to study the mean convergence of Fourier series [13].

Further, we shall consider the "harmonic" function (Poisson integral) associated to (1.1):

$$f(r,x) = \sum_{n=0}^{\infty} r^n \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x), 0 \le r < 1, \tag{1.4}$$

and the "conjugate harmonic" function (conjugate Poisson integral) of (1.4)

$$\tilde{f}(r,x) = \frac{-1}{2\alpha+2}\sum_{n=1}^{\infty} r^n n\hat{f}(n)\omega_n^{(\alpha,\beta)} R_{n-1}^{(\alpha+1,\beta+1)}(x)\sqrt{1-x^2}, \tag{1.5}$$

(see [6]).

Denote by $S_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (1.2), and by $\tilde{S}_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (1.3), where $x = \cos\theta$. If $\alpha = \beta = -\frac{1}{2}$, the corresponding Fourier-Jacobi series becomes Fourier-Chebyshev series, so by $S_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)$ we denote the n-th partial sum of the Fourier-Chebyshev series of f. Also, throughout this paper we use the following general notations: $L[a,b]$ is the space of integrable functions on $[a,b]$ and $C[a,b]$ is the space of continuous function on $[a,b]$ with the uniform norm $\|\cdot\|_{C[a,b]}$. $W[a,b]$ is the space of functions on $[a,b]$ which may have discontinuities only of the first kind and which are normalized by the condition $f(x) = \frac{1}{2}(f(x+) + f(x-))$.

In this paper first we give a review of the results on determination of jump discontinuities for functions of generalized bounded variation by differentiated and integrated Fourier series, and then we prove a new result on determination of jump discontinuities by the n-th order tails of the conjugate Fourier-Jacobi series. After that, we prove the equiconvergence related to "harmonic" function (Poisson integral) and conjugate "harmonic" function (conjugate Poisson integral) of the Fourier-Chebyshev series.

2. Determination of jumps by Fourier series

The knowledge of the precise location of the discontinuity points is essential for many of the methods aiming at obtaining exponential convergence of the Fourier series of a piecewise smooth function, avoiding the well-known Gibbs phenomenon, the oscillatory behavior of the Fourier partial sums of a discontinuous function.

If a function f is integrable on $[-\pi, \pi]$, then it has a Fourier series with respect to the trigonometric system $\{1, \cos nx, sin\, nx\}_{n=1}^{\infty}$, and we denote the n-th partial sum of the Fourier series of f by $S_n(x, f)$, i.e.,

$$S_n(x,f) = \frac{a_0(f)}{2} + \sum_{k=1}^{n} (a_k(f)\cos(kx) + b_k(f)\sin(kx)),$$

where $a_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\cos(kt)\,dt$ and $b_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\sin(kt)\,dt$ are the $k-$th Fourier coefficients of the function f. By $\tilde{S}_n(x, f)$ we denote the $n-$th partial sum of the conjugate series, i.e.,

$$\tilde{S}_n(x,f) = \sum_{k=1}^{n} (a_k(f)\sin(kx) - b_k(f)\cos(kx)).$$

The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time. Let $f(x)$ be a function of bounded variation with period 2π, and $S_n(x, f)$ be the partial sum of order n of its Fourier series. By the classical theorem of Fejer [13] the identity

$$\lim_{n\to\infty} \frac{S_n'(x,f)}{n} = \frac{1}{\pi}(f(x+0) - f(x-0)) \tag{2.1}$$

holds at any point x.

The classes of functions of bounded variation of higher were firstly introduced by N. Wiener [12].

A function f is said to be of bounded p-variation, $p \geq 1$, on the segment $[a, b]$ and to belong to the class $\mathcal{V}_p[a, b]$ if

$$V_{a\ p}^{b}(f) = \sup_{\Pi_{a,b}} \left\{ \sum_i |f(x_i) - f(x_{i-1})|^p \right\}^{\frac{1}{p}} < \infty,$$

where $\Pi_{a,b} = \{a = x_0 < x_1 < ... < x_n = b\}$ is an arbitrary partition of the segment $[a, b]$. $V_{a\ p}^{b}(f)$ is the p-variation of f on $[a, b]$.

B. I. Golubov [3] has shown that identity (2.1) is valid for classes $\mathcal{V}_p$.

Theorem 2.1. *Let $f(x) \in \mathcal{V}_p$, $(1 \leq p < \infty)$ and $r \in \mathbb{N}_0$. Then for any point x one has the equation*

$$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(x,f)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(f(x+0) - f(x-0)).$$

Another type of generalization of the class BV on everywhere convergence of Fourier series, for every change of variable, was introduced by D. Waterman in [11].

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that the series $\sum \frac{1}{\lambda_n}$ diverges and $\{I_n\}$ be a sequence of nonoverlapping segments $I_n = [a_n, b_n] \subset [a, b]$. A function f is said to be of Λ-bounded variation on $I = [a, b]$ $(f \in \Lambda BV)$ if

$\sum \frac{|f(b_n) - f(a_n)|}{\lambda_n} < \infty$ for every choice of $\{I_n\}$. The supremum of these sums is called the Λ-variation of f on I. In the case $\Lambda = \{n\}$, one speaks of harmonic bounded variation (HBV).

The class HBV contains all Wiener classes. M. Avdispahić has shown in [1] that HBV is the limiting case for validity of the identity (2.1).

G. Kvernadze in [5] generalized Theorem A for ΛBV classes.

Theorem 2.2. *Let $r \in \mathbb{Z}_+$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then*

(a) *the identity*

$$\lim_{n\to\infty} \frac{((S_n(g;\theta))^{(2r+1)}}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(g(\theta+) - g(\theta-)).$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi, \pi]$ if and only if $\Lambda BV \subseteq HBV$.

(b) *there is no way to determine the jump at the point $\theta \in [-\pi, \pi]$ of an arbitrary function $g \in \Lambda BV$ by means of the sequence $((S_n(g;\theta))^{(2r)}, n \in \mathbb{N}_0$.*

Also, here we note the result for conjugate Fourier series [5]:

Theorem 2.3. *Let $r \in \mathbb{N}$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then the identity*

$$\lim_{n\to\infty} \frac{(\widetilde{S}_n(g;\theta))^{(2r)}}{n^{2r}} = \frac{(-1)^{(r+1)}}{2r\pi}(g(\theta+) - g(\theta-))$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi, \pi]$ if and only if condition $\Lambda BV \subseteq HBV$ holds.

Similar identities hold if we consider the integrated rather than the differentiated Fourier series [4]. By $R_n(x, f)$ we denote the n-th order tails of the Fourier series of the function f, i.e.,

$$R_n(x, f) = \sum_{k=n}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx),$$

for $n \in \mathbb{N}$.

For any function f, integrable on $[-\pi, \pi]$, $f^{(-r)}$, $r \in \mathbb{N}_0$, is defined as follows

$$f^{(-r-1)} \equiv \int f^{(-r)},$$

where $f^{(0)} \equiv f$, and the constants of integration are successively determined by the condition

$$\int_{-\pi}^{\pi} f^{(-r)}(t)dt = 0.$$

Such results find their application in recovering edges in piecewise smooth functions with finitely many jump discontinuities [2].

Here we note two results of Kvernadze, Hagstrom and Shapiro [4] which determine the jumps of a $2\pi-$periodic function of $f \in \mathcal{V}_p$, $1 \le p < 2$, class, with a finite number of discontinuities, by means of the tails of its integrated Fourier series:

Theorem 2.4. *Let $r \in \mathbb{N}_0$ and suppose the function $f \in \mathcal{V}_p$, $1 \le p < 2$, has a finite number of discontinuities. Then:*

1. *the identity*

$$\lim_{n\to\infty} n^{2r+1} R_n^{(-2r-1)}(f;x) = \frac{(-1)^{r+1}}{(2r+1)\pi}(f(x+) - f(x-))$$

is valid for each fixed $x \in [-\pi, \pi]$*;*

2. *there is no way to determine the jump at the point* $x \in [-\pi, \pi]$ *of an arbitrary function* $f \in \mathcal{V}_p$*,* $p \geq 1$*, by means of the sequence* $(R_n^{(-2r-2)}(f;.)), n \in \mathbb{N}$.

Theorem 2.5. *Let* $r \in \mathbb{N}$ *and suppose the function* $f \in \mathcal{V}_p$*,* $1 \leq p < 2$*, has a finite number of discontinuities. Then:*

1. *the identity*

$$\lim_{n\to\infty} n^{2r} \widetilde{R}_n^{(-2r)}(f;x) = \frac{(-1)^{r+1}}{2r\pi}(f(x+) - f(x-))$$

is valid for each fixed $x \in [-\pi, \pi]$*;*

2. *there is no way to determine the jump at the point* $x \in [-\pi, \pi]$ *of an arbitrary function* $f \in \mathcal{V}_p$*,* $p \geq 1$*, by means of the sequence* $(\widetilde{R}_n^{(-2r-1)}(f;.)), n \in \mathbb{N}$.

Similar identity which determines the jump of a periodic function of $\mathcal{V}_p$, $1 \leq p < 2$ class with a finite number of discontinuities, by means of the tails of its integrated Fourier-Jacobi series, was proved in [8].

3. Main Results

Theorem 3.1. *Let* $\alpha > -1$, $\beta > -1$ *and* $f \in HBV$*. Then the identity*

$$\lim_{n\to\infty} \frac{\tilde{R}_n^{(\alpha,\beta)}(f,x)}{\log n} = \frac{f(x+0) - f(x-0)}{\pi}$$

is valid for each fixed $x \in (-1, 1)$.

Proof. In [9] it was proved that for $f \in HBV$, and $\alpha > -1$, $\beta > -1$ relation:

$$\lim_{n\to\infty} \frac{\tilde{S}_n^{(\alpha,\beta)}(f,x)}{\log n} = -\frac{f(x+0) - f(x-0)}{\pi}$$

is valid for each fixed $x \in (-1, 1)$.

If we now use the well-known identity (see [10]),

$$\tilde{S}_n^{(\alpha,\beta)}(f,x) = f(x) - \tilde{R}_n^{(\alpha,\beta)}(f,x)\,,$$

we get

$$\lim_{n\to\infty} \frac{f(x) - \tilde{R}_n^{(\alpha,\beta)}(f,x)}{\log n} = -\frac{f(x+0) - f(x-0)}{\pi}\,.$$

Further, we have

$$\lim_{n\to\infty} \frac{f(x)}{\log n} - \lim_{n\to\infty} \frac{\tilde{R}_n^{(\alpha,\beta)}(f,x)}{\log n} = -\frac{f(x+0) - f(x-0)}{\pi}\,.$$

As the first member on the left side of the last equality is tending to zero as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{\tilde{R}_n^{(\alpha,\beta)}(f,x)}{\log n} = \frac{f(x+0) - f(x-0)}{\pi}\,.$$

□

Theorem 3.2. *Let $f \in L_1(-\frac{1}{2},-\frac{1}{2})$ and $0 \le r < 1$. If $f_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)$ denotes the n-th partial sum of the "harmonic" function (Poisson integral) of Fourier-Chebyshev series and $\tilde{f}_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)$ denotes the n-th partial sum of the conjugate "harmonic" function (conjugate Poisson integral) of Fourier-Chebyshev series, then for $-1 < x < 1$,*

$$\lim_{n\to\infty}\left(\frac{\partial f_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)}{\partial x}+\frac{r}{\sqrt{1-x^2}}\frac{\partial \tilde{f}_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)}{\partial r}\right)=0.$$

Proof. Differentiating the expression for the n-th partial sum of the "harmonic" function (Poisson integral) of Fourier-Chebyshev series:

$$f_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)=\sum_{k=0}^{n} r^k\hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})}R_k^{(-\frac{1}{2},-\frac{1}{2})}(x)\,,$$

with respect to x, we get

$$\frac{\partial f_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)}{\partial x}=\sum_{k=0}^{n}\hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})}\frac{d}{dx}(R_k^{(-\frac{1}{2},-\frac{1}{2})}(x))\,.$$

Here we will give a procedure for the general (α,β), so using [10, 4.21.7.] i.e.:

$$\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)\right]=\frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

we have:

$$\begin{aligned}\frac{\partial f_n^{(\alpha,\beta)}(r,x)}{\partial x} &= \sum_{k=0}^{n} r^k\hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{d}{dx}\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)}=\\ &= \sum_{k=0}^{n} r^k\hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{d}{dx}(P_k^{(\alpha,\beta)}(x))\frac{1}{P_k^{(\alpha,\beta)}(1)}=\\ &= \sum_{k=0}^{n} r^k\hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{(k+\alpha+\beta+1)}{2}\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_k^{(\alpha,\beta)}(1)}=\\ &= -\frac{1}{2}\sum_{k=0}^{n}(k+\alpha+\beta+1)r^k\hat{f}(k)\omega_k^{(\alpha,\beta)}\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_k^{(\alpha,\beta)}(1)}\,.\end{aligned}$$

As $P_{k-1}^{(\alpha+1,\beta+1)}(1)=\begin{pmatrix}k+\alpha\\k-1\end{pmatrix}$, we get

$$\frac{P_{k-1}^{(\alpha+1,\beta+1)}(1)}{P_k^{(\alpha,\beta)}(1)}=\frac{\begin{pmatrix}k+\alpha\\k-1\end{pmatrix}}{\begin{pmatrix}k+\alpha\\k\end{pmatrix}}=\frac{k}{\alpha+1}.$$

So $P_k^{(\alpha,\beta)}(1)=\dfrac{P_{k-1}^{(\alpha+1,\beta+1)}(1)}{\frac{k}{\alpha+1}}$, and we obtain

$$\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_k^{(\alpha,\beta)}(1)}=\frac{P_{k-1}^{(\alpha+1,\beta+1)}(x)}{P_{k-1}^{(\alpha+1,\beta+1)}(1)}\cdot\frac{k}{\alpha+1}=R_{k-1}^{(\alpha+1,\beta+1)}(x)\cdot\frac{k}{\alpha+1}\,.$$

Now, we have

$$\frac{\partial f_n^{(\alpha,\beta)}(r,x)}{\partial x} = \sum_{k=1}^{n} \frac{k\cdot(k+\alpha+\beta+1)}{2\alpha+2} r^k \hat{f}(k)\omega_k^{(\alpha,\beta)} \cdot R_{k-1}^{(\alpha+1,\beta+1)}(x)\,,$$

and for $\alpha=\beta=-\frac{1}{2}$

$$\frac{\partial f_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)}{\partial x} = \sum_{k=1}^{n} k^2 r^k \hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})} R_{k-1}^{(\frac{1}{2},\frac{1}{2})}(x)\,. \tag{3.1}$$

According to (1.5)

$$\tilde{f}_n^{(\alpha,\beta)}(r,x) = \frac{-1}{2\alpha+2}\sum_{k=1}^{n} k\cdot \hat{f}(k)\omega_k^{(\alpha,\beta)}\cdot R_{k-1}^{(\alpha+1,\beta+1)}(x)\sqrt{1-x^2}\,,$$

so, for $\alpha=\beta=-\frac{1}{2}$ we get

$$\tilde{f}_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x) = -\sum_{k=1}^{n} k\cdot\ r^k \hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})}\cdot R_{k-1}^{(\frac{1}{2},\frac{1}{2})}(x)\sqrt{1-x^2}\,. \tag{3.2}$$

Now, differentiating the identity (3.2) with respect to r we obtain

$$\frac{\partial \tilde{f}_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)}{\partial r} = -\sum_{k=1}^{n} k^2\cdot\ r^{k-1}\hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})}\cdot R_{k-1}^{(\frac{1}{2},\frac{1}{2})}(x)\sqrt{1-x^2}\,. \tag{3.3}$$

Multiplying by r the identity (3.3) and dividing by $\sqrt{1-x^2}$ we get

$$\frac{r}{\sqrt{1-x^2}}\frac{\partial \tilde{f}_n^{(-\frac{1}{2},-\frac{1}{2})}(r,x)}{\partial r} = -\sum_{k=1}^{n} k^2\cdot\ r^k\hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})}\cdot R_{k-1}^{(\frac{1}{2},\frac{1}{2})}(x)\,. \tag{3.4}$$

Finally, adding (3.1) and (3.4), and letting $n\to\infty$ proves the result. □

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