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COMPUTATIONAL AND ANALYTICAL SOLUTION OF NON-LINEAR SYSTEM OF 2-DIMENSIONAL TIME-FRACTIONAL NAVIER-STOKES EQUATION

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ABSTRACT. In this paper, we introduce an analytical and approximate technique to obtain the solution of two dimensional time fractional order non linear Navier Stokes equation in the Cartesian coordinates with a variable pressure, which is based on Fractional Sumudu Transform method and its differential and integral properties. We give an illustrative application to demonstrate the effectiveness and accuracy of the proposed method, where numerical solutions and graphical representations show that the proposed method performs extremely well in terms of efficiency and simplicity and it can be utilised to solve more problems in the field of non-linear fractional differential equations. The results obtained by the proposed technique indicate that the approach is easy to implement and computationally very attractive. The small size of computation contrary to the other schemes is its strength.

1. INTRODUCTION

Fractional Calculus deals with the differential and integral operators with non-integral powers. Noting that the integer-order differential operator is a local operator while the fractional order differential operator is non-local, it means that the next state of a system depends not only upon its current state but also upon all of its previous states. It is more realistic and is one of the main reasons why the fractional calculus has become so popular. However in recent years extensive notice in fractional differential equations has been motivated due to its abundant applications in the areas of science, physics, engineering [1][2]. Numerous significant models are well described by fractional differential equations in fluid mechanics, electro-chemistry, electromagnetics, viscoelasticity, life sciences and financial market [3][4][6][5][7]. Recently, numerous methods have drawn special attention such as Homotopy Perturbation method [8], Adomian Decomposition method [9], Reduced Differential transform method [10], Homotopy Analysis method [11] and Modified Laplace Decomposition method [12]. In this given work, an analytical study of nonlinear time-fractional model of Navier-Stokes equation of order q, $0 < q \leq 1$, is

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presented. The nonlinear time-fractional model of Navier-Stokes equation for an incompressible fluid flow of kinematic viscosity μ_0 and constant density ρ is given as follows [13]:

$$D_\tau^q \Psi + (\Psi \cdot \nabla)\Psi = -(\frac{1}{\rho})\nabla \cdot p + \mu_0 \nabla^2 \cdot \Psi \quad \text{in} \quad \Omega \times (0,T], 0 < q \leq 1.$$

This system is subject to the following conditions [14]:

$$\Psi \cdot \nabla = 0 \quad \text{in} \quad \Omega \times (0,T] \quad \text{with} \quad x \in \Omega \quad \text{the compressibility condition};$$

$$\Psi = \Psi(x), \quad \text{on} \quad \Gamma_{rigid} \times (0,T] \quad \text{is the boundary condition};$$

$$\Psi(x,0) = \Psi_0(x) \quad \text{in} \quad \Omega \times (0,T] \quad \text{is the initial condition};$$

where $\Psi(\psi,\nu)$ is is the fluid velocity vector field with components ψ and ν at the point (x,y) and time τ, $(x,y) \in \Omega \subseteq \mathbb{R}^2$, Γ_{rigid} is the boundary of Ω, $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is gradient operator and $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the Laplacian operator, p is the pressure, ρ is the density, μ_0 is the kinematic viscosity which is equal to the ratio $\mu_0 = \dfrac{\eta}{\rho}$. The objective of this paper is to spread the application of the Fractional Sumudu Transform Method (FSTM) [3][5] to acquire a solution of the 2-dimensional time-fractional nonlinear Navier-Stokes equation in the cartesian coordinates with a variable pressure. This equation describes many physical things such as ocean currents, liquid flow in pipes, blood flow and air flow around the wings of an aircraft. The Fractional Sumudu Transform Method (FSTM) is a combination of the Sumudu Transform method and its differential and integral properties. The benefits of this technique is its capability for attaining exact and approximate analytical solutions. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as matched to the classical methods while still keeping the high accurateness of the numerical outcome, the size reduction amounts to a perfection of the performance of the approach. This paper is organised as follows. In Section 2 some definitions regarding fractional calculus and Sumudu transform are given. In Section 3 the solution of the time fractional order nonlinear Navier-Stokes equation with a variable pressure is constructed using the FSTM method. In Section 4 our technique is applied on a time fractional Navier-Stokes model, and graphical and numerical results are presented. And in Section 5 conclusions are given.

2. Basic Definitions of Fractional Calculus and Sumudu Transform

In this section, we give some basic definitions and properties of Fractional Calculus and Sumudu transforms, which will be used in this paper.

Definition 2.1. *Let $\mu \in \mathbb{R}$ and $m \in \mathbb{N}$. A real valued function $\psi : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathbb{C}_\mu$ if there exists $\lambda \in \mathbb{R}$, $\lambda > \mu$ and $g \in \mathbb{C}[0,\infty)$ such that $\psi(x) = x^\lambda g(x)$, for all $x \in \mathbb{R}^+$. Moreover, $\psi \in \mathbb{C}_\mu^m$ if $\psi^{(m)} \in \mathbb{C}_\mu$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order q of a function $\psi(x) \in \mathbb{C}_\mu, \quad \mu \geq -1$ is given with:*

$$J^q\psi(x) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^x (x-\tau)^{q-1}\psi(\tau)d\tau, & q > 0, x > 0 \\ \psi(x), & q = 0 \end{cases}.$$

The operator J^q has some properties for q and K is a real constant.

- $J^q K = \frac{K}{\Gamma(q+1)} x^q$

- $J^0\psi(x) = \psi(x)$

Definition 2.3. *The Caputo Fractional derivatives D^q of a function $\psi(x)$ of any real number q such that $m-1 < q \leq m, \quad m \in \mathbb{N}$, for $x > 0$ and $\psi \in \mathbb{C}^m_{-1}$ as*

$$D^q\psi(x) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^x (x-\tau)^{m-q-1}\psi^{(m)}(\tau)d\tau, \\ \frac{\partial^m \psi(x)}{\partial x^m}, \quad q = m \end{cases}$$

and has the following properties for $m-1 < q \leq m, m \in \mathbb{N}, \quad \mu \geq -1$ and $\psi \in \mathbb{C}^m_\mu$

- $D^q J^q \psi(x) = \psi(x)$
- $J^q D^q \psi(x) = \psi(x) - \sum_{k=0}^{m-1} \psi^{(k)}(0)\frac{x^k}{k!}$, *for x>0.*

Definition 2.4. *In* [15] *a new integral transform, first proposed by Watugala in 1998, called Sumudu Transform defined for functions of exponential order to solve engineering problems. We consider functions in the set A defined by*

$$A = \{\psi(\tau) : \exists M, t_1, t_2 > 0, |\psi(\tau)| < Me^{\frac{\tau}{t_j}}, \quad \textit{if} \quad \tau \in (-1)^j \times [0,\infty)\}.$$

For a given function in the set A, the constant M must be finite, while t_1 and t_2 need not simultaneously exist, and each may be infinite. Instead of being used as a power to the exponential, the variable u in the Sumudu transform is used to factor the variable τ in the argument of the function ψ. Specifically, for $\psi(\tau)$ in A, the Sumudu transform is defined by

$$F(u) = S[\psi(\tau)] = \int_0^\infty \frac{1}{u} e^{\frac{-\tau}{u}} \psi(\tau) d\tau.$$

The existence and uniqueness of Sumudu transform was given in [16]. For more information and properties of Sumudu transform and its derivatives see [17]. The Sumudu transform $S[\psi(\tau)]$ has some differential and integral properties as:

- The Sumudu transform $S[\psi(\tau)]$ of Riemann-Liouville fractional integral is given as: (see [18])
$$S[J^q\psi(\tau)] = u^q S[\psi(\tau)].$$
- The Sumudu transform $S[\psi(\tau)]$ of the Caputo fractional derivative is given as: (see [19])
$$S[D^q\psi(\tau)] = u^{-q}S[\psi(\tau)] - \sum_{k=0}^{m-1} u^{-q+k}\psi^{(k)}(0), \quad m-1 < q \leq m.$$
- The inverse Sumudu transform is given by [19]
$$S^{-1}[\sum_{k=0}^{m-1} u^k \psi^{(k)}(0)] = \sum_{k=0}^{m-1} \frac{\tau^k \psi^{(k)}(0)}{\Gamma(k+1)}.$$

Definition 2.5. *G. M. Mittag-Leffler has developed a function for two parameters using series expansion given by*

$$E_{q,r}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(qn+r)}, \quad (q > 0, r > 0).$$

3. Analysis of Fractional Sumudu Transform method on 2- dimensional Time Fractional order Nonlinear Navier Stokes equation

In this section, the solution of nonlinear system of 2-dimensional time-fractional order Navier-Stokes equation is presented. The time-fractional model of Navier-Stokes equation for an incompressible fluid flow is the principal equation of computational fluid dynamics, relating pressure and external forces acting on a fluid to the reaction of the fluid flow. The Navier-Stokes and continuity equations are given as: The nonlinear system of time-fractional model for 2-dimensional Navier-Stokes equations of order q, $0 < q \leq 1$ in Cartesian co-ordinates are given as

$$D_\tau^q \psi + \psi\psi_x + \nu\psi_y = -\frac{1}{\rho}P_x + \mu_0(\psi_{xx} + \psi_{yy})\,; \tag{3.1}$$

$$D_\tau^q v + \psi\nu_x + \nu\nu_y = -\frac{1}{\rho}P_y + \mu_0(\nu_{xx} + \nu_{yy})\,. \tag{3.2}$$

Subject to the following conditions:

$$\frac{\partial\psi}{\partial x} + \frac{\partial\nu}{\partial y} = 0, \quad \text{incompressibility condition}$$

$$\Psi(x,y,\tau) = \Psi(b), \quad ,(x,y) \in \Gamma \text{ the boundary condition}$$

$$\Psi(x,y,0) = f_i(x,y), \quad \text{initial condition} \tag{3.3}$$

where $\Psi = (\psi,\nu) = (\psi(x,y,\tau),\nu(x,y,\tau)), \quad (x,y) \in \Omega \subseteq \mathbb{R}^2$ also Γ is the boundary condition of Ω for $i = 1,2,\ldots$ and $0 < q \leq 1$. D_τ^q represents the Caputo fractional derivative of order q, where q is the parameter telling the order of the time fractional derivatives. In the case $q = 1$ the fractional equation reduces to the standard Navier-Stokes equation. The Fractional Sumudu Transform Method FSTM shows a fractional power series solution about the initial point $\tau = 0$, defined as:

$$\psi(x,y,\tau) = \sum_{n=0}^{\infty} f_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,; \tag{3.4}$$

$$\nu(x,y,\tau) = \sum_{n=0}^{\infty} g_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,; \tag{3.5}$$

$$P(x,y,\tau) = \sum_{n=0}^{\infty} P_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,,$$

where $0 < q \leq 1$, x and $y \in \Omega$. It is clear that $\psi(x,y,\tau)$ and $\nu(x,y,\tau)$ satisfy the initial conditions (3.3) which can be rewritten as :

$$\psi(x,y,0) = f(x,y)\,,$$
$$\nu(x,y,0) = g(x,y)\,.$$

Hence, we can obtain the initial guess approximation of $\psi(x,y,\tau)$ and $\nu(x,y,\tau)$ as :

$$\psi_0(x,y,0) = f_0(x,y) = f(x,y)\,,$$
$$\nu_0(x,y,0) = g_0(x,y) = g(x,y)\,.$$

So equations (3.4) and (3.5) could be reformulated as:

$$\psi(x,y,\tau) = f(x,y) + \sum_{n=1}^{\infty} f_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,,$$

$$\nu(x,y,\tau) = g(x,y) + \sum_{n=1}^{\infty} g_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,.$$

Now operating with the proposed fractional Sumudu transform technique [3] [5] for equations (3.1) and (3.2) as follow by using both sides of the Sumudu transform (3.1) and (3.2) we get:

$$\begin{aligned} S[D_\tau^q\psi(x,y,\tau)] &= S[-\frac{1}{\rho}P_x + \mu_0(\psi_{xx}+\psi_{yy}) - \psi\psi_x - \nu\psi_y]\,,\\ S[D_\tau^q\nu(x,y,\tau)] &= S[-\frac{1}{\rho}P_y + \mu_0(\nu_{xx}+\nu_{yy}) - \psi\nu_x - \nu\nu_y]\,. \end{aligned}$$

Operating the differential property of Sumudu transform:

$$S[\psi(x,y,\tau)] = \psi(x,y,0) + u^q S[-\frac{1}{\rho}P_x + \mu_0(\psi_{xx}+\psi_{yy}) - \psi\psi_x - \nu\psi_y]\,, \tag{3.6}$$

$$S[\nu(x,y,\tau)] = \nu(x,y,0) + u^q S[-\frac{1}{\rho}P_y + \mu_0(\nu_{xx}+\nu_{yy}) - \psi\nu_x - \nu\nu_y]\,. \tag{3.7}$$

Using both sides, the inverse Sumudu (3.6) and (3.7) we have:

$$\begin{aligned} \psi(x,y,\tau) &= \psi(x,y,0) + S^{-1}[u^q S[-\frac{1}{\rho}P_x + \mu_0(\psi_{xx}+\psi_{yy}) - \psi\psi_x - \nu\psi_y]]\,,\\ \nu(x,y,\tau) &= \nu(x,y,0) + S^{-1}[u^q S[-\frac{1}{\rho}P_y + \mu_0(\nu_{xx}+\nu_{yy}) - \psi\nu_x - \nu\nu_y]]\,. \end{aligned}$$

Operating with the integral property of Sumudu transform

$$\psi(x,y,\tau) = \psi(x,y,0) + J_\tau^q[-\frac{1}{\rho}P_x + \mu_0(\psi_{xx}+\psi_{yy}) - \psi\psi_x - \nu\psi_y]\,, \tag{3.8}$$

$$\nu(x,y,\tau) = \nu(x,y,0) + J_\tau^q[-\frac{1}{\rho}P_y + \mu_0(\nu_{xx}+\nu_{yy}) - \psi\nu_x - \nu\nu_y]\,. \tag{3.9}$$

The Sumudu transform decomposition admits a solution in the form

$$\psi(x,y,\tau) = \sum_{n=0}^{\infty}\psi_n(x,y,\tau)\,, \tag{3.10}$$

$$\nu(x,y,\tau) = \sum_{n=0}^{\infty}\nu_n(x,y,\tau)\,, \tag{3.11}$$

$$P(x,y,\tau) = \sum_{n=0}^{\infty}P_n(x,y,\tau)\,. \tag{3.12}$$

Substituting (3.10), (3.11) and (3.12) in (3.8) and (3.9) we get

$$\begin{aligned} \sum_{n=0}^{\infty}\psi_n(x,y,\tau) &= \psi(x,y,0) + J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{nx} + \mu_0(\sum_{n=0}^{\infty}\psi_{nxx} + \sum_{n=0}^{\infty}\psi_{nyy}) - \sum_{n=0}^{\infty}\psi_n\psi_{nx} - \sum_{n=0}^{\infty}\nu_n\psi_{ny}]\\ \sum_{n=0}^{\infty}\nu_n(x,y,\tau) &= \nu(x,y,0) + J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{ny} + \mu_0(\sum_{n=0}^{\infty}\nu_{nxx} + \sum_{n=0}^{\infty}\nu_{nyy}) - \sum_{n=0}^{\infty}\psi_n\nu_{nx} - \sum_{n=0}^{\infty}\nu_n\nu_{ny}]\,, \end{aligned}$$

where $\psi_n\psi_{nx} = \phi_n(\psi)$, $\nu_n\psi_{ny} = \eta_n(\psi)$, $\psi_n\nu_{nx} = \beta_n(\nu)$ and $\nu_n\nu_{ny} = \gamma_n(\nu)$ are nonlinear terms, and the above equations become:

$$\begin{aligned}
\sum_{n=0}^{\infty}\psi_n(x,y,\tau) &= \psi(x,y,0) + J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{nx} + \mu_0(\sum_{n=0}^{\infty}\psi_{nxx} + \sum_{n=0}^{\infty}\psi_{nyy}) - \sum_{n=0}^{\infty}\phi_n(\psi) - \sum_{n=0}^{\infty}\eta_n(\psi)] \\
\sum_{n=0}^{\infty}\nu(x,y,\tau) &= \nu(x,y,0) + J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{ny} + \mu_0(\sum_{n=0}^{\infty}\nu_{nxx} + \sum_{n=0}^{\infty}\nu_{nyy}) - \sum_{n=0}^{\infty}\beta_n(\nu) - \sum_{n=0}^{\infty}\gamma_n(\nu)]\,,
\end{aligned}$$

where the nonlinear terms decomposed by the method of [20]. The technique shows a series solution for $\psi(x,y,\tau)$ and $\nu(x,y,\tau)$, we obtain

$$\begin{aligned}
\psi(x,y,\tau) &= \sum_{n=0}^{\infty}\psi_n(x,y,\tau) \\
&= \psi(x,y,0) + J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{nx} + \mu_0(\sum_{n=0}^{\infty}\psi_{nxx} + \sum_{n=0}^{\infty}\psi_{nyy}) - \sum_{n=0}^{\infty}\phi_n(\psi) - \sum_{n=0}^{\infty}\eta_n(\psi)] \\
&= \sum_{n=0}^{\infty}f_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,; \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
\nu(x,y,\tau) &= \sum_{n=0}^{\infty}\nu_n(x,y,\tau) \\
&= \nu(x,y,0) + J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{ny} + \mu_0(\sum_{n=0}^{\infty}\nu_{nxx} + \sum_{n=0}^{\infty}\nu_{nyy}) - \sum_{n=0}^{\infty}\beta_n(\nu) - \sum_{n=0}^{\infty}\gamma_n(\nu)] \\
&= \sum_{n=0}^{\infty}g_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,. \qquad (3.14)
\end{aligned}$$

Equating the terms on both sides of (3.13) and (3.14), we get the following relation

$$\begin{aligned}
\psi_0(x,y,\tau) &= \psi(x,y,0) = f_0(x,y) \\
\psi_{n+1}(x,y,\tau) &= J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{nx} + \mu_0(\sum_{n=0}^{\infty}\psi_{nxx} + \sum_{n=0}^{\infty}\psi_{nyy}) - \sum_{n=0}^{\infty}\phi_n(\psi) - \sum_{n=0}^{\infty}\eta_n(\psi)] \\
&= \sum_{n=1}^{\infty}f_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,;
\end{aligned}$$

$$\begin{aligned}
\nu_0(x,y,\tau) &= \nu(x,y,0) = g_0(x,y) \\
\nu_{n+1}(x,y,\tau) &= J_\tau^q[-\frac{1}{\rho}\sum_{n=0}^{\infty}P_{ny} + \mu_0(\sum_{n=0}^{\infty}\nu_{nxx} + \sum_{n=0}^{\infty}\nu_{nyy}) - \sum_{n=0}^{\infty}\beta_n(\nu) - \sum_{n=0}^{\infty}\gamma_n(\nu)] \\
&= \sum_{n=1}^{\infty}g_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,;
\end{aligned}$$

and the functions $(f_n)_{n=0\ldots}$ and $(g_n)_{n=0\ldots}$ are given by:

$$f_0 = \psi_0(x,y,\tau) = \psi(x,y,0)$$
$$f_1 = -\frac{1}{\rho}P_{0x} + \mu_0(f_{0xx} + f_{0yy}) - f_0 f_{0x} - g_0 f_{0y}$$
$$f_2 = -\frac{1}{\rho}P_{1x} + \mu_0(f_{1xx} + f_{1yy}) - f_1 f_{0x} - f_0 f_{1x} - g_1 f_{0y} - g_0 f_{1y}$$
$$f_3 = -\frac{1}{\rho}P_{2x} + \mu_0(f_{2xx} + f_{2yy}) - f_2 f_{0x} - f_1 f_{1x} - f_0 f_{2x} - g_2 f_{0y} - g_1 f_{1y} - g_0 f_{2y}$$
$$f_4 = -\frac{1}{\rho}P_{3x} + \mu_0(f_{3xx} + f_{3yy}) - f_3 f_{0x} - f_2 f_{1x} - f_1 f_{2x} - f_0 f_{3x} - g_3 f_{0y} - g_2 f_{1y} - g_1 f_{2y} - g_0 f_{3y}$$
$$\vdots$$
$$g_0 = \nu_0(x,y,\tau) = \nu(x,y,0)$$
$$g_1 = -\frac{1}{\rho}P_{0y} + \mu_0(g_{0xx} + g_{0yy}) - f_0 g_{0x} - g_0 g_{0y}$$
$$g_2 = -\frac{1}{\rho}P_{1y} + \mu_0(g_{1xx} + g_{1yy}) - f_1 g_{0x} - f_0 g_{1x} - g_1 g_{0y} - g_0 g_{1y}$$
$$g_3 = -\frac{1}{\rho}P_{2y} + \mu_0(g_{2xx} + g_{2yy}) - f_2 g_{0x} - f_1 g_{1x} - f_0 g_{2x} - g_2 g_{0y} - g_1 g_{1y} - g_0 g_{2y}$$
$$g_4 = -\frac{1}{\rho}P_{3y} + \mu_0(g_{3xx} + g_{3yy}) - f_3 g_{0x} - f_2 g_{1x} - f_1 g_{2x} - f_0 g_{3x} - g_3 g_{0y} - g_2 g_{1y} - g_1 g_{2y} - g_0 g_{3y}\,.$$

Now applying the boundary condition for finding the value of P_{nx} and P_{ny}. The solution of $P(x,y,\tau)$ in series form is defined as :

$$P(x,y,\tau) = P_0 + P_1\frac{\tau^q}{\Gamma(q+1)} + P_2\frac{\tau^{2q}}{\Gamma(2q+1)} + P_3\frac{\tau^{3q}}{\Gamma(3q+1)} + P_4\frac{\tau^{4q}}{\Gamma(4q+1)} + \ldots + P_n\frac{\tau^{nq}}{\Gamma(nq+1)}\,,$$

and that the solution $\psi(x,y,\tau)$ and $\nu(x,y,\tau)$ in series form is defined as:

$$\begin{aligned}
\psi(x,y,\tau) &= f_0 + f_1\frac{\tau^q}{\Gamma(q+1)} + f_2\frac{\tau^{2q}}{\Gamma(2q+1)} + f_3\frac{\tau^{3q}}{\Gamma(3q+1)} + f_4\frac{\tau^{4q}}{\Gamma(4q+1)} + \ldots + f_n\frac{\tau^{nq}}{\Gamma(nq+1)}\,,\\
\nu(x,y,\tau) &= g_0 + g_1\frac{\tau^q}{\Gamma(q+1)} + g_2\frac{\tau^{2q}}{\Gamma(2q+1)} + g_3\frac{\tau^{3q}}{\Gamma(3q+1)} + g_4\frac{\tau^{4q}}{\Gamma(4q+1)} + \ldots + g_n\frac{\tau^{nq}}{\Gamma(nq+1)}\,.
\end{aligned}$$

4. Application and Numerical Results

We discuss the implementation of our proposed method and investigate its accuracy on 2-dimensional time-fractional order nonlinear Navier-Stokes equation. The simplicity and accurateness of the proposed technique is shown through the following example.

4.1. **Application.** Our problem consists of the system (3.1) and (3.2), where $0 \le x \le \pi$, $0 \le y \le \pi$ and subject to the following conditions:

$$\begin{aligned}
\psi(x,y,0) &= -\cos(x)\sin(y) = f_0(x,y)\\
\nu(x,y,0) &= \sin(x)\cos(y) = g_0(x,y)\,,
\end{aligned}$$

with the boundary conditions:

$$\psi(x,0,\tau) = 0 \tag{4.1}$$

$$\nu(0,y,\tau) = 0\,. \tag{4.2}$$

Now, operating with the proposed Fractional Sumudu transform technique, we obtained the following relation:

$$\begin{aligned}
\psi_0(x,y,\tau) &= \psi(x,y,0) = f_0(x,y) = -\cos(x)\sin(y) \\
\psi_{n+1}(x,y,\tau) &= J_\tau^q[-\frac{1}{\rho}P_{0x} + \mu_0(\psi_{nxx}+\psi_{nyy}) - \phi_n(\psi) - \eta_n(\psi)] \\
&= \sum_{n=1}^{\infty} f_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,,
\end{aligned}$$

$$\begin{aligned}
\nu_0(x,y,\tau) &= \nu(x,y,0) = g_0(x,y) = \sin(x)\cos(y) \\
\nu_{n+1}(x,y,\tau) &= J_\tau^q[-\frac{1}{\rho}P_{0y} + \mu_0(\nu_{nxx}+\nu_{nyy}) - \beta_n(\nu) - \gamma_n(\nu)] \\
&= \sum_{n=1}^{\infty} g_n(x,y)\frac{\tau^{nq}}{\Gamma(qn+1)}\,,
\end{aligned}$$

and the functions $(f_n)_{n=0\ldots}$ and $(g_n)_{n=0\ldots}$ are given by:

$$\begin{aligned}
f_0 &= \psi_0(x,y,\tau) = \psi(x,y,0) = -\cos(x)\sin(y) \\
g_0 &= \nu_0(x,y,\tau) = \nu(x,y,0) = \sin(x)\cos(y)\,;
\end{aligned}$$

$$\begin{aligned}
f_1 &= -\frac{1}{\rho}P_{0x} + \mu_0(f_{0xx}+f_{0yy}) - f_0f_{0x} - g_0f_{0y} \qquad (4.3)\\
&= -\frac{1}{\rho}P_{0x} + 2\mu_0\cos(x)\sin(y) + \sin(x)\cos(x)
\end{aligned}$$

$$\begin{aligned}
g_1 &= -\frac{1}{\rho}P_{0y} + \mu_0(g_{0xx}+g_{0yy}) - f_0g_{0x} - g_0g_{0y} \qquad (4.4)\\
&= -\frac{1}{\rho}P_{0y} - 2\mu_0\sin(x)\cos(y) + \sin(y)\cos(y)\,.
\end{aligned}$$

Now we apply boundary conditions (4.1) and (4.2) for finding the value of P_{0x} and P_{0y}. The boundary conditions are:

$$\psi_1(x,0,\tau) = 0 = f_1 \tag{4.5}$$

$$\nu_1(0,y,\tau) = 0 = g_1\,. \tag{4.6}$$

Substituting (4.5) and (4.6) in (4.3) and (4.4) we have:

$$P_{0x} = \rho\sin(x)\cos(x) \tag{4.7}$$

$$P_{0y} = \rho\sin(y)\cos(y)\,. \tag{4.8}$$

Integrating equation (4.7) with respect to x, we get:

$$P_0(x,y) = -\frac{\rho}{4}\cos(2x) + \theta_1(y)\,, \tag{4.9}$$

where $\theta_1(y)$ is a function of y only and to find it, and differentiate the last equation with respect to y, we get:

$$P_{0y} = \theta_1(y)\,.$$

Substitute this equation in (4.8), then integrate the resultant equation with respect to y, we get

$$\theta_1(y) = -\frac{\rho}{4}\cos(2y)\,.$$

Now substitute this equation in (4.9), we get

$$P_0(x,y) = -\frac{\rho}{4}\cos(2x) + \cos(2y)\,,$$

and substitute (4.7) and (4.8) in (4.3) and (4.4) we have:

$$\begin{aligned} f_1 &= 2\mu_0 \cos(x)\sin(y) \\ g_1 &= -2\mu_0 \sin(x)\cos(y)\,. \end{aligned}$$

Now, for $f_2, f_3, f_4, \ldots, f_n$ and $g_2, g_3, g_4, \ldots, g_n$ we repeat the same process, we get

$$\begin{aligned} f_2 &= -\frac{1}{\rho}P_{1x} + \mu_0(f_{1xx} + f_{1yy}) - f_1 f_{0x} - f_0 f_{1x} - g_1 f_{0y} - g_0 f_{1y} \\ &= -\frac{1}{\rho}P_{1x} - 4\mu_0^2\cos(x)\sin(y) - 4\mu_0\sin(x)\cos(x) \\ &= -(-2\mu_0)^2\cos(x)\sin(y)\,; \end{aligned}$$

$$\begin{aligned} f_3 &= -\frac{1}{\rho}P_{2x} + \mu_0(f_{2xx} + f_{2yy}) - f_2 f_{0x} - f_1 f_{1x} - f_0 f_{2x} - g_2 f_{0y} - g_1 f_{1y} - g_0 f_{2y} \\ &= -\frac{1}{\rho}P_{2x} + 8\mu_0^3\cos(x)\sin(y) + 12\mu_0^2\sin(x)\cos(x) \\ &= -(-2\mu_0)^3\cos(x)\sin(y)\,; \end{aligned}$$

$$\begin{aligned} f_4 &= -\frac{1}{\rho}P_{3x} + \mu_0(f_{3xx} + f_{3yy}) - f_3 f_{0x} - f_2 f_{1x} - f_1 f_{2x} - f_0 f_{3x} - g_3 f_{0y} - g_2 f_{1y} - g_1 f_{2y} - g_0 f_{3y} \\ &= -\frac{1}{\rho}P_{3x} - 16\mu_0^4\cos(x)\sin(y) - 32\mu_0^3\sin(x)\cos(x) \\ &= -(-2\mu_0)^4\cos(x)\sin(y) \\ &\vdots \\ f_n &= -(-2\mu_0)^n\cos(x)\sin(y), \quad \forall n \geq 0 \end{aligned}$$

$$\begin{aligned} g_2 &= -\frac{1}{\rho}P_{1y} + \mu_0(g_{1xx} + g_{1yy}) - f_1 g_{0x} - f_0 g_{1x} - g_1 g_{0y} - g_0 g_{1y} \\ &= -\frac{1}{\rho}P_{1y} + 4\mu_0^2\sin(x)\cos(y) - 4\mu_0\sin(y)\cos(y) \\ &= (-2\mu_0)^2\sin(x)\cos(y)\,; \end{aligned}$$

$$\begin{aligned} g_3 &= -\frac{1}{\rho}P_{2y} + \mu_0(g_{2xx} + g_{2yy}) - f_2 g_{0x} - f_1 g_{1x} - f_0 g_{2x} - g_2 g_{0y} - g_1 g_{1y} - g_0 g_{2y} \\ &= -\frac{1}{\rho}P_{2y} - 8\mu_0^3\sin(x)\cos(y) + 12\mu_0^2\sin(y)\cos(y) \\ &= (-2\mu_0)^3\sin(x)\cos(y)\,; \end{aligned}$$

$$\begin{aligned} g_4 &= -\frac{1}{\rho}P_{3y} + \mu_0(g_{3xx} + g_{3yy}) - f_3 g_{0x} - f_2 g_{1x} - f_1 g_{2x} - f_0 g_{3x} - g_3 g_{0y} - g_2 g_{1y} - g_1 g_{2y} - g_0 g_{3y} \\ &= -\frac{1}{\rho}P_{3y} + 16\mu_0^4 \sin(x)\cos(y) - 32\mu_0^3 \sin(y)\cos(y) \\ &= (-2\mu_0)^4 \sin(x)\cos(y) \\ &\vdots \\ g_n &= (-2\mu_0)^n \sin(x)\cos(y), \quad \forall n \geq 0 \end{aligned}$$

Applying boundary conditions (4.1) and (4.2), the above mention same process for finding the values of P_{nx} and P_{ny}, then the term $P(x, y, \tau)$ in series form are given as:

$$\begin{aligned} P(x,y,\tau) &= -\frac{\rho}{4}(\cos(2x) + \cos(2y)) + \mu_0\rho(\cos(2x) + \cos(2y))\frac{\tau^q}{\Gamma(q+1)} \\ &\quad -3\mu_0^2\rho(\cos(2x) + \cos(2y))\frac{\tau^{2q}}{\Gamma(2q+1)} + 8\mu_0^3\rho(\cos(2x) + \cos(2y))\frac{\tau^{3q}}{\Gamma(3q+1)} + \ldots \end{aligned}$$

So that the solutions $\psi(x, y, \tau)$ and $\nu(x, y, \tau)$ in series form are defined as:

$$\begin{aligned} \psi(x,y,\tau) &= f_0 + f_1\frac{\tau^q}{\Gamma(q+1)} + f_2\frac{\tau^{2q}}{\Gamma(2q+1)} + f_3\frac{\tau^{3q}}{\Gamma(3q+1)} + f_4\frac{\tau^{4q}}{\Gamma(4q+1)} + \ldots + f_n\frac{\tau^{nq}}{\Gamma(nq+1)}, \\ \nu(x,y,\tau) &= g_0 + g_1\frac{\tau^q}{\Gamma(q+1)} + g_2\frac{\tau^{2q}}{\Gamma(2q+1)} + g_3\frac{\tau^{3q}}{\Gamma(3q+1)} + g_4\frac{\tau^{4q}}{\Gamma(4q+1)} + \ldots + g_n\frac{\tau^{nq}}{\Gamma(nq+1)}. \end{aligned}$$

Now, the solution $\psi(x, y, \tau)$ and $\nu(x, y, \tau)$ in closed form is defined as:

$$\begin{aligned} \psi(x,y,\tau) &= \sum_{n=0}^{\infty} \psi_n(x,y,\tau) = -\cos(x)\sin(y)\sum_{n=0}^{\infty}\frac{(-2\mu_0\tau^q)^n}{\Gamma(nq+1)} \\ &= -\cos(x)\sin(y)E_{q,1}(-2\mu_0\tau^q); \end{aligned}$$

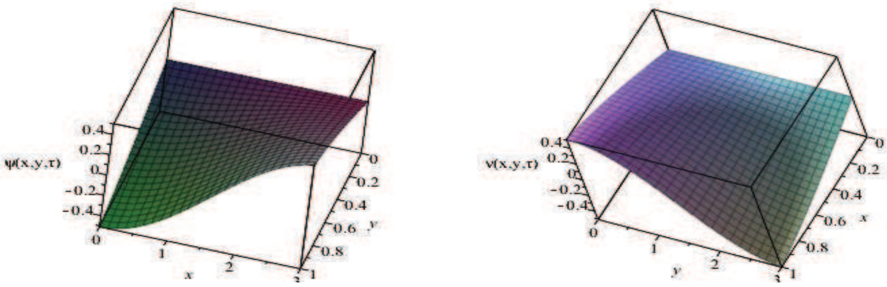
$$\begin{aligned} \nu(x,y,\tau) &= \sum_{n=0}^{\infty} \nu_n(x,y,\tau) = \sin(x)\cos(y)\sum_{n=0}^{\infty}\frac{(-2\mu_0\tau^q)^n}{\Gamma(nq+1)} \\ &= \sin(x)\cos(y)E_{q,1}(-2\mu_0\tau^q); \end{aligned}$$

where $E_{q,r}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(qn+r)}$, $(q > 0, r > 0)$, denotes the Mittag-Leffler function of two parameters. Setting for special condition $q = 1$, notice that $E_{1,1}(x) = e^x$, we have:

$$\begin{aligned} \psi(x,y,\tau) &= -e^{-2\mu_0\tau}\cos(x)\sin(y) \\ \nu(x,y,\tau) &= e^{-2\mu_0\tau}\sin(x)\cos(y), \end{aligned}$$

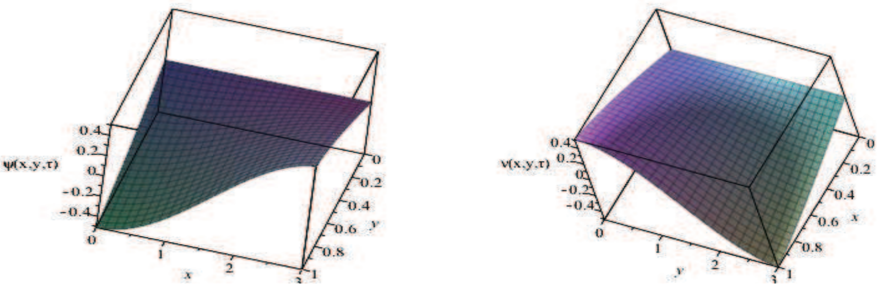
which is an exact solution of the classical Navier-Stokes equation for the velocity field. The solution of velocity field of the classical Navier Stokes equation is given in Fig: 1 and the solution of 2-dimensional nonlinear Navier-Stokes equation with time-fractional order $q = 1, 0.8, 0.5, 0.2$ is given in Figs: 2, 3, 4 and 5, respectively. The solution of $P(x, y, \tau)$ of the Navier-Stokes equation is given in Fig: 6, with time-fractional order $q = 1, 0.8, 0.5, 0.2$.

4.2. **Numerical solution for the time-fractional Nonlinear Navier-Stokes equation.** In this section we will study the solutions of the time fractional nonlinear Navier-Stokes equation numerically in order to validate the efficiency and accuracy of the Fractional Sumudu Transform Method FSTM, at first we will demonstrate the plots of solutions of



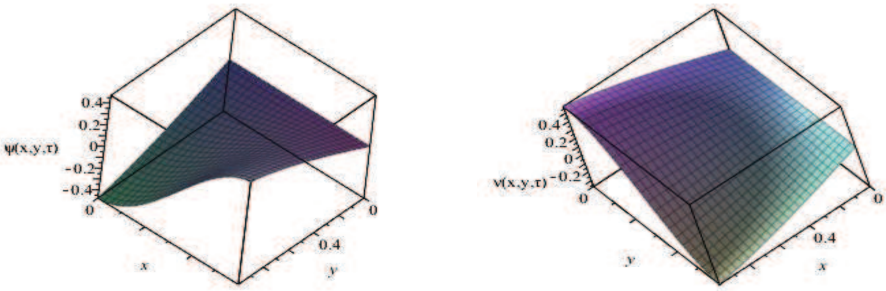
(A) 1-a (B) 1-b

FIGURE 1. The exact solution of ψ and ν of the classical Navier-Stokes equation at $\tau = 0.5$ with the parameter $\mu_0 = 0.5$.



(A) 2-a (B) 2-b

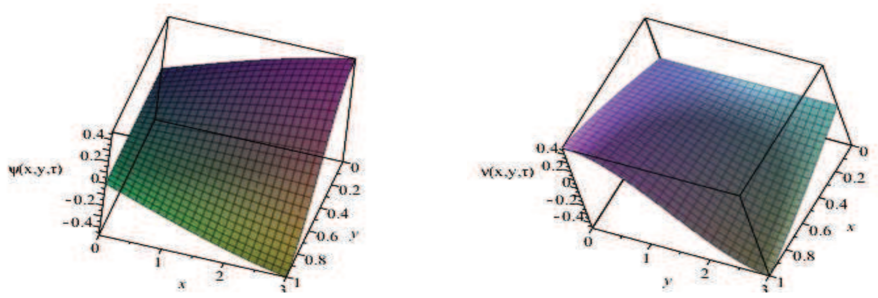
FIGURE 2. Solution of ψ and ν of Navier-Stokes equation at $\tau = 0.5$ with the parameters $q = 1$ and $\mu_0 = 0.5$.



(A) 3-a (B) 3-b

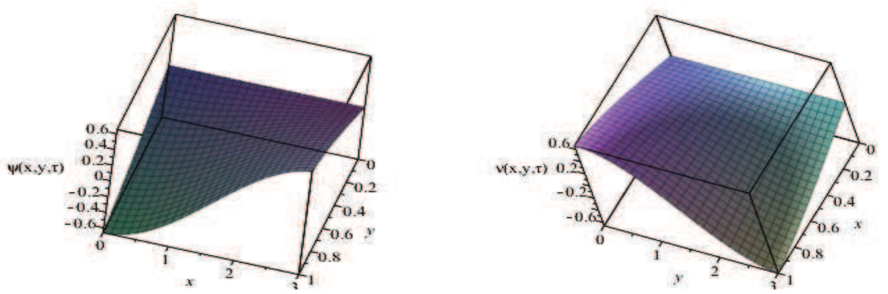
FIGURE 3. Solution of ψ and ν of Navier-Stokes equation at $\tau = 0.5$ with the parameters $q = 0.8$ and $\mu_0 = 0.5$.

$\psi(x, y, \tau)$ and $\nu(x, y, \tau)$ for the exact solutions at $q = 1$ in Fig. 1 then for the approximate solution of $\psi(x, y, \tau)$, $\nu(x, y, \tau)$ and $P(x, y, \tau)$ at $q = 1, 0.8, 0.5, 0.2$, in Figs. 2-6. And secondly we will show the absolute errors E_ψ and E_ν between exact solutions and



(A) 4-a (B) 4-b

FIGURE 4. Solution of ψ and ν of Navier-Stokes equation at $\tau = 0.5$ with the parameters $q = 0.5$ and $\mu_0 = 0.5$.



(A) 5-a (B) 5-b

FIGURE 5. Solution of ψ and ν of Navier-Stokes equation at $\tau = 0.5$ with the parameters $q = 0.2$ and $\mu_0 = 0.5$.

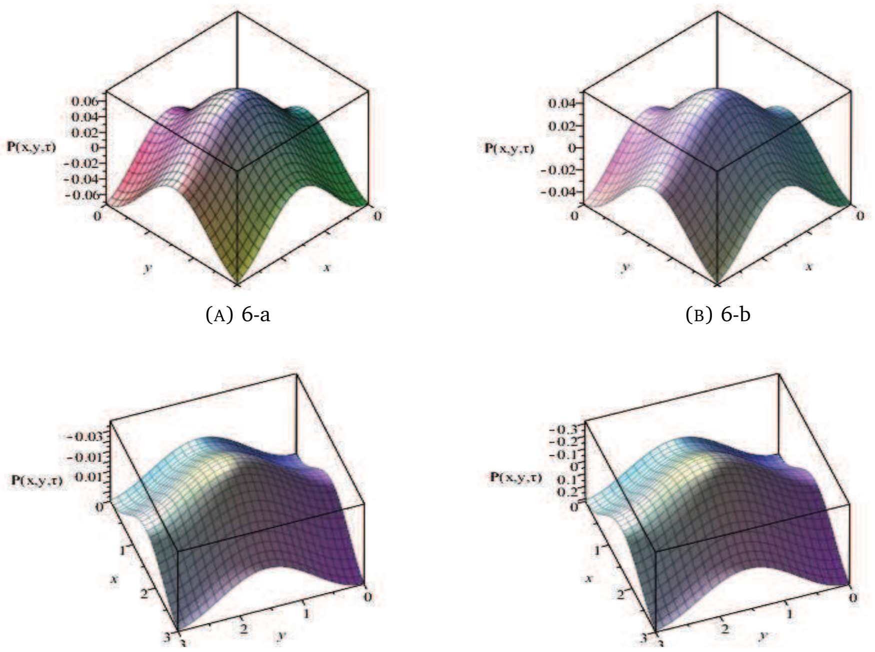
approximate solutions at $q = 1$ and for different values of x, y and τ in Table 1. Table 1 shows the obtained errors for the approximate solutions of the fractional Navier-Stokes equation compared with the exact solution at q = 1, where

$$\begin{aligned} E_\psi &= |\psi_{exact} - \psi| \\ E_\nu &= |\nu_{exact} - \nu| \end{aligned}$$

for the different values of x, y and τ. It is obvious that for short time interval there are small errors.

5. CONCLUSION

In this paper, a concept of the Sumudu transform and its derivative properties is successfully applied for 2-dimensional time-fractional order nonlinear Navier-Stokes equation with variable pressure by using Fractional Sumudu Transform Method FSTM [3][5]. The proposed method are free from discretization, perturbation or restrictive conditions. However, the proposed method need very small size of computation in comparison with RPS method [14], FRDTM [13] and HPTM [21]. We anticipate that this work is a step towards extending applications of the FSTM method to solve fractional problems with



(A) 6-a

(B) 6-b

(C) 6-c

(D) 6-d

FIGURE 6. Solution of P of Navier-Stokes equation at $\tau = 0.5$ with the parameters $q = 1.0, 0.8, 0.5, 0.2$ and $\mu_0 = 0.5, \rho = 0.5$.

TABLE 1

$\mathbf{x}$	$\mathbf{y}$	τ	$\mathbf{E}_\psi$	$\mathbf{E}_\nu$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	0.2	0.4093666666	$0.12900e-5$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	0.5	0.3033854166	$0.1200868e-3$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	0.9	0.2054187500	$0.21339202e-2$
$\frac{\pi}{4}$	$\frac{2\pi}{3}$	0.2	0.5013697257	$0.9123e-6$
$\frac{\pi}{4}$	$\frac{2\pi}{3}$	0.5	0.3715697333	$0.849144e-4$
$\frac{\pi}{4}$	$\frac{2\pi}{3}$	0.9	0.2515855606	$0.15089095e-2$
$\frac{2\pi}{3}$	$\frac{\pi}{4}$	0.2	0.2894659460	$0.15801e-5$
$\frac{2\pi}{3}$	$\frac{\pi}{4}$	0.5	0.2145258855	$0.1470759e-3$
$\frac{2\pi}{3}$	$\frac{\pi}{4}$	0.9	0.1452529911	$0.26135078e-2$
$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	0.2	0.3545219330	$0.11173e-5$
$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	0.5	0.26277394782	$0.1039984e-3$
$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	0.9	0.1778978560	$0.18480291e-2$

boundary conditions. The achieved outcomes are calculated using the symbolic calculus software Maple 16. This scheme FSTM [3][5] is clearly very efficient and powerful technique in finding the analytical solutions as well as numerical solutions of nonlinear system of fractional differential equations.

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