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PARTIAL SUM FOR UNIVALENT MEROMORPHIC FUNCTIONS OF COMPLEX ORDER BASED ON BESSEL FUNCTION

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ABSTRACT. By considering the Bessel function, a new class of meromorphically univalent functions is defined. The coefficient estimates, extreme points, radii properties and partial sum concept on this class are obtained.

1. INTRODUCTION

Let $\sum$ donote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} \alpha_k z^{k-1} \tag{1.1}$$

which are analytic in the puntured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Gasper and Rahman [3] defined the q-derivative of $f(z)$ introduced by (1.1) as follow:

$$D_q f(z) := \frac{f_q(z) - f(z)}{(q-1)z}, \qquad z \in \Delta^*, 0 < q < 1, \tag{1.2}$$

where $f_q(z) = f(q.z)$.
From (1.2) and (1.1) we get

$$D_q f(z) = -\frac{1}{q^{z^2}} + \sum_{k=1}^{+\infty} [k-1]_q a_k z^{k-2}, \qquad z \in \Delta^*,$$

where,

$$[k-1]_q := \frac{1-q^{k-1}}{1-q} = 1 + q + ... + q^{k-2}.$$

As $q \to 1^-$, we conclude that $[k-1]_q \to k-1$ and so $\lim_{q \to 1^-} D_q f(z) = f'(z)$. If $\alpha \in \mathbb{C}$, then the q-shifted factorials are defined by

$$(\alpha; q) := 1, \qquad (\alpha; q)_n := \prod_{k=0}^{n-1} (1 - \alpha q^k), \qquad n \in \mathbb{N}.$$

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If $|q| < 1$, the above definition remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty}(1 - \alpha q^j).$$

According to the q-analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1-q)^n}{\Gamma_q(\alpha)}, \qquad n > 0,$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \qquad 0 < q < 1.$$

Also, we note that,

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n,$$

where,

$$(\alpha)_n = \left\{ \begin{array}{lr} 1 \qquad\qquad\qquad\qquad\quad , & n = 0 \\ \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1), & n \in \mathbb{N}. \end{array} \right\}.$$

The q-analogue of Bessel function is defined by

$$\mathcal{J}_\nu(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{h=0}^{+\infty} \frac{(-1)^k}{(q;q)_k (q^{\nu+1};q)_k} (\frac{z}{2})^{2k+\nu}, \ 0 < q < 1.$$

Mostafa et al. in [4] introduced

$$\begin{aligned} \mathcal{L}_\nu(z;q) &:= \frac{2^\nu (q;q)_\infty}{(q^{\nu+1};q)_\infty (1-q)^\nu z^{\nu/2+1}} \mathcal{J}_\nu(z^{1/2}(1-q);q) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k (1-q)^{2k}}{4^k (q;q)_k (q^{\nu+1};q)_k} z^{k-1}, \qquad z \in \Delta^*. \end{aligned}$$

In the same paper by using the familiar Hadamard product (convolution), they introduced and studied the linear operator

$$\mathcal{L}_{q,\nu} : \sum \to \sum$$

defined by

$$\begin{aligned} (\mathcal{L}_{q,\nu} f)(z) &:= \mathcal{L}_\nu(z;q) * f(z) \\ &= \frac{1}{z} + \sum_{k=1}^{+\infty} \frac{(-1)^k (1-q)^{2k}}{4^k (q;q)_k (q^{\nu+1};q)_k} a_k z^{k-1}, \ z \in \Delta^*, \end{aligned} \tag{1.3}$$

where $f \in \sum$ has the form (1.1).

As $q \to 1^-$, the operator $\mathcal{L}_{q,\nu}$ reduces to operator $\mathcal{L}_\nu$ which was studied by Aoof et al. [1] (see also [2]). It is convenient to write $(\mathcal{L}_{q,\nu} f)(z) = \mathcal{L}(f)$.

A function $f(z)$ belonging to the class $\sum$ is in the class $\sum(b, \mathcal{L})$ if it satisfies the condition

$$\Big| \frac{z\mathcal{L}(f)}{z^2[\mathcal{L}(f)]' + (\frac{1-b}{2}) z^3 [\mathcal{L}(f)]''} + \frac{1}{b} \Big| < 1, \tag{1.4}$$

where $b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\mathcal{L}(f) = (\mathcal{L}_{q,\nu} f)(z)$ is defined by (1.3).

2. Coefficient bounds and extreme points

In this section we obtain coefficient inequality and extreme points for functions in $\sum(b,\mathcal{L})$.

Theorem 2.1. *The function* $f(z)$ *of the form* (1.1) *belongs to* $\sum(b,\mathcal{L})$ *if and only if,*

$$\sum_{k=1}^{+\infty} \frac{\frac{1}{2}(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\left[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\right]a_k < b^2. \tag{2.1}$$

The result is sharp for the function $F(z)$ *given by*

$$F(z)=\frac{1}{z}+\frac{b^2(4^k)(q;q)_k(q^{\nu+1};q)_k}{\frac{1}{2}(-1)^k(1-q)^2k\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big]}z^{k-1},$$

k=1,2,... .

Proof. Let $f(z)\in\sum(b,\mathcal{L})$. Then the inequality (2.1) or equivalently

$$\left|\frac{bz\mathcal{L}(f)+z^2[\mathcal{L}(f)]'+\left(\frac{1-b}{2}\right)z^3[\mathcal{L}(f)]''}{b\{z^2[\mathcal{L}(f)]'+\left(\frac{1-b}{2}\right)z^3[\mathcal{L}(f)]''\}}\right|<1$$

holds true. Therefore, by making use of (1.3) and (1.4) we have,

$$\left|\frac{\sum_{k=1}^{+\infty}\frac{(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\Big[(1-b)\big(\frac{1}{2}(k-1)(k-2)-1\big)+k\Big]a_kz^k}{-b^2+\sum_{k=1}^{+\infty}\frac{(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\Big[b(k-1)\Big(1+\left(\frac{1-b}{2}\right)(k-2)\Big)\Big]a_kz^k}\right|<1$$

Since $Re\{z\}\le|z|$ for all z,

$$Re\left\{\frac{\sum_{k=1}^{+\infty}\frac{(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\Big[(1-b)\big(\frac{1}{2}(k-1)(k-2)-1\big)+k\Big]a_kz^k}{-b^2+\sum_{k=1}^{+\infty}\frac{(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\Big[b(k-1)\Big(1+\left(\frac{1-b}{2}\right)(k-2)\Big)\Big]a_kz^k}\right\}<1.$$

By letting $z\to1^-$ through real values, we get

$$\sum_{k=1}^{+\infty}\frac{\frac{1}{2}(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\Big[(k-1)(b+1)\big[(k-2)(1-b)+1\big]+b\Big]a_k<b^2.$$

Conversely, let (2.1) holds true. If we let $z\in\partial\Delta^*$, where $\partial\Delta^*$ denotes the boundary of Δ^*, then we have:

$$\left|\frac{bz\mathcal{L}(f)+z^2[\mathcal{L}(f)]'+\frac{(1-b)}{2}z^3[\mathcal{L}(f)]''}{b\Big\{z^2[\mathcal{L}(f)]'+\frac{(1-b)}{2}z^3[\mathcal{L}(f)]''\Big\}}\right|$$
$$\le\frac{\sum_{k=1}^{+\infty}\frac{(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\Big[(1-b)\big(\frac{1}{2}(k-1)(k-2)-1\big)+k\Big]|a_k|}{b^2-\sum_{k=1}^{+\infty}\frac{(-1)^k(1-q)^{2k}}{4^k(q;q)_k(q^{\nu+1};q)_k}\Big[b(k-1)\big(1+(\frac{1-b}{2})(k-2)\big)\Big]|a_k|}.$$

Thus, by the maxiamum modulus theorem, we conclude $f(z)\in\sum(b,\mathcal{L})$. □

Remark 2.1. *Theorem 2.1 shows that if* $f(z)\in\sum(b,\mathcal{L})$*, then*

$$|a_k|\le\frac{4b^2(q;q)_1(q^{\nu+1};q)_1}{-\frac{1}{2}(1-q)^2b},\qquad k=1,2,...,$$

or equivalently,

$$|a_k| \leq \frac{-8b(1-q^{\nu+1})}{(1-q)}, \qquad k=1,2,\dots . \tag{2.2}$$

Next we obtain extreme points for the class $\sum(b,\mathcal{L})$.

Theorem 2.2. *The function $f(z)$ of the form (1.1) belongs to $\sum(b,\mathcal{L})$ if and only if, it can be expressed by*

$$f(z)=\sum_{k=0}^{\infty}\lambda_k f_k(z), \qquad \lambda_k \geq 0, k=0,1,...,$$

where $f_0(z)=\dfrac{1}{z}$, $\sum_{k=0}^{\infty}\lambda_k=1$ and

$$f_k(z)=\frac{1}{z}+\frac{2b^2 4^k (q;q)_k (q^{\nu+1};q)_k}{(-1)^k(1-q)^{2k}\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big]a_k}z^k,$$

$k=1,2,...,\sum_{k=0}^{\infty}\lambda_k=1.$

Proof. Let

$$\begin{aligned} f(z)=&\sum_{k=0}^{+\infty}\lambda_k f_k(z)=\lambda_0 f_0(z)+\\ &\sum_{k=1}^{+\infty}\lambda_k\left\{\frac{1}{z}+\frac{2b^2 4^k (q;q)_k (q^{\nu+1};q)_k}{(-1)^k(1-q)^{2k}\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big]a_k}z^k\right\}\\ &=\frac{1}{z}+\sum_{k=1}^{+\infty}\frac{2b^2 4^k (q;q)_k (q^{\nu+1};q)_k}{(-1)^k(1-q)^{2k}\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big]a_k}\lambda_k z^k. \end{aligned}$$

Now, by using Theorem 2.1 we conclude that $f(z)\in\sum(b,\mathcal{L})$. Conversely, if $f(z)$ given by (1.1) belong to $\sum(b,\mathcal{L})$, by letting $\lambda_0=1-\sum_{k=1}^{+\infty}\lambda_k$, where

$$\lambda_k=\frac{\frac{1}{2}(-1)^k(1-q)^{2k}\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big]}{2b^2 4^k (q;q)_k (q^{\nu+1};q)_k}a_k, \qquad k=1,2,...$$

we conclude the required result. □

3. Partial sum and Radii properties

In this last section we show property of partial sum and obtain radii of starlikeness and convexity.

Theorem 3.1. *Let $f(z)\in\sum$ and define*

$$S_1(z)=\frac{1}{z}, \qquad S_m(z)=\frac{1}{z}+\sum_{k=1}^{m-1}a_k z^{k-1}.$$

If $\sum_{k=1}^{+\infty} \sigma_k a_k \leq 1$*, where*

$$\sigma_k = \frac{\frac{1}{2}(-1)^k(1-q)^{2k}\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big]}{2b^2 4^k (q;q)_k (q^{\nu+1};q)_k} a_k \,,$$

then:

$$\text{(3.1)} \qquad Re\left\{\frac{f(z)}{S_m(z)}\right\} > 1 - \frac{1}{\sigma_m}, \qquad Re\left\{\frac{S_m(z)}{f(z)}\right\} > \frac{\sigma_m}{1+\sigma_m}.$$

Proof. Since $\sum_{k=1}^{+\infty} \sigma_k a_k \leq 1$, then by Theorem 2.1, $f(z) \in \sum(b, \mathcal{L})$. Also, by $k \geq 1$ it is easy to see that $\{\sigma_k\}$ is an increasing sequence. Therefore we get

$$\text{(3.2)} \qquad \sum_{k=1}^{m-1} a_k + \sigma_m \sum_{k=m}^{+\infty} a_k \leq 1.$$

Now, by putting

$$\sigma_m\Big[\frac{f(z)}{S_m(z)} - \big(1 - \frac{1}{\sigma_m}\big)\Big] = \mathcal{A}_m(z)\,,$$

and making use of (3.2) we get:

$$\begin{aligned} Re\left\{\frac{\mathcal{A}_m(z)-1}{\mathcal{A}_m(z)+1}\right\} &\leq \Big|\frac{\mathcal{A}_m(z)-1}{\mathcal{A}_m(z)+1}\Big| \\ &= \Big|\frac{\sigma_m f(z) - \sigma_m S_m(z)}{\sigma_m f(z) - \sigma_m S_m(z) + 2S_m(z)}\Big| \\ &= \Big|\frac{\sigma_m \sum_{k=m}^{+\infty} a_k z^{k-1}}{\sigma_m \sum_{k=m}^{+\infty} a_k z^{k-1} + 2\big(\frac{1}{z} + \sum_{k=1}^{m-1} a_k z^{k-1}}\Big| \\ &\leq \frac{\sigma_m \sum_{k=m}^{+\infty} |a_k|}{2 - \sum_{k=1}^{m-1} - \sigma_m \sum_{k=m}^{+\infty} |a_k|} \leq 1. \end{aligned}$$

By a simple calculation we conclude

$$Re\{\mathcal{A}_M(z)\} > 0, \qquad \text{therefore} \qquad Re\Big\{\frac{\mathcal{A}_m(z)}{\sigma_m}\Big\} > 0,$$

or equivalently $Re\left\{\frac{f(z)}{S_m(z)} - \big(1 - \frac{1}{\sigma_m}\big)\right\} > 0$. This gives the first inequality in (3.1).

For the second inquality we consider

$$\mathcal{B}_m(z) = (1+\sigma_m)\Big[\frac{S_m(z)}{f(z)} - \frac{\sigma_m}{1+\sigma_m}\Big],$$

and by using (3.2) we have $\Big|\frac{\mathcal{B}_m(z)-1}{\mathcal{B}_m(z)+1}\Big| \leq 1$. Hence $Re\{\mathcal{B}_m(z)\} > 0$, and therefore $\Big\{\frac{\mathcal{B}_m(z)-1}{1+\sigma_m}\Big\} > 0$ or equivalently $Re\Big\{\frac{S_m(z)}{f(z)} - \frac{\sigma_m}{1+\sigma_m}\Big\} > 0$. This shows the second inequality in (3.1). □

Theorem 3.2. *If* $f(z) \in \sum(b, \mathcal{L})$*, then*

(i) f *is meromerphically univalent starlike of order* η $(0 < \eta < 1)$ *in the disk* $|z| < R_1$*, where*

$$R_1 = \inf_k \Big\{\frac{(-1)^k(1-q)^{2k}[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b](1-\eta)}{2b^2 4^k (q;q)_k (q^{\nu+1};q)_k (k+1-\eta)}\Big\}^{\frac{1}{k-1}}$$

(ii) *f is meromerphically univalent convex of order $\xi(0<\xi<1)$ in $|z|<R_2$, where*

$$R_2=\inf_k\left\{\frac{(-1)^k(1-q)^{2k}[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b](1-\xi)}{2b^2 4^k(q;q)_k(q^{\nu+1};q)_k(k+1-\xi)}\right\}^{\frac{1}{k-1}}$$

Proof. (i) It is enough to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right|<1-\eta.$$

But

$$\left|\frac{zf'(z)}{f(z)}+1\right|=\left|\frac{\sum_{k=1}^{+\infty}ka_kz^{k-1}}{1+\sum_{k=1}^{+\infty}a_kz^{k-1}}\right|\le\frac{\sum_{k=1}^{+\infty}ka_k|z|^{k-1}}{1-\sum_{k=1}^{+\infty}a_k\|z|^{k-1}}\le 1-\eta,$$

or

$$\sum_{k=1}^{+\infty}ka_k|z|^{k-1}\le(1-\eta)-(1-\eta)\sum_{k=1}^{+\infty}a_k|z|^{k-1},$$

or

$$\sum_{k=1}^{+\infty}\frac{k+1-\eta}{1-\eta}a_k|z|^{k-1}\le 1.$$

By using (2.2) we have

$$\sum_{k=1}^{+\infty}\frac{k+1-\eta}{1-\eta}a_k|z|^{k-1}\le$$
$$\sum_{k=1}^{+\infty}\frac{2b^2 4^k(q;q)_k(q^{\nu+1};q)_k(k+1-\eta)}{(-1)^k(1-q)^{2k}\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big](1-\eta)}|z|^{k-1}\le 1.$$

So it is enough to suppuse

$$|z|^{k-1}\le\frac{(-1)^k(1-q)^{2k}\Big[(k-1)(b+1)\big((k-2)(1-b)+1\big)+b\Big](1-\eta)}{2b^2 4^k(q;q)_k(q^{\nu+1};q)_k(k+1-\eta)}.$$

(ii) Since $f(z)$ is convex if, and only if, $zf'(z)$ be starlike, by easy calculation we concude the required result (ii). Hence the proof is complete. □

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