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THE NEUBERGER SPECTRUM OF A CERTAIN NONLINEAR SUPERPOSITION OPERATOR

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ABSTRACT. In the present paper we consider the nonlinear superposition operator F in l_p spaces of sequences ($1 \leq p \leq \infty$), generated by the function $f(s,u) = a(s) + \frac{u}{u^2+1}$. We find out the Neuberger spectrum $\sigma_N(F) \subseteq \mathbb{R}$ of this operator and its spectral radius. We make a comparison between this spectrum and the point and Rhodius spectrum of considering operator F.

1. INTRODUCTION AND PRELIMINARIES

The spectral theories for nonlinear operators are relatively new research topics and they are still in developing process. At the beginning the notion spectrum has been defined in a way that the spectrum of a considered nonlinear operator has contained the set of eigenvalues i.e. the point spectrum. Different notions of spectra and related spectral theories for nonlinear operators have arisen by now (see [1, 3, 9, 10, 11]).
J. W. Neuberger introduced in 1969. the notion of spectrum ([9]) for the class $\mathfrak{C}^1(X)$ of continuously Fréchet differentiable operators F on Banach space X.

Definition 1.1. *([1, 2]) An operator $F : X \to Y$ is called Fréchet differentiable at $x_0 \in X$ if there is an operator $L : X \to Y$ such that*

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|} \|F(x_0+h) - F(x_0) - Lh\| = 0 \quad (h \in X).$$

In this case the linear operator L is called Fréchet derivative of F at x_0 and denoted by $F'(x_0)$. The value $F'(x_0)\, x \in Y$ for arbitrary $x \in X$, is called Fréchet derivative of operator F at x_0 along x.

If F is differentiable at each point $x \in X$ and the map $x \mapsto F'(x)$ is continuous, we write $F \in \mathfrak{C}^1(X,Y)$ and call F continuously differentiable.

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Theorem 1.1. *(*[4]*) Let $f(s,u)$ be a Carathéodory function and operator F generated by the function $f(s,u)$ acts from l_p to l_q. If operator F is differentiable at $x_0 \in l_p$, then its (Fréchet) derivative in x_0 has the form*

$$F'(x_0)h(s) = a(s)h(s) \tag{1.1}$$

where $a \in l_q/l_p$ is given by

$$a(s) = \lim_{u\to 0} \frac{f(s,x_0(s)+u) - f(s,x_0(s))}{u}.$$

If superposition operator G, generated by the function

$$g(s,u) = \begin{cases} \frac{1}{u}[f(s,x(s)+u) - f(s,x(s))]; & u \neq 0 \\ a(s); & u = 0. \end{cases}$$

acts from l_p to l_q/l_p and it is continuous in 0, then F is differentiable at x_0 and formula (1.1) *holds.*

In the sequel $\mathbb{K}$ is the field of real or complex numbers ($\mathbb{R}$ or $\mathbb{C}$).

Definition 1.2. *(*[1]*) For an operator $F : X \to X$ which is continuously Fréchet differentiable, the set*

$$\rho_N(F) = \left\{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}^1(X)\right\}$$

is called the Neuberger resolvent and the set

$$\sigma_N(F) = \mathbb{K} \setminus \rho_N(F)$$

is the Neuberger spectrum of F.

We call the number

$$r_N(F) = \sup\{|\lambda| : \lambda \in \sigma_N(F)\}$$

Neuberger spectral radius of $F \in \mathfrak{C}^1(X)$. One see that $\lambda \in \mathbb{K}$ belongs to $\rho_R(F)$ if and only if $\lambda I - F$ is a diffeomorphism on X. The Neuberger spectrum is always nonempty (in the complex case), but it need be neither closed nor bounded and it may be useful in solvability of some nonlinear operator equations and eigenvalue problems ([9]).

In 1984., A.Rhodius has considered the notion of spectrum for the class $\mathfrak{C}(X)$ of all continuous operators F on Banach space X. Namely, a point $\lambda \in \mathbb{K}$ belongs to $\rho_R(F)$ if and only if $\lambda I - F$ is a homeomorphism on X ([1, 11]).

The Rhodius and Neuberger spectra of some nonlinear superposition operators in the spaces of sequences may be found in [5], [6], [7], [8]. Since $\mathfrak{C}^1(X) \subseteq \mathfrak{C}(X)$, the following inclusions hold: $\rho_N(F) \subseteq \rho_R(F)$ and $\sigma_N(F) \supseteq \sigma_R(F)$, for $F \in \mathfrak{C}^1(X)$ ([3]).

Definition 1.3. *The set of all eigenvalues of the operator F*

$$\sigma_p(F) = \{\lambda \in \mathbb{K} : Fx = \lambda x \text{ for some } x \neq 0\}$$

is called the point spectrum of F.

It is known that if F is a nonlinear operator F with $F0 = 0$ then

$$\sigma_p(F) \subseteq \sigma_R(F).$$

Let Ω denotes an arbitrary set and $f = f(s,u)$ be a function defined on $\Omega \times \mathbb{R}$ and taking values in $\mathbb{R}$. For a given function $x = x(s)$ on Ω one can define another function $y(s) = f(s,x(s))$ for $s \in \Omega$. In this way, the function f generates an operator

$$Fx(s) = f(s,x(s)), \tag{1.2}$$

This operator F is usually called a superposition operator, Nemytskij operator or composition operator ([2, 4]). The nonlinear superposition operators appear in many mathematics and physical problems and have vast applications.

We will consider a certain operator of superposition, defined in the Banach spaces of sequences l_p $(1 \leq p \leq \infty)$.

Several results about the conditions for acting, continuity and differentiability of superposition operators on the sequence spaces l_p for $1 \leq p \leq \infty$ have been given in [4]:

Theorem 1.2. *Let $1 \leq p, q < \infty$. Then the following properties are equivalent:*

(i) *the operator F acts from l_p to l_q;*
(ii) *there are functions $a(s) \in l_q$ and constants $\delta > 0, n \in \mathbb{N}, b \geq 0$, for which $|f(s,u)| \leq a(s) + b|u|^{\frac{p}{q}}$ $(s \geq n, |u| < \delta)$;*
(iii) *for any $\varepsilon > 0$ there exists a function $a_\varepsilon \in l_q$ and constants $\delta_\varepsilon > 0$, $n_\varepsilon \in \mathbb{N}, b_\varepsilon \geq 0$, for which $\|a_\varepsilon(s)\|_q < \varepsilon$ and*

$$|f(s,u)| \leq a_\varepsilon(s) + b_\varepsilon |u|^{\frac{p}{q}} \;\; (s \geq n_\varepsilon, \;\; |u| \leq \delta_\varepsilon).$$

Theorem 1.3. *Let $1 \leq p, q < \infty$ and let the superposition operator* (1.2)*, generated by the function $f(s,u)$, acts from l_p to l_q. Then this operator is continuous if and only if each of the functions is continuous for every $s \in \mathbb{N}$.*

Theorem 1.4. *Let $1 \leq p, q < \infty$ and the operator F generated by the function $f(s,u)$ acts from l_p into l_q. The operator F is differentiable at $x_0 \in l_p$ if and only if $f'_u(s,\cdot)$ is continuous at x_0 for almost all $s \in \mathbb{N}$.*

2. Main results

We consider the superposition operator F, generated by the function

$$f(s,u) = a(s) + \frac{u}{u^2+1},$$

where $a = (a(s))_{s\in\mathbb{N}}$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. Let us denote $a = (a_1, a_2, \ldots)$, then for arbitrary $x = (x_1, x_2, \ldots) \in l_p$ we have

$$F(x_1, x_2, ...) = (a_1 + \frac{x_1}{x_1^2+1}, a_2 + \frac{x_2}{x_2^2+1}, ...).$$

We have shown in [5] that this operator acts from the space l_p to the space l_p $(1 \leq p \leq \infty)$. For every $s \in \mathbb{N}$ the function $f(s,u) = a(s) + \dfrac{u}{u^2+1}$, is continuous, so in virtue of the Theorem 1.3, the operator F is a continuous one, i.e. $F \in \mathfrak{C}(l_p)$. It is not hard to see that this operator is a continuously differentiable and its Fréchet derivative at arbitrary $x_0 = (x_1, x_2, \ldots)$ along $h = (h_1, h_2, \ldots)$ is

$$F'(x_0)\,h = \left(\frac{1-x_1^2}{(x_1^2+1)^2} \cdot h_1, \frac{1-x_2^2}{(x_2^2+1)^2} \cdot h_2, \ldots \right)$$

and $F'(0)h = h$. Hence, according to the Theorem 1.1 the operator F belongs to the class $\mathfrak{C}^1(l_p)$ and we may find out its Neuberger spectrum.

Theorem 2.1. *Let the superposition operator* $F : l_p \to l_p$, *be generated by the function* $f(s,u) = a(s) + \dfrac{u}{u^2+1}$, *where* $(a(s))_s$ *is a sequence from the space* l_q $(1 \le q \le p \le \infty)$. *Then the Neuberger spectrum of* F *is* $\sigma_N(F) = \left[-\dfrac{1}{8}, 1\right]$.

Proof. We consider the operator

$$(\lambda I - F)x = \left(\lambda x_1 - a_1 - \frac{x_1}{x_1^2+1}, \lambda x_2 - a_2 - \frac{x_2}{x_2^2+1}, \dots\right).$$

In our paper [5] we have shown that this operator $\lambda I - F$ is bijective for $\lambda \in (-\infty, -\frac{1}{8}] \cup [1, \infty)$ and it is not bijective for $\lambda \in (-\frac{1}{8}, 1)$. So, from the Definition 1.2, we have

$$\left(-\frac{1}{8}, 1\right) \subseteq \sigma_N(F) \tag{2.1}$$

and we need to find out for $\lambda \in (-\infty, -\frac{1}{8}] \cup [1, \infty)$ whether the operator $(\lambda I - F)^{-1}$ is a continuously differentiable operator. The operator $\lambda I - F$ is generated by the function

$$f_1(s,u) = \lambda u - a(s) - \frac{u}{u^2+1}$$

and the Fréchet derivative of the operator $\lambda I - F$ at x_0 is

$$(\lambda I - F)'(x_0)\,h = \left(\frac{\lambda x_1^4 + (2\lambda+1)x_1^2 + \lambda - 1}{(x_1^2+1)^2} \cdot h_1, \frac{\lambda x_2^4 + (2\lambda+1)x_2^2 + \lambda - 1}{(x_2^2+1)^2} \cdot h_2, \dots\right).$$

For $\lambda = -\frac{1}{8}$ the operator $-\frac{1}{8}I - F$ is generated by the function

$$f_2(s,u) = -\frac{1}{8}u - a(s) - \frac{u}{u^2+1}.$$

The partial derivative of f_2 with respect to u is

$$(f_2)'_u(s,u) = -\frac{(u^2-3)^2}{8(u^2+1)^2}.$$

Now, for every $u \neq \pm\sqrt{3}$ there exists

$$(f_2^{-1})'(s,u) = -\frac{8(u^2+1)^2}{(u^2-3)^2}$$

but for $u \pm \sqrt{3}$ the function f_2^{-1} is not differentiable. That is why we conclude from the Theorem 1.4 and Theorem 1.1 the operator $(-\frac{1}{8}I - F)^{-1}$ (generated by the function f_2^{-1}) is not a continuously differentiable operator (it is not differentiable at points x, where $x(s) = \pm\sqrt{3}$, for almost all $s \in \mathbb{N}$) and

$$-\frac{1}{8} \in \sigma_N(F). \tag{2.2}$$

For $\lambda = 1$ the operator $I - F$ is generated by the function

$$f_3(s,u) = u - a(s) - \frac{u}{u^2+1}.$$

Then we find

$$(f_3)'_u(s,u) = \frac{u^2(u^2+3)}{(u^2+1)^2}.$$

Now, for every $u \neq 0$ we have

$$(f_3^{-1})'(s,u) = \frac{(u^2+1)^2}{u^2(u^2+3)}$$

and this function $(f_3^{-1})'$ is discontinuous one (the point $u = 0$ is a point of discontinuity). Thus the operator $(I - F)^{-1}$ is not continuously differentiable operator and

$$1 \in \sigma_N(F). \tag{2.3}$$

Generally, for given λ, the partial derivative of f_1 with respect to u is

$$(f_1)'_u(s,u) = \frac{\lambda u^4 + (2\lambda+1)u^2 + \lambda - 1}{(u^2+1)^2}$$

and we may notice it is a function of one variable u. If $\lambda > 1$ then $2\lambda+1 > 0$ and $\lambda - 1 > 0$, so $(f_1)'_u > 0\,(\forall s \in \mathbb{N})$. Hence this function $(f_1)'_u$ is a strictly positive, continuous function and there exists $(f_1^{-1})'_u$ ([12]), for every u,

$$(f_1^{-1})'_u = \frac{(u^2+1)^2}{\lambda u^4 + (2\lambda+1)u^2 + \lambda - 1}.$$

We find out that the function f_1^{-1} is a continuously differentiable for $\lambda > 1$ (for every $s \in \mathbb{N}$) and from the Theorem 1.4 it follows that the operator $(\lambda I - F)^{-1}$ (generated by the function f_1^{-1}) is a continuously differentiable operator for $\lambda > 1$, so

$$(1, \infty) \subseteq \rho_N(F). \tag{2.4}$$

In case when $\lambda < -\frac{1}{8}$ we get

$$(f_1)'_u(s,u) = \frac{\lambda u^4 + (2\lambda+1)u^2 + \lambda - 1}{(u^2+1)^2} <$$

$$< \frac{-\frac{1}{8}u^4 + \frac{3}{4}u^2 - \frac{1}{8} - 1}{(u^2+1)^2} = -\frac{(u^2-3)^2}{8(u^2+1)^2} < 0.$$

Hence, $(f_1)'_u$ is a strictly negative, continuous function, so there exists

$$(f_1^{-1})'_u = \frac{(u^2+1)^2}{\lambda u^4 + (2\lambda u+1)u^2 + \lambda - 1}.$$

The function $(f_1^{-1})'_u$ is continuous for every $s \in \mathbb{N}$ and from the Theorem 1.4 it follows that $(\lambda I - F)^{-1}$ is a continuously differentiable operator for $\lambda < -\frac{1}{8}$, i.e. $(\lambda I - F)^{-1} \in \mathfrak{C}^1(l_p)$ and we get

$$\left(-\infty, -\frac{1}{8}\right) \subseteq \rho_N(F). \tag{2.5}$$

Finally, when we summarize all above, from (2.1), (2.2), (2.3), (2.4) and (2.5), we find the Neuberger spectrum of this operator F:

$$\sigma_N(F) = \left[-\frac{1}{8}, 1\right]. \tag{2.6}$$

$\square$

We see that the Neuberger spectrum (2.6) of the considering operator F is nonempty, bounded, and closed, compact set and the Neuberger spectral radius is

$$r_N(F) = \sup\{|\lambda| : \lambda \in \sigma_N(F)\} = \sup\{|\lambda| : \lambda \in [-\frac{1}{8}, 1]\} = 1.$$

In our previous paper [5] we have found the Rhodius spectrum of this operator and now we may compare these two different spectra (of the same operator F):

$$\sigma_R(F) = \left(-\frac{1}{8}, 1\right) \subseteq \sigma_N(F) = \left[-\frac{1}{8}, 1\right].$$

The point spectrum of this operator F in case when $a(s) = 0, \forall s \in \mathbb{N}$, ($F0 = 0$) is $\sigma_p(F) = (0,1)$ (see Theorem 2.2. in [5]). So, we have inclusion

$$\sigma_p(F) = (0,1) \subseteq \sigma_N(F) = \left[-\frac{1}{8}, 1\right].$$

In other cases, when $F0 \neq 0$, we do not have this inclusion, i.e. the point spectrum $\sigma_p(F)$ is not a subset of the Neuberger spectrum.

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