

## A NOTE ON CESÀRO OF ORDER $K$ SUMMABILITY OF DILUTED $P$ -FABER SERIES

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**ABSTRACT.** In this study, some relationships between the overconvergence of  $p$ -Faber series and the existence of an elongation of the sequence of the partial sums of the  $p$ -Faber series, whose Cesàro means of order  $k$  is convergent, are investigated.

### 1. INTRODUCTION

Let  $G \subset \mathbb{C}$  be a Jordan domain, that is, its boundary  $\partial G := L$  is a Jordan curve, and  $\Phi$  be the conformal mapping of the domain  $\Omega := \mathbb{C}_\infty \setminus \overline{G}$  where  $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ , onto  $\Delta := \{w : |w| > 1\}$  with the usual normalization at infinity:

$$(1.1) \quad w = \Phi(z) = \alpha z + a_0 + \frac{a_1}{z} + \cdots, \quad \alpha > 0, \quad z \in \Omega.$$

Let  $\Psi := \Phi^{-1} : \Delta \rightarrow \Omega$  denote the inverse conformal map. Then,

$$z = \Psi(w) = \beta w + b_0 + \frac{b_1}{w} + \cdots, \quad |w| > 1,$$

where  $\beta = 1/\alpha$  gives the capacity  $\text{cap}(L)$  of  $L$ .

Let  $0 < p < \infty$ . The  $p$ -Faber polynomials  $F_{n,p}$  associated the domain  $G$  are defined as the polynomial part of the Laurent expansion of

$$\Phi^n(z) (\Phi'(z))^{1/p}, \quad n = 0, 1, 2, \dots,$$

in a neighborhood of the infinity. Therefore, from (1.1), we have the  $p$ -Faber polynomial of degree  $n$

$$F_{n,p}(z) := \alpha^{n+1/p} z^n + \dots$$

Also, the  $p$ -Faber polynomials associated with  $G$  can be defined by the generating function

$$g(w) := \frac{(\Psi'(w))^{1-1/p}}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_{n,p}(z)}{w^{n+1}}, \quad z \in G, \quad |w| > 1.$$

If  $p$  tends to infinity the  $p$ -Faber polynomials coincide with the usual Faber polynomials  $F_n$ .

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For every  $R > 1$ , the level curve  $L_R := \{z : |\Phi(z)| = R\}$  is a Jordan curve. We denote the interior of  $L_R$  by  $G_R$ , and the exterior of  $L_R$  by  $\Omega_R$ . A  $p$ -Faber series

$$(1.2) \quad f(z) = \sum_{n=0}^{\infty} a_n F_{n,p}(z) \quad \text{with} \quad \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}, \quad \text{and } R > 1$$

is uniformly convergent to the holomorphic function  $f$  on each compact subset of  $G_R$  and divergent for all  $z \in \Omega_R$ . On the other hand, given a function  $f$  which is holomorphic in  $G_R$  with  $R > 1$ , then the representation (1.2) holds with the  $p$ -Faber coefficients

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_s} \frac{f(\Psi(w))(\Psi'(w))^{1/p}}{w^{n+1}} dw, \quad 1 < s < R.$$

Let us denote the partial sums of (1.2) by

$$(1.3) \quad S_n(z) = \sum_{k=0}^n a_k F_{k,p}(z).$$

From the above results, it is easily seen that the sequence  $\{S_n\}_{n=0}^{\infty}$  converges compactly in  $G_R$ , and for every  $z \in \Omega_R$ , we have

$$(1.4) \quad \limsup_{n \rightarrow \infty} |S_n(z)|^{\frac{1}{n}} = \frac{|\Phi(z)|}{R} > 1.$$

For more details on  $p$ -Faber polynomials, we refer the reader to ([4], [12] and [13]).

It is known that it can be constructed such a Faber series with the property that a certain subsequence of  $\{S_n\}$  converges to  $f$  on the sets in  $\Omega_R$ , where the function  $f$  is regular. This is the phenomenon of overconvergence. A Faber series in (1.2) is called *compactly overconvergent* if there exists an open set  $U \subset \Omega_R$  and a monotone increasing sequence of positive integers  $\{n_k\}$  such that  $\{S_{n_k}\}$  converges on every compact subset of  $U$ . Examples of overconvergent Faber series are given in ([5], [8], [14]).

Let  $m = \{m_n\}_{n=0}^{\infty}$  be an arbitrary sequence of positive integers. It is called that a sequence  $\{s_n\}_{n=0}^{\infty}$  is being *elongated* with respect to the sequence  $m$  if for each  $n$  the term  $s_n$  is listed  $m_n$ -times, i.e. if it is written by the following way:

$$(1.5) \quad (\underbrace{s_0, s_0, \dots, s_0}_{m_0\text{-times}}, \underbrace{s_1, s_1, \dots, s_1}_{m_1\text{-times}}, \dots, \underbrace{s_n, s_n, \dots, s_n}_{m_n\text{-times}}, \dots).$$

The sequence (1.5) is called  $m$ -elongation of  $\{s_n\}$ . It is obvious that the sequence  $\{s_n\}$  is convergent if and only if any  $m$ -elongation of  $\{s_n\}$  is convergent with the same limit.

Let  $A = (a_{n,k})(n, k = 0, 1, 2, 3, \dots)$  be an infinite matrix of real (or complex) numbers. A sequence  $\{s_n\}$  of real (or complex) numbers is said to be *summable* to a number  $S$  by the method  $A = (a_{n,k})$ , shortly *A-summable* to  $S$ , if the limit relation

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k} S_k = S$$

holds, and it is written as  $A - \lim_{n \rightarrow \infty} S_n = S$ . The matrix  $A = (a_{n,k})$  is called regular if it transforms convergent sequences to convergent sequences with the same limit. It is well known that the matrix  $A = (a_{n,k})$  is regular if and only if it satisfies the following Silverman-Toeplitz conditions (see [7, p. 142], also [9]):

- (i) There exists a constant  $M > 0$  such that  $\sum_{k=1}^{\infty} |a_{n,k}| \leq M$ , for each  $n = 1, 2, 3, \dots$ ;

(ii) For each positive integer  $k$ ,

$$(1.6) \quad \lim_{n \rightarrow \infty} a_{n,k} = 0;$$

$$(iii) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{n,k} = 1.$$

Here, we mainly deal with a special regular method, namely with the Cesàro means  $(C, k)$  of order  $k \in \mathbb{N}$  which transforms the given sequence  $\{s_n\}$  into the sequence

$$\sigma_n^{(k)} := \frac{1}{\binom{n+k}{n}} \sum_{i=0}^n \binom{n-i+k-1}{n-i} s_i.$$

For  $k = 1$  we especially obtain the arithmetical means

$$\sigma_n := \sigma_n^{(1)} = \frac{1}{n+1} \sum_{i=0}^n s_i.$$

If  $\left\{ \sigma_n^{(k)} \right\}_{n=0}^{\infty}$ ,  $k \in \mathbb{N}$ , converges, then  $\left\{ \frac{s_n}{n^k} \right\}$  tends to 0, as  $n \rightarrow \infty$  [3].

In this paper, our aim is to investigate the equivalence between the overconvergence of the  $p$ -Faber series (1.2) and the existence of an elongation of  $\{S_n\}$  the sequence of the partial sums of (1.2), whose  $(C, k)$  transformation converges. The obtained results can be used in the theory of universal series.

## 2. MAIN RESULTS

Let  $f$  be a  $p$ -Faber series as in (1.2) with partial sums  $\{S_n\}$  as in (1.3), and let  $A = (a_{n,k})$  be any regular matrix transformation. The matrix  $A$  transforms the sequence  $\{S_n(z)\}$  defined by (1.3) into the sequence  $\{A_n(z)\}$ , where

$$A_n(z) = \sum_{k=0}^{\infty} a_{n,k} S_k(z).$$

By the regularity of the matrix  $A$ , the sequence  $\{A_n\}$  converges compactly to  $f$  on  $G_R$ . Especially, for the matrix transformation  $(C, k)$ ,  $k \in \mathbb{N}$ , of the sequence of the partial sums  $\{S_n\}$ ,

$$\sigma_n^{(k)}(z) := \frac{1}{\binom{n+k}{n}} \sum_{i=0}^n \binom{n-i+k-1}{n-i} S_i(z),$$

it is well known that  $\{\sigma_n^{(k)}\}$  converges to  $f$  on every compact subsets of  $G_R$  and diverges for all  $z \in \Omega_R$ . But, it is also known that, there may be convergent subsequences of  $\{\sigma_n^{(k)}(z)\}$  in  $\Omega_R$  (see e.g. [2], [10]).

The following two theorems extend the results of Luh and Nieß in [6] on the overconvergence of Faber series.

**Theorem 2.1.** *Let  $A = (a_{n,k})$  be any regular matrix transformation and  $U \subset \Omega_R$  be an open set. If the  $p$ -Faber series (1.2) is compactly overconvergent to a limit function  $F$  in  $U$ , then there exists an elongation of the sequence (1.3) which is compactly  $A$ -summable on  $U$  to the function  $F$ .*

*Proof.* It is known that there exists an increasing sequence of compact sets  $\{K_n\}$  with the property that if  $K \subset U$  is a compact set, then there exists a positive integer  $n_0 = n_0(K)$  such that  $K \subset K_{n_0}$ , since  $U$  is an open set in complex plane [11, p. 267].

Assume that there exists a monotone increasing sequence  $\{n_k\}$  of positive integers such that  $\{S_{n_k}\}$  is compactly convergent in  $U$  to a limit function  $F$ , that is,

$$\lim_{k \rightarrow \infty} \sup_{z \in K_m} |S_{n_k}(z) - F(z)| = 0 \quad \text{for all } m \in \mathbb{N}_0.$$

Let  $\{m_k\}$  be a sequence of positive integers which will be determined later. We now elongate the sequence  $\{S_n\}$  to the sequence  $\{\tilde{S}_n\}$  where the terms  $S_{n_k}$  for  $k \geq 1$  are listed  $m_k + 1$  times while the others remain unchanged, i.e.

$$\{\tilde{S}_n\} = (S_0, S_1, \dots, S_{n_0-1}, \underbrace{S_{n_0}, S_{n_0}, \dots, S_{n_0}}_{m_0+1\text{-times}}, S_{n_0+1}, \dots, S_{n_k-1}, \underbrace{S_{n_k}, S_{n_k}, \dots, S_{n_k}}_{m_k+1\text{-times}}, \dots).$$

If we denote the  $n$ th term of the transformation of the elongated sequence  $\{\tilde{S}_n\}$  under the matrix  $A$  by  $A_n$ , then

$$\begin{aligned} A_n(z) &:= \sum_{i=0}^{\infty} a_{n,i} \tilde{S}_i(z) = \sum_{i=0}^{n_0} a_{n,i} S_i(z) + \sum_{i=n_0+1}^{n_0+m_0} a_{n,i} S_{n_0}(z) \\ &\quad + \sum_{i=n_0+1}^{n_1} a_{n,i+m_0} S_i(z) + \sum_{i=n_1+1}^{n_1+m_1} a_{n,i+m_0} S_{n_1}(z) + \dots \\ &\quad + \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}} S_i(z) + \sum_{i=n_k+1}^{n_k+m_k} a_{n,i+M_{k-1}} S_{n_k}(z) + \dots \\ &= \sum_{k=0}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}} (S_i(z) - S_{n_k}(z)) + \sum_{i=0}^{\infty} a_{n,i} g_i(z), \end{aligned}$$

where  $M_k = \sum_{i=0}^k m_i$ ,  $k \in \mathbb{N}$ ,  $n_{-1} := M_{-1} := 0$  and  $\{g_i\}$  is an elongation of  $\{S_{n_i}\}$ , that is

$$\{g_i\} = (\underbrace{S_{n_0}, S_{n_0}, \dots, S_{n_0}}_{n_0+m_0\text{-times}}, \underbrace{S_{n_1}, S_{n_1}, \dots, S_{n_1}}_{n_1-n_0+m_1\text{-times}}, \dots, \underbrace{S_{n_k}, S_{n_k}, \dots, S_{n_k}}_{n_k-n_{k-1}+m_k\text{-times}}, \dots).$$

Since the subsequence  $\{S_{n_k}\}$  of  $\{S_n\}$  converges compactly to the function  $F$  on  $U$ , then the  $r$ -elongation of  $\{S_{n_k}\}$ ,  $\{g_k\}$  is so, where  $r = \{r_k\} = \{n_k - n_{k-1} + m_k\}$ . By the regularity of the matrix  $A = (a_{n,k})$ , the sequence  $\{h_n\}$  defined by

$$h_n(z) := \sum_{i=0}^{\infty} a_{n,i} g_i(z)$$

converges compactly to the function  $F$  on  $U$ . If we prove that

$$\sum_{k=0}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}} (S_i(z) - S_{n_k}(z))$$

tends to zero on each compact subset  $K_k$ , when  $n \rightarrow \infty$ , the assertion follows.

Let  $\{\gamma_n\}$  be a sequence of nonnegative numbers which tends to zero. By the Silverman-Toeplitz conditions for the regularity of matrix transformations, we can choose the natural

numbers  $m_k, k \in \mathbb{N}_0$ , such that the inequality

$$(2.1) \quad \sum_{i=n_{k-1}+1}^{n_k} |a_{n,i+M_{k-1}}| \leq \frac{\gamma_n}{N_k 2^k},$$

holds, where

$$N_k = \max_{n_{k-1}+1 \leq i \leq n_k} \sup_{z \in K_k} |S_i(z)|.$$

Consequently, by the inequality (2.1) it is obtained that

$$\begin{aligned} \left| \sum_{k=0}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}} (S_i(z) - S_{n_k}(z)) \right| &\leq \sum_{k=0}^{\infty} 2N_k \sum_{i=n_{k-1}+1}^{n_k} |a_{n,i+M_{k-1}}| \\ &\leq \sum_{i=0}^{n_0} 2N_0 |a_{n,i}| + \sum_{k=1}^{\infty} \frac{\gamma_n}{2^{k-1}} \\ &= \sum_{i=0}^{n_0} 2N_0 |a_{n,i}| + \gamma_n. \end{aligned}$$

Using the fact that  $\gamma_n$  tends to zero, when  $n \rightarrow \infty$  and (1.6), we have proven the theorem.  $\square$

The other result gives a partially converse of the above-mentioned theorem.

**Theorem 2.2.** *Let  $U \subset \Omega_R$  be an open set and  $k \in \mathbb{N}$ . If there exists an elongation of the sequence (1.3) such that its sequence of  $(C, k)$ -means converges compactly to a function  $F$  in  $U$ , then, the  $p$ -Faber series (1.2) is compactly overconvergent to the function  $F$  in  $U$ .*

*Proof.* Suppose that there exists an  $m = \{m_n\}_{n=0}^{\infty}$  elongation of the sequence  $\{S_n(z)\}$  whose the  $(C, k)$ -means converges compactly in  $U$ , that is, the sequence  $\{\tilde{\sigma}_n^{(k)}(z)\}$  defined by

$$\tilde{\sigma}_n^{(k)}(z) := \frac{1}{\binom{n+k}{n}} \sum_{i=0}^n \binom{n-i+k-1}{n-i} \tilde{S}_i(z),$$

where

$$\{\tilde{S}_k(z)\} = (\underbrace{S_0, S_0, \dots, S_0}_{m_0\text{-times}}, \underbrace{S_1, S_1, \dots, S_1}_{m_1\text{-times}}, \dots, \underbrace{S_n, S_n, \dots, S_n}_{m_n\text{-times}}, \dots),$$

is convergent uniformly on every compact subset of  $U$ .

Let us consider a special subsequence of the sequence  $\{\tilde{\sigma}_n^{(k)}(z)\}$  which has the form

$$\sigma_n(z) := \tilde{\sigma}_{M_n}^{(k)}(z) = \frac{1}{\binom{M_n+k}{M_n}} \sum_{i=0}^n \left( \sum_{j=M_{i-1}+1}^{M_i} \binom{M_n-j+k-1}{M_n-j} \right) S_i(z)$$

where

$$M_k = \sum_{i=0}^k m_i, \quad M_{-1} := m_{-1} = 0.$$

If we set

$$\beta_j := \sum_{i=M_{j-1}+1}^{M_j} \binom{M_n-j+k-1}{M_n-j} \quad j = 0, 1, 2, \dots \quad \text{and} \quad B_n := \binom{M_n+k}{M_n},$$

then, we have  $B_n = \sum_{j=0}^n \beta_j$ , and therefore the relation

$$S_n(z) = \frac{B_n}{\beta_n} \{ \sigma_n(z) - \sigma_{n-1}(z) \} + \sigma_{n-1}(z).$$

holds. If we show that the sequence  $\left\{ \frac{B_n}{\beta_n} \right\}$  has a bounded subsequence, then the desired result will be clear. Let us now show that.

For any  $z \in U$ , the  $(C, k)$ -means of  $\{\tilde{S}_n(z)\}$  is convergent; hence we obtain, as already mentioned in section 1, that  $\left\{ \frac{\tilde{S}_n(z)}{n^k} \right\}$  is bounded. Since  $\tilde{S}_{B_n}(z) = S_n(z)$ , there is a constant  $c > 0$  with

$$(2.2) \quad |S_n(z)| \leq c B_n^k$$

for every  $n \in \mathbb{N}_0$ . Assume that  $\left\{ \frac{B_n}{\beta_n} \right\}$  does not have a bounded subsequence, i.e.  $\frac{B_n}{\beta_n} \rightarrow \infty$ , as  $n \rightarrow \infty$ . This implies

$$\frac{B_{n-1}}{B_n} = \frac{B_n - \beta_n}{B_n} = 1 - \frac{\beta_n}{B_n} \rightarrow 1, \quad \text{as } n \rightarrow \infty,$$

thus

$$(B_n)^{\frac{1}{n}} \rightarrow 1, \quad \text{as } n \rightarrow \infty.$$

From (2.2), we get

$$\limsup_{n \rightarrow \infty} |S_n(z)|^{\frac{1}{n}} \leq 1,$$

which contradicts (1.4). This completes the proof of the theorem.  $\square$

Combining these two theorems we get the following:

**Corollary 2.1.** *Let  $U \subset \Omega_R$  be an open set and  $k \in \mathbb{N}$ . There exists an elongation of the sequence (1.3) that  $(C, k)$ -convergent compactly to a function  $F$  in  $U$  iff the  $p$ -Faber series (1.2) is compactly overconvergent to the function  $F$  in  $U$ .*

**Remark 2.1.** *For the case  $k = 1$  and  $G$  is an open disc, we obtain the results of Drobot [1] and Gharibyan and Luh [2] from Corollary 2.1 when  $p$  tends to infinity.*

**Remark 2.2.** *For the case  $k = 1$ , we obtain the result of Luh and Nieß in [6] from Corollary 2.1 when  $p$  tends to infinity.*

**Remark 2.3.** *In [15], it was obtained the similar result to Corollary 2.1 with the power series instead of  $p$ -Faber series and regular Riesz means instead of  $(C, k)$ -means.*

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