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GERŠGORIN-TYPE LOCALIZATION SETS OF SOME POLYNOMIAL EIGENVALUE PROBLEMS

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ABSTRACT. Diagonalization and separation of variables, the problem of resonance, asymptotics and stability are key factors of popularity of eigenvalues of matrices in science and engineering. In that way, localization of spectra can simplify solving complex polynomial eigenvalue problems arising in applications. In some cases, matrices might describe one spectral pattern in complex plane and quite another if small perturbations are added. In such cases, the localization of pseudospectra can be used for analysing transient behavior, sensitivity of the eigenvalues of a matrix polynomials to perturbations, etc. Polynomial eigenvalue problems are highly represented in modern science, especially in the field of vibro-acoustics, fluid mechanics, Markov chains and control theory. Here we give the technique for constructing localization sets of spectra and pseudospectra for matrix polynomials based on famous Geršgorin theorem. Such Geršgorin-type spectra and pseudospectra localization sets for matrix polynomials will be presented and their applicability will be analysed on several examples arised from problems in engineering.

1. INTRODUCTION

Studying acoustics of a car, quantum mechanics, design of buildings, ecology, stability and bifurcation analysis of dynamical systems, stationary distribution of random processes (of birth and death), control theory, stochastic models in telecommunications are among numerous applications of eigenvalues in science and engineering [8]. The behavior of these and similar phenomena is depicted by systems of differential equations of second and higher degree, which are approximated with matrix polynomials of the appropriate order. In these applications, we are interested in studying behavior of one or more eigenvalues inside or at the end of the spectrum, or number of eigenvalues on a specific interval. In solving eigenvalue problems, it is enough to localize the spectrum, which is computationally cheaper problem. Therefore, the localization of spectra has an important role in solving eigenvalue problems.

In some cases, such as in the domains of structures with physical damping (viscoelastic, hysteretic damping), rotational structures, vibro-acoustics, transient behaviour of non normal matrices, pipe flow in hydrodynamic stability problem etc., when small perturbations are included, spectra analysis can be misleading. In these cases pseudospectra

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analysis may reveal more than spectra about certain aspects of the behavior of matrices and operators, because it takes into account small perturbations of the observed system [8]. Therefore, it is of the greatest importance to develop computationally cheap technique that constructs pseudospectra localization sets.

Inspired by elegant and computationally cheep localization technique developed for SEP and GEP and described and used in [5, 6], and NLEP [4] here we present it for PEP and apply to several examples that arise from applications.

2. Notations and definitions

The polynomial eigenvalue problem (or, shortly, PEP) for analytic and regular matrix polynomial $P_m : \Omega \to \mathbb{C}^{n,n}$ on a simply connected domain $\Omega \subseteq \mathbb{C}$ is to find discrete set of solutions (z, v) for which holds:

$$P_m(z)v = 0, \quad v \neq 0, \tag{2.1}$$

where $P_m(z)$ is matrix polynomial of the form:

$$P_m(z) = z^m A_m + z^{m-1} A_{m-1} + ... + A_0,$$

where $A_i \in \mathbb{C}^{n,n}$, $i = 0, \dots m$, and $z \in \Omega$ and $v \in Ker(P_m(z))$ are eigenvalues and corresponding (right) eigenvector, respectively. $\mathcal{N}_n(\Omega)$ is the family of all such analytic and regular matrix polynomials P_m on simply connected domain Ω. The set $\Lambda(z) := \{z \in \mathbb{C} \ : \ \det(P_m(z)) = 0\}$ is called the spectrum of a matrix polynomial P_m, and, in the case when small perturbations are included, the set of eigenvalues of a perturbed matrix polynomial P_m is called the pseudospectrum of P_m. Analogously, the pseudoeigenvalues are eigenvalues of a perturbed matrix polynomial P_m. When Ω is unbounded in $\mathbb{C}$, let $\mathbb{C}_\infty$ be one-point-compactification of the complex plane $\mathbb{C}$ whose geometrical representation is the Riemann sphere. Having this in mind, we can present eigenvalues in infinity by using the Moebius transform $z \to 1/z$. We define ∞ as an eigenvalue of P_m if there exists $\varphi \in \mathcal{N}_n(\Omega)$ and singular $M \in \mathbb{C}^{n,n}$, such that

$$\lim_{k\to\infty} \frac{P_m(z_k)}{\varphi(z_k)} = M \neq 0$$

for all unbounded sequences $\{z_k\}_{k\in N} \subseteq \Omega$.

In this setting, we define multiplicities od eigenvalues of P as multiplicities of zero eigenvalue of $\hat{P}_m : \hat{\Omega} \to \mathbb{C}^{n,n}$, where $\hat{\Omega} := \{0\} \cup \{1/z : z \in \Omega\}$ and

$$\hat{P}_m(z) := \frac{P_m(1/z)}{\varphi(1/z)}.$$

In other words, ∞ is an eigenvalue of P_m if and only if 0 is an eigenvalue of $\hat{P}_m$, and the multiplicities coincide.

In the case when small perturbations are included, the pseudospectrum localization gives more precise answers to some (mentioned) questions. There are few definitions of the pseudospectra and we use the one defined in [8]. Given $\epsilon > 0$, the pseudospectrum for matrix polynomial P_m is the set $\Lambda_{q,\epsilon}(P_m) = \{z \in \mathbb{C} : ||P_m(z)^{-1}||^{-1} \leq \epsilon q(|z|)\}$, where $q(z) = \sum_{k=0}^{m} \alpha_k z^k$, and α_k are nonnegative parameters defined depending on the measure of perturbations ($\alpha_k \equiv 1$, if the perturbations are measured in absolute sense, and $\alpha_k = ||A_k||$, if the perturbations are measured in relative sense). For singular P_m, we use the convention $||P_m^{-1}||^{-1} = 0$ and assume P_m has only finite (pseudo)eigenvalues. For calculating with infinite eigenvalues, one should use the technique defined in [7].

In the following, we will only consider regular PEP's of the form (2.1), such that $det(P_m(z)) \neq 0$. If P_m is singular, then the pseudospectra set for such P_m is $\mathbb{C} \cup \{\infty\}$, and no useful information for the pseudospectra of P_m can be obtained [3].

As our goal is to construct the spectra and the pseudospectra localization sets for matrix polynomial $P_m \in \mathcal{N}_n(\Omega)$, we use an elegant connection between famous Geršgorin theorem (Theorem 1.1, [9]) and the properties of some well-known classes of DD-matrices (defined, for example in [2] and [9]). The localization technique we use in this paper is based on a SDD property of a matrix:

Definition 2.1. *The matrix $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ is called an SDD matrix if and only if for every (row) index $i \in N := \{1, 2, \dots, n\}$*

$$|a_{i,i}| > r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{i,j}|.$$

Now we define DD-type class of matrices $\mathcal{K}$ as the class of matrices for which all nonzero diagonal entries, which is closed for increasing of moduli of diagonal entries and decreasing of moduli of nondiagonal entries and arbitrary change of a complex argument of an arbitrary entry [4]. Then, we define SDD-type class of matrices, also defined in [4], as a class that is open DD-type class and $\forall \alpha > 0, \alpha \in \mathbb{R}$ is $\alpha A \in \mathcal{K}$. (An open class is if for every matrix $A \in \mathcal{K}$, there exists an arbitrary small $\varepsilon > 0$, such that for every matrix $B \in \mathbb{C}^{n,n}$, $|(A - B)_{i,j}| < \varepsilon$, for all $i, j \in N$, implies $B \in \mathcal{K}$.) Using the properties of some well known SDD-type classes of matrices and the Geršgorin theorem, we construct some (pseudo)spectra localization sets for P_m.

Let $\mathbb{C}^n$ (as in the above) is a complex n-dimensional vector space of column vectors $x = [x_1, x_2, \dots, x_n]^T$, where $x_i \in \mathbb{C}$, $i = 1, 2, \dots, n$, for an arbitrary $n \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$, $\mathbb{C}^{n,n}$ is the collection of all $n \times n$ matrices with complex entries. $A = [a_{ij}]$ is a matrix $A \in \mathbb{C}^{n,n}$, with entries $a_{ij} := (A)_{ij} \in \mathbb{C}$, for all $1 \leq j \leq n$. diag$(A) :=$ diag$(a_{11}, a_{22}, \dots, a_{nn})$, is the diagonal part of A and $|A| = [|a_{ij}|]$ is a matrix of the moduli of the elements of a matrix A. $\langle A \rangle := [m_{i,j}] \in \mathbb{R}^{n,n}$ denotes the comparison matrix of A, i.e. $m_{i,j} = |a_{i,i}|$, for $i = j$, and $m_{i,j} = -|a_{i,j}|$. N is a set of indices, I is identity matrix and O is zero matrix. For given matrix polynomial $P_m = [p_{ij}]$, $i, j \in N$, the sum of moduli of i-th deleted row of P_m is $r_i(z) = \sum_{\substack{j \neq i \\ i,j \in N}} |p_{ij}|$ the sum of moduli of i-th deleted column of P_m is $c_i(z) := \sum_{j \in \mathbb{N} \setminus \{i\}} |p_{ji}(z)|$, while $s_i(z) := \sum_{j \neq i} \frac{|p_{ij}(z)| + |p_{ji}(z)|}{2}$. Also, for an arbitrary set of indices $\emptyset \neq S \subseteq N$, $r_i^S(z) := \sum_{j \in S \setminus \{i\}} |p_{ij}(z)|$. Using these preliminaries, in the following section we will construct Geršgorin-type localization sets for eigenvalues of (2.1).

3. Geršgorin-type localization sets

Given a class $\mathcal{K}$ of square complex matrices of an arbitrary size, and matrix polynomial $P_m \in \mathcal{N}_n(\Omega)$, we define the set of complex numbers

$$\Theta^{\mathcal{K}}(P_m) := \{z \in \Omega \, : \, P_m(z) \notin \mathcal{K}\} \, .$$

Having in mind the setup of one-point-compactification $\mathbb{C}_\infty$, the set $\Theta^{\mathcal{K}}(P_m)$ can be defined on $\mathbb{C}_\infty$. Namely, if Ω is unbounded, $\infty \in \Theta^{\mathcal{K}}(P_m)$ if and only if for every unbounded sequence $\{z_k\}_{k \in \mathbb{N}} \subset \Omega$, there exists $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$, $P_m(z_k) \notin \mathcal{K}$.

Using the characterization of $\mathbb{S}$, $\mathbb{O}$ and $\mathbb{H}$ classes of matrices (given in [2] and [5]), for given matrix polynomial $P_m(z) = [p_{i,j}(z)] \in \mathcal{N}_n(\Omega)$, Geršgorin-type localization sets for P_m, the sets $\Theta^{\mathbb{S}}(P_m)$, $\Theta^{\mathbb{O}}(P_m)$ and $\Theta^{\mathbb{H}}(P_m)$, are characterized in the following way, as in [4]:

$$\Theta^{\mathbb{S}}(P_m) = \bigcup_{i\in N} \Theta_i^{\mathbb{S}}(P_m) = \bigcup_{i\in N} \{z \in \mathbb{C} : |p_{i,i}(z)| \le r_i(z)\} \tag{3.1}$$

$$\Theta^{\mathbb{O}}(P_m) = \bigcap_{\alpha\in[0,1]} \bigcup_{i\in N} \Theta_i^{\mathbb{O}}(P_m) = \bigcap_{\alpha\in[0,1]} \bigcup_{i\in N} \{z \in \mathbb{C} : |p_{i,i}(z)| \le (r_i(z)^{\alpha}(c_i(z))^{1-\alpha}\} \tag{3.2}$$

$$\Theta^{\mathbb{H}}(P_m) := \{z \in \mathbb{C} : \mu(P(z)) \ge 0\}, \tag{3.3}$$

where μ is left-most eigenvalue of $\langle P_m(z)\rangle$.

When small perturbations are included, combining Geršgorin theorem and the properties of some SDD-type classes, we get localization sets for pseudospectrum of P_m. Here we construct these sets in infinity norm, which is very useful in some applications. Regarding Lemma 1 (localization principle) in [6] and Lemma 2.1 in [7] and equality 2, given $\mu : \mathbb{C}^{n,n} \to \mathbb{R}$ such that for arbitrary matrix polynomial P_m holds

$$||P_m(z)^{-1}||_{\infty}^{-1} \ge \mu(z),$$

the corresponding pseudospectra localization set for P_m is:

$$\Lambda_{q,\epsilon}(z) \subseteq \Theta^{\mu}_{q,\epsilon}(P_m) := \{z \in \mathbb{C} : \mu(z) \le \epsilon q(|z|)\} \tag{3.4}$$

where $q(z) = \sum_{k=0}^{m} \alpha_k z^k$, and α_k are nonnegative parameters defined in previous section.

Using the inequality (3.4), it is possible to consider different lower bounds μ for $||P_m(z)^{-1}||_{\infty}^{-1}$ based on the properties of DD classes of matrices and construct the corresponding pseudospectra localization sets in infinity norm. The first one is:

$$||P_m(z)^{-1}||_{\infty}^{-1} \ge \mu_1(z) := \min_{i\in\mathbb{N}}\{|p_{ii}(z)| - \sum_{j\neq i} |p_{ij}(z)|\} \tag{3.5}$$

Having in mind Lemma 2 and Theorem 1 in [6], given matrix polynomial $P_m(z) = [p_{ij}(z)] \in \mathbb{C}^{n,n}$, and (3.5), the corresponding pseudospectra localization for P_m is:

$$\Theta^{\mu_1}_{q,\epsilon}(P_m) := \bigcup_{i\in N} \Theta^{\mu_1,i}_{q,\epsilon}(P_m) = \bigcup_{i\in N} \{z \in \mathbb{C} : |p_{ii}(z)| \le \sum_{j\neq i} |p_{ij}(z)| + \epsilon q(|z|)\} \tag{3.6}$$

and $\mu_1(P_m(z)) \le \epsilon q(|z|)$ holds true.

Next localization result uses the properties of doubly diagonal dominant (dSDD) matrices (defined in [2, 9]), Lemma 3 and Theorem 2 in [6]:

$$||P_m(z)^{-1}||_{\infty}^{-1} \ge \mu_2(z) = \min \frac{|p_{ii}(z)||p_{jj}(z)| - \sum_{k\neq i} |p_{ik}(z)| \sum_{l\neq j} |p_{jl}(z)|}{|p_{ii}(z)| + \sum_{l\neq j} |p_{jl}(z)|} \tag{3.7}$$

where minimum is counting trough every $i \neq j$ for which $|p_{ii}(z)| + \sum_{l\neq j} |p_{lj}(z)| \neq 0$, and, in the trivial case, for $P_m(z) = 0$, $\mu_2(z)$ is defined to be zero.

The corresponding pseudospectra localization set for P_m is of the form:

$$\Theta^{\mu_2}_{q,\epsilon}(P) := \bigcup_{i\neq j} \Theta^{\mu_2,i}_{q,\epsilon}(z) = \bigcup_{i\neq j}\{z \in \mathbb{C} : |p_{ii}(z)|(|p_{jj}(z)| - \epsilon q(|z|)) \leq \sum_{j\neq l} |p_{jl}(z)|(\sum_{k\neq i} |p_{ik}| + \epsilon q(|z|)\}$$

and $\mu_2(P(z)) \leq \epsilon q(|z|)$ holds true.
The set $\Theta^{\mu_2}_{q,\epsilon}(P_m)$ localizes the pseudospectra of P_m, i.e., $\Lambda_{q,\epsilon}(z) \subseteq \Theta^{\mu_2}_{q,\epsilon}(P_m)$.

3.1. **The properties of the set** $\Theta^{\mathcal{K}}(P_m)$**.** The first property of Geršgorin-type localization sets for matrix polynomial $P_m \in \mathcal{N}_n(\Omega)$ comes from the definition of diagonal dominance and the set $\Theta^{\mathcal{K}}(P_m)$. Originally, it is known as monotony property and is defined and proved for nonlinear eigenvalue problem (NLEP) in [4]:

Theorem 3.1 (Monotony property for PEP)**.** *For given DD-type classes $\mathcal{K}_1$ and $\mathcal{K}_2$ and matrix polynomial $P_m \in \mathcal{N}_n(\Omega)$, $m \geq 2$, the corresponding localization sets $\Theta^{\mathcal{K}_1}(P_m)$ and $\Theta^{\mathcal{K}_2}(P_m)$ are standing in reverse inclusion than the inclusion of the classes, i.e., the narrower DD-type class $\mathcal{K}$ is, the wider is the corresponding localization set $\Theta^{\mathcal{K}}(P_m)$:*

$$\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \Theta^{\mathcal{K}_1}(P_m) \supseteq \Theta^{\mathcal{K}_2}(P_m).$$

It is well known that $\mathbb{S} \subseteq \mathbb{O} \subseteq \mathbb{H}$, so, using Theorem 3.1 the relation:

$$\Theta^{\mathbb{H}}(P_m) \subseteq \Theta^{\mathbb{O}}(P_m) \subseteq \Theta^{\mathbb{S}}(P_m)$$

holds for every polynomial $P_m \in \mathcal{N}_n(\Omega)$.

Other properties, known as equivalence property (that claims that all the matrices from a class $\mathcal{K}$ are regular and that for arbitrary $P_m \in \mathcal{K}$, $\Theta^{\mathcal{K}}(P_m) \supseteq \Lambda(P_m)$), isolation property (that gives the number of eigenvalues in each component of the localization set), the boundedness property (that gives the information about the boundedness of the set $\Theta^{\mathcal{K}}(P_m)$ are also defined and proved for NLEP in [4]. Here, we give their formulaton for PEP:

Theorem 3.2. *Let $P_m \in \mathcal{N}_n(\Omega)$ is analytic and regular matrix polynomial on a simply connected domain $\Omega \subseteq \mathbb{C}$, that defines the observed NLEP. For the set $\Theta^{\mathbb{S}}(P_m)$, constructed on the base of the properties of the class $\mathcal{K}$, next conditions are fulfilled:*

1. $\Lambda(P_m) \subseteq \Theta^{\mathbb{S}}(P_m)$.
2. *If there exist closed connected sets $U, V \subseteq \Omega$, such that $\Omega \setminus (U \cup V)$ is connected and for the corresponding localization set $\Theta^{\mathbb{S}}(P_m)$, holds:*

$$\Theta^{\mathbb{S}}(P_m) \subseteq U \ \cup \ V,$$

then the number of eigenvalues of P_m in the set U and the number of the solutions of the equations $p_{ii}(z) = 0$, $i \in N$ coincide.

3. *For given positive homogeneity SDD-type class $\mathcal{K}$, the set $\Theta^{\mathbb{S}}(P_m)$, is closed in $\mathbb{C}$. Moreover, if the domain Ω is unbounded, then $A_m \in \mathcal{K}$ implies $\Theta^{\mathcal{K}}(P_m)$ is compact in $\mathbb{C}$.*

Remark 3.1. *The converse of the Theorem 3.2, item 3, is not always valid-see* [4]*, page 183.*

The property of positive homogeneity allows the properties of the localization sets (equivalence, isolation, monotony and boundedness) to be transferred to the localization sets $\Theta^{\mathbb{S}}(P_m)$, $\Theta^{\mathbb{O}}(P_m)$ and $\Theta^{\mathbb{H}}$, as direct corollaries of Theorem 3.2. Having this in mind, Theorem 3.2 holds for the sets $\Theta^{\mathbb{O}}(P_m)$ and $\Theta^{\mathbb{H}}(P_m)$, too.

4. Examples

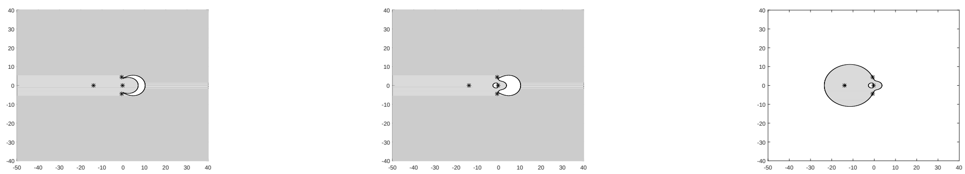
Example 1. A bicycle problem *is quadratic PEP arised from the dynamic behavior of a Whipple bicycle model, a dynamical system where one of its key properties is due to a feedback mechanism that is created by the design of the front fork* [10]. *The equations of motion can be derived by keeping track of the balance of forces acting on the system:*

$$J\frac{d^2\phi}{dt^2} - \frac{Dv}{b}\frac{d\delta}{dt} = mgh\sin\phi + \frac{mv^2h}{b}\delta,$$

where v is the forward speed of the bicycle. The Whipple model of the bicycle can be computed by using the rigid-body dynamics of the front fork and frame. The linearized model (around the upright position, with steering wheel straight) is given by:

$$C\begin{bmatrix}\ddot{\phi}\\ \ddot{\delta}\end{bmatrix} + Bv\begin{bmatrix}\dot{\phi}\\ \dot{\delta}\end{bmatrix} + (K_0 + K_2v^2)\begin{bmatrix}\phi\\ \delta\end{bmatrix} = \begin{bmatrix}O\\ T\end{bmatrix}$$

where C, B, $A = K_0 + K_2v^2$ are 2×2 matrices that depend on the geometry and mass distribution of the bicycle [10]. *C is symmetric mass matrix, B is nonsymmetric damping matrix, and the default forward speed $v = 5m/s$. The associated quadratic matrix polynomial is $P_2(z) = z^2C + zvB + A$, with $A, B, C \in \mathbb{C}^{2,2}$ given in* [1].



(a) *(b)* *(c)*

FIGURE 1. Localization sets $\Theta^{\mathbb{S}}(P_2)$ (a), $\Theta^{\mathbb{O}}(P_2)$ (b) and $\Theta^{\mathbb{H}}(P_2)$ (c) for NLEP bicycle

The sets $\Theta^{\mathbb{S}}(P_2)$, $\Theta^{\mathbb{O}}(P_2)$ and $\Theta^{\mathbb{H}}(P_2)$ are shown in Figure 1, respectively. The leading matrix $C \notin \mathbb{S}$ and $C \notin \mathbb{O}$, so the sets $\Theta^{\mathbb{S}}(P_2)$ and $\Theta^{\mathbb{O}}(P_2)$ are unbounded in $\mathbb{C}$ and we cannot apply item 3 of the Theorem 3.2. On the other side, $C \in \mathbb{H}$, so the set $\Theta^{\mathbb{H}}(P_2)$ is bounded.These sets can be calculated analytically, because of the small format of the matrices that describe this problem. The set $\Theta^{\mathbb{S}}(P_2) = \Theta^{\mathbb{S}}_1(P_2) \cup \Theta^{\mathbb{S}}_2(P_2)$ (3.1), where:

$$\Theta^{\mathbb{S}}_1(P_2) = \{z \in \mathbb{C} : |z| \cdot |z - 9.83| \le |0.29z^2 + 2.1z + 23.4|\ \},$$

$$\Theta^{\mathbb{S}}_2(P_2) = \{z \in \mathbb{C} : |z^2 + 28.3z + 196.4| \le |7.8z^2 - 14.3z - 85.6|\ \}.$$

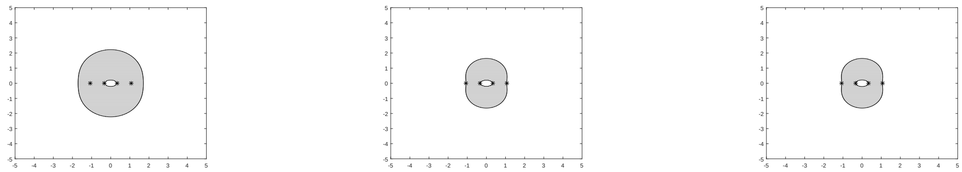
The sets $\Theta^{\mathbb{O}}(P_2)$ (3.2) and $\Theta^{\mathbb{H}}(P_2)$ (3.3)are more complicated in computational sense, but can be calculated analogously, following the definitions of these sets.

(a) *(b)*

FIGURE 2. The set Θ^{μ_i}, $i = 1, 2$, for NLEP bicycle, measured with absolute (a) and relative (b) perturbations, $\epsilon = 0.005$

The pseudospectra localization sets (defined with (3.6) and (3.7)) are shown in Figure 1, for perturbations measured in absolute (Figure 1) and relative sense (Figure 1), and $\epsilon = 0.005$.

Example 2. Closed loop system *is quadratic PEP associated with closed-loop control system with feedback gains* 1 *and* $1+\alpha$*, designed to automatically achieve and maintain the desired output condition by comparing it with the actual condition. These systems are used in various industry applications (agriculture, chemical plants, quality control, nuclear power plants, water treatment plants and environmental control)* [11]. *The associated quadratic matrix polynomial* $P_2(z) = z^2C + zB + A$, $A, B, C \in \mathbb{C}^{2,2}$*, where default forward speed in m/s is* $v = 5$*. The default value* $\alpha = 1$*. The matrices defining this PEP are given in* [1].



(a) *(b)* *(c)*

FIGURE 3. Sets $\Theta^{\mathbb{S}}(P_2)$ (a), $\Theta^{\mathbb{O}}(P_2)$ (b), $\Theta^{\mathbb{H}}(P_2)$ (c) for NLEP closed loop

Localization sets $\Theta^{\mathbb{S}}(P_2)$, $\Theta^{\mathbb{O}}(P_2)$ *and* $\Theta^{\mathbb{H}}(P_2)$ *are shown in Figures 2, 2 and 2, respectively. The leading matrix* C *and matrix* A *of this QEP are diagonal, i.e. SDD matrices, while* $B \notin \mathbb{S}$*. As* $C \subseteq \mathbb{S} \subseteq \mathbb{O} \subseteq \mathbb{H}$*, we can conclude (due to Theorem 3.2, item 3) that these localization sets are compact in* $\mathbb{C}$.

As this problem is defined by 2×2 *matrices, it is easy to present analiticallly the Geršgorin-type localization set* $\Theta^{\mathbb{S}}(P_2) = \bigcup_{i=1}^{2} \Theta_i^{\mathbb{S}}(P_2)$*, where:*

$$\Theta_1^{\mathbb{S}}(P_2) = \{z \in \mathbb{C} : |z - 0.71i| \cdot |z + 0.71i| \leq 2|z| \},$$
$$\Theta_2^{\mathbb{S}}(P_2) = \{z \in \mathbb{C} : |z - 0.5| \cdot |z + 0.5| \leq |z|\}.$$

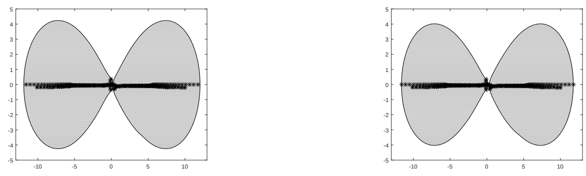
On the other hand, the sets $\Theta^{\mathbb{O}}(P_2)$ *and* $\Theta^{\mathbb{H}}(P_2)$ *are complicated in computational sense.*

(a) *(b)*

FIGURE 4. The set Θ^{μ_i}, $i = 1, 2$, for NLEP closed loop, measured with absolute (a) and relative (b) perturbations, $\epsilon = 0.1$

The pseudospectra localization sets in infinity norm are shown in Figure 2, for perturbations measured in absolute (Figure 2) and relative sense (Figure 2), and $\epsilon = 0.1$.

Example 3. Plasma drift *is cubic PEP, arose from the problem of design of Tokamak reactor, a device that uses a powerful magnetic field to confine a hot plasma in the shape of a torus in the process of modelling instabilities of drifts on the edge of plasma inside the reactor. From 2016, it is the leading candidate for a practical fusion reactor* [12].



(a) *(b)*

FIGURE 5. The sets $\Theta^{\mathbb{S}}(P_3) = \Theta^{\mathbb{O}}(P_3)$ (a), $\Theta^{\mathbb{H}}(P_3)$ (c) for plasma drift NLEP

The associated cubic polynomial is: $P_3(z) = z^3 D + z^2 C + zB + A$*, for given* $A, B, C, D \in \mathbb{C}^{n,n}$*. Only two values of* n *are allowed:* $n = 128$ *(default value) and* $n = 512$*. We will use the default value of* n *for constructing the localization sets for* P_3*. The matrices* A, B, C, D *are given in* [1]*, where* D *and* C *are diagonal matrices (i.e. SDD), and* A*,* B *are not SDD matrices.*

(a) *(b)*

FIGURE 6. The sets $\Theta^{\mu_i}_{q,\epsilon}(P_3)$, $i = 1, 2$ for NLEP plasma drift, measured with absolute (a) and relative (b) perturbations, $\epsilon = 0.1$

As D *is SDD-type, we expect that all sets (*$\Theta^{\mathbb{S}}(P_3)$*,* $\Theta^{\mathbb{O}}(P_3)$*,* $\Theta^{\mathbb{H}}(P_3)$*) give good compact approximations of the eigenvalues of* P_3*. This is illustrated in Figure 5. Matrix* $D \in \mathbb{S} \subseteq \mathbb{O} \subseteq \mathbb{H}$*, so, according to Theorem 3.2, item 3, we can conclude that these sets are compact in* $\mathbb{C}$*, as Figure 5 suggests. As* $D \in \mathbb{S} \subseteq \mathbb{O}$ *, it follows that* $\Theta^{\mathbb{S}}(P_3) = \Theta^{\mathbb{O}}(P_3)$ *(Figure 3). Having in mind the format of matrices that define this problem (*$n = 128$*), the sets* $\Theta^{\mathbb{S}}(P_3)$*,* $\Theta^{\mathbb{O}}(P_3)$ *and* $\Theta^{\mathbb{H}}(P_3)$ *must be treated numerically. The pseudospectra localization sets in infinity norm are shown in Figure 6, for perturbations measured in absolute and relative sense and* $\epsilon = 0.1$*.*

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