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A NEW MULTIPLIER DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper, we obtain a new multiplier differential transformation for the analytic univalent functions of the form $z+\sum_{n=2}^{\infty} a_n z^n$ and we give some inclusion properties for the operator so obtained using the principle of subordination between analytic functions.

1. INTRODUCTION AND PRELIMINARIES

Let A denote the class of normalized univalent functions of the form:

$$z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

Definition 1.1. *Let $g(z)$ be analytic and univalent in U and $f(z)$ is analytic in U, then, f is said to be subordinate to g if there exists a Schwartz function $w(z)$ which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$.*

This is expressed as $f \prec g$. Moreover, suppose g is univalent in U, then the following equivalence holds [1, 4, 5, 8]:

$$f \prec g \iff f(0) = g(0) \text{ and } f(U) \subset g(U)\,.$$

For the function f of the form (1.1), the following results are well known: f is said to be starlike respectively convex with respect to the origin, if, and only if,

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0,\ |z| < 1,$$

and

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ |z| < 1\,.$$

f is said to be starlike, respectively convex, of order γ if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, 0 \le \gamma < 1, |z| < 1$$

and

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma,\ 0 \le \gamma < 1,\ |z| < 1.$$

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Remark 1.1. *From the above, it is clear that f is convex if and only if $zf'(z)$ is starlike.*

Definition 1.2. *Let $f \in A$ and g is starlike of order γ i.e.$g \in S^*(\gamma)$, then $f \in K(\beta, \gamma)$, if, and only if, Re$\left\{\frac{zf'(z)}{g(z)}\right\} > \beta,\ z \in U$. This functions are called close-to-convex function of order β type γ.*

For $f \in A$, the following subclasses of starlike, convex and close-to-convex functions $S^*(\xi, \phi), C(\xi, \phi)$ and $K(\xi, \rho; \phi, \varphi)$ of order ξ, are studied by several authors [5, 7, 8] and are respectively defined by:

$$\begin{aligned} S^*(\mu, \psi) &= \left\{ f \in A : \frac{1}{1-\mu}\left(\frac{zf'(z)}{f(z)} - \mu\right) \prec \psi(z), z \in U \right\} \\ C(\mu, \psi) &= \left\{ f \in A : \frac{1}{1-\mu}\left(1 + \frac{zf''(z)}{f'(z)} - \mu\right) \prec, \psi(z), z \in U \right\} \\ k(\mu, \zeta; \psi\varphi) &= \left\{ f \in A : \frac{1}{1-\zeta}\left(\frac{zf'(z)}{g(z)} - \zeta\right) \prec \varphi(z), z \in U, g(z) \in S^*(\mu, \psi) \right\} \end{aligned}$$

For the function of the form (1.1),we have

$$\begin{aligned} (f(z))^\alpha &= \left(z + \sum_{n=2}^{\infty} a_n z^n\right)^\alpha, \alpha \geq 1 \\ &= z^\alpha + \alpha a_2 z^{\alpha+1} + \left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2!} a_2^2\right) z^{\alpha+2} \\ &+ \left(\alpha a_4 + \frac{\alpha(\alpha-1)}{2!} 2a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} a_2^3\right) z^{\alpha+3} \\ &+ \left(\alpha a_5 \frac{\alpha(\alpha-1)}{2!}(2a_2 a_4 + a_3^2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} 3a_2^2 a_3 + 6\right. \\ &+ \left.\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!} a_2^4\right) z^{\alpha+4} .definition \end{aligned}$$

We represent $(f(z))^\alpha$ by :

$$f^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} \alpha a_n z^{n+\alpha-1}. \tag{1.2}$$

Thus as $\alpha \to 1$, we obtain:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Now, using (1.2) we define a new differential operator $D_\alpha^m f$:

Definition 1.3. *Let $0 \leq \lambda \leq 1, \alpha \geq 1, m \in \mathbb{N} \cup 0$. Then for $f \in A$, we define the operator $D_\alpha^m f : A \to A$ by:*

$$D_\alpha^0 f(z) = f(z)$$

$$D_\alpha f(z) = (1-\lambda) f(z) + z\lambda z f'(z) \tag{1.3}$$

$$D_\alpha^m f(z) = D_\alpha(D_\alpha^{m-1} f(z)) \tag{1.4}$$

$$D_\alpha^{m+1} f(z) = (1-\lambda) D_\alpha^m f(z) + z\lambda(D_\alpha^m f(z))'. \tag{1.5}$$

From equation (1.5) we have:

$$z\lambda(D_\alpha^m f(z))' = D_\alpha^{m+1} f(z) - (1-\lambda) D_\alpha^m f(z)\,. \tag{1.6}$$

From (1.3) and (1.4), we obtain:

$$D_\alpha^m f(z) = z + \sum_{n=2}^{\infty} \alpha \left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)} \right)^m a_n z^n.$$

We denote by $\mathcal{H}$, the class of all functions ψ which are analytic and univalent in U for which $\psi(U)$ is convex such that $\psi(0)=1$ and $\text{Re}(\psi(z))>0, z\in U$.

In this paper, we introduce a new multiplier differential operator in relation to (1.1) and (1.2) and we shall employ the principle of subordination between analytic functions to introduce the subclasses of starlike, convex and close-to-convex functions $S^*(\mu,\psi), C(\mu,\psi)$ and $K(\mu,\zeta;\varphi,\psi)$ of order μ respectively,for the function $\psi,\varphi\in\mathbb{H}$ which are defined by:

$$\begin{aligned} S_\alpha^m(\mu,\psi) &= \{f\in A : D_\alpha^m f(z) \in S^*(\mu,\psi)\}\,, \\ C_\alpha^m(\mu,\psi) &= \{f\in A : D_\alpha^m f(z) \in C(\mu,\psi)\} \\ K_\alpha^m(\mu,\zeta;\psi\varphi) &= \{f\in A : D_\alpha^m f(z) \in K(\mu,\zeta;\varphi,\psi)\} \end{aligned}$$

Next, we give the preliminary results that we shall employ in the proof of the main results of this paper.

Lemma 1.1. [3, 2, 6, 8] *Let ϕ be convex, univalent in U with $\phi(0)=1$ and Re$\{k\phi(z)+\gamma\}\geq 0, k,\gamma\in C$. If p is analytic in U with $p(0)=1$, then*

$$p(z) + \frac{zp'(z)}{kp(z)+\gamma} \prec \phi(z), z\in U, \textit{ implies } p(z)\prec\phi(z),\ z\in U$$

Lemma 1.2. [5, 8] *Let ϕ be convex, univalent in U and w be analytic in U with Re$(w(z))\geq 0$. If p is analytic in U with $p(0)=\phi(0)$, then*

$$p(z) + w(z)zp'(z) \prec \phi(z), z\in U \textit{ implies } p(z)\prec\phi(z),\ z\in U\,.$$

In what follows, we give some inclusion properties of the operator $D_\alpha^m f$, using the principle of subordination.

2. Main Results

Theorem 2.1. *Let f belongs to the analytic function of the form (1.1) and let $\psi\in\mathcal{H}$ with Re$((1-\mu)\psi(z)+\mu+\frac{1-\lambda}{\lambda})>0$. Then,*

$$S_\alpha^{m+1}(\mu,\psi)\subset S_\alpha^m(\mu,\psi)\,.$$

Proof. Let $f(z)$ belongs to the class $S_\alpha^{m+1}(z)$ and let

$$p(z) = \frac{1}{1-\mu}\left(\frac{z\,(D_\alpha^m f(z))'}{D_\alpha^m f(z)} - \mu \right). \tag{2.1}$$

Applying (1.6) in (2.1), we obtain:

$$\frac{D_\alpha^{m+1} f(z) - D_\alpha^m f(z) + \lambda D_\alpha^m f(z)}{\lambda D_\alpha^m f(z)} = (1-\mu)p(z)+\mu\,.$$

Now we have:

$$\frac{1}{\lambda}\frac{D_\alpha^{m+1} f(z)}{D_\alpha^m f(z)} = (1-\mu)p(z)+\mu+\frac{1-\lambda}{\lambda}\,, \tag{2.2}$$

and from (2.2), we obtain:

$$\frac{(D_\alpha^{m+1} f(z))'}{D_\alpha^{m+1} f(z)} = \frac{(D_\alpha^m f(z))'}{D_\alpha^m f(z)} + \frac{(1-\mu)p'(z)}{(1-\mu)p(z)+\mu+\frac{1-\lambda}{\lambda}}\,. \tag{2.3}$$

On the other hand:

$$\frac{(D_\alpha^m f(z))'}{D_\alpha^m f(z)} = \frac{(1-\mu)p(z)+\mu}{z}. \tag{2.4}$$

Using (2.3) and (2.4), we obtain:

$$\frac{1}{1-\mu}\left(\frac{z\left(D_\alpha^{m+1} f(z)\right)'}{D_\alpha^{m+1} f(z)} - \mu\right) = p(z) + \frac{zp'(z)}{(1-\mu)p(z)+\mu+\frac{1-\lambda}{\lambda}}\,. \tag{2.5}$$

Applying Lemma 1.1 to (2.5) shows that:

$$p(z) \prec \psi(z), i.e. f \in D_\alpha^{m+1} f(z)\,.$$

Thus,

$$S_\alpha^{m+1}(\mu,\psi) \subset S_\alpha^m(\mu,\psi)\,,$$

proving the theorem. □

Theorem 2.2. *Let f belongs to the analytic function of the form (1.1) and let $\psi \in \mathcal{H}$ with Re$\{((1-\mu)\psi(z)+\mu+\frac{1-\lambda}{\lambda})\} > 0$. Then,*

$$C_\alpha^{m+1}(\mu,\psi) \subset C_\alpha^m(\mu,\psi)\,.$$

Proof. From Remark 1.1, we have:

$$f \in C_\alpha^{m+1}(\mu,\psi) \Leftrightarrow zf' \in S_\alpha^{m+1}(\mu,\psi)\,,$$

and from Theorem 2.1, we have:

$$\begin{aligned} f \in C_\alpha^{m+1}(\mu,\psi)(\mu,\psi) \quad &\Leftrightarrow \quad zf' \in S_\alpha^{m+1} \subset S_\alpha^m(\mu,\psi) \\ &\Rightarrow \quad zf' \in S_\alpha^m(\mu,\psi) \\ &\Rightarrow \quad f \in C_\alpha^m(\mu,\psi)(\mu,\psi)\,. \end{aligned}$$

Thus,

$$C_\alpha^{m+1}(\mu,\psi) \subset C_\alpha^m(\mu,\psi)\,.$$

□

The function $\psi(z) = \dfrac{1-Az}{1+Bz}$ is analytic and satisfies $\psi(0) = 1$. Thus, we have the following corollaries:

Corollary 2.1. *Let $f \in A$ and $\psi = \dfrac{1+Az}{1-Bz}, -1 \le B \le A \le 1$ in Theorem 2.1. Then:*

$$S_\alpha^{m+1}(\mu,A,B) \subset S_\alpha^m(\mu,A,B)\,.$$

Corollary 2.2. *Let $f \in A$ and $\psi = \dfrac{1+Az}{1-Bz}, -1 \le B \le A \le 1$ in Theorem 2.1. Then:*

$$K_\alpha^{m+1}(\mu,A,B) \subset K_\alpha^m(\mu,A,B)\,.$$

Theorem 2.3. *Let $f \in A$ and let $\psi, \varphi \in \mathcal{H}$ with Re$\{(1-\mu)\psi(z)+\mu+\frac{1-\lambda}{\lambda}\} > 0$. Then:*

$$K_\alpha^{m+1}(\mu,\zeta;\psi,\varphi) \subset K_\alpha^m(\mu,\zeta;\psi,\varphi)\,.$$

Proof. Let $f \in K_{\alpha}^{m+1}(\mu, \zeta; \psi, \varphi)$, then there exist a function $g \in S_{\alpha}^{m+1}(\mu, \psi)$ such that:

$$Re\left\{\frac{z(D_{\alpha}^{m+1}f(z))'}{D_{\alpha}^{m+1}g(z)}\right\} > \zeta,\ z \in U\,.$$

That is, we should have:

$$\frac{1}{1-\zeta}\left(\frac{z(D_{\alpha}^{m+1}f(z))'}{D_{\alpha}^{m+1}g(z)} - \zeta\right) \prec \varphi,\ z \in U\,.$$

Let

$$p(z) = \frac{1}{1-\zeta}\left(\frac{z(D_{\alpha}^{m}f(z))'}{D_{\alpha}^{m}g(z)} - \zeta\right). \tag{2.6}$$

From (1.6), we have:

$$z(D_{\alpha}^{m}f(z))' = \frac{D_{\alpha}^{m+1}f(z) - (1-\lambda)D_{\alpha}^{m}f(z)}{\lambda}\,.$$

Now, from (2.6) we have:

$$\frac{1}{\lambda}D_{\alpha}^{m+1}f(z) = \left(\frac{1-\lambda}{\lambda}\right)(D_{\alpha}^{m}f(z)) + ((1-\zeta)p(z)+\zeta)\,D_{\alpha}^{m}g(z)\,.$$

This implies that

$$\begin{aligned}\frac{1}{\lambda}z\left(D_{\alpha}^{m+1}f(z)\right)' = \left(\frac{1-\lambda}{\lambda}\right)z\left(D_{\alpha}^{m}f(z)\right)' + ((1-\zeta)zp'(z))\left(D_{\alpha}^{m}g(z)\right)\\ + ((1-\zeta)p(z)+\zeta)z[D_{\alpha}^{m}g(z)]'.\end{aligned} \tag{2.7}$$

Also, by Theorem 2.1, $g \in S_{\alpha}^{m+1}(\mu, \psi) \Rightarrow g \in S_{\alpha}^{m}(\mu, \psi)$.

Now, let

$$q(z) = \frac{1}{1-\mu}\left(\frac{z(D_{\alpha}^{m}g(z))'}{D_{\alpha}^{m}g(z)} - \mu\right)\,. \tag{2.8}$$

Using (1.6) in (2.8), we obtain:

$$\frac{1}{\lambda}\frac{D_{\alpha}^{m+1}g(z))}{D_{\alpha}^{m}g(z))} = (1-\mu)q(z) + \mu + \frac{1-\lambda}{\lambda}\,, \tag{2.9}$$

and further, from (2.7) and (2.9), we obtain:

$$\frac{z\left(D_{\alpha}^{m+1}f(z)\right)'}{D_{\alpha}^{m+1}g(z)} = (1-\zeta)p(z) + \zeta + \frac{(1-\zeta)zp'(z)}{(1-\mu)q(z) + \mu + \left(\frac{1-\lambda}{\lambda}\right)}\,. \tag{2.10}$$

Algebraic manipulation in (2.10) gives:

$$\frac{1}{1-\zeta}\left(\frac{z\left(D_{\alpha}^{m+1}f(z)\right)'}{D_{\alpha}^{m+1}g(z)} - \zeta\right) = p(z) + \frac{zp'(z)}{(1-\mu)q(z) + \mu + \left(\frac{1-\lambda}{\lambda}\right)}\,.$$

Thus, making

$$\frac{1}{1-\mu)q(z) + \mu + \left(\frac{1-\lambda}{\lambda}\right)} = w(z)\,.$$

and apply Lemma 1.2, we have that $p(z) \prec \varphi(z)$ which implies that $f \in K_{\alpha}^{m+1}(\mu, \zeta; \psi, \varphi)$, proving the theorem. □

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