ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **8** (2019), no.1, 47–61
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

PERIODIC OSCILLATION FOR A FOUR-NODE NEURAL NETWORK MODEL WITH DISCRETE AND DISTRIBUTED DELAYS

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ABSTRACT. This paper investigates the existence of periodic oscillations for a four-node neural recurrent network with discrete and distributed delays. Three theorems are provided to guarantee the existence of periodic oscillations for this model by using Chafee's limit cycle criterion. The criteria for selecting of the parameters in this network are derived. Some simulation examples are presented to demonstrate the correctness of the results.

1. INTRODUCTION

It is well known that the studies of the dynamics behavior such as oscillations, chaos and bifurcations on neural systems are very important, and various interesting results have been reported. In 2007, Zhao and Wang have considered the Hopf bifurcation for the following two-neuron Cohen-Grossberg system with distributed delays [1]:

$$\begin{cases} x_1'(t) = -a_1(x_1(t))[b_1(x_1(t)) - \sum_{j=1}^{2} t_{1j} \int_0^{+\infty} S_j(s)x_j(t-s)ds + J_1], \\ x_2'(t) = -a_2(x_2(t))[b_2(x_2(t)) - \sum_{j=1}^{2} t_{2j} \int_0^{+\infty} S_j(s)x_j(t-s)ds + J_2]. \end{cases}$$

The stability of bifurcating periodic solutions and the direction of Hopf bifurcation are investigated by using the normal form theory and the center manifold theorem. Liao et al. have studied the bifurcation of a two-neuron system with distributed delays in the frequency domain as follows [2]:

$$\begin{cases} y_1'(t) = -y_1(t) + a_1 f(y_2(t)) - b_2 \int_0^{+\infty} F(r) f(y_2(t-r))dr, \\ y_2'(t) = -y_2(t) + a_2 f(y_1(t)) - b_1 \int_0^{+\infty} F(r) f(y_1(t-r))dr. \end{cases} \tag{1.1}$$

where $F(r) = \mu^2 r e^{-\mu r} (\mu > 0)$ is a strong kernel, $f(y_i)(i = 1, 2)$ are activation functions. By applying the frequency domain method and analyzing the associated characteristic equation, the existence of bifurcation parameter for system (1.1) is determined. Huang et al., [3] considered a two-neuron with four delays network modeled by the following nonlinear differential system

$$\begin{cases} x'(t) = -x(t) + a_{11} f(x(t-\tau_1)) + a_{12} f(y(t-\tau_2)), \\ y'(t) = -y(t) + a_{21} f(x(t-\tau_3)) + a_{22} f(y(t-\tau_4)). \end{cases} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34K11.
Key words and phrases. Four-node network model, strong kernel, delay, instability.

By using the normal form method and the center manifold theory, the linear stability and Hopf bifurcation including its direction and stability of the model (1.2) were established by the authors. Ding et al., [4] extended two-neuron system to a three-node recurrent neural network model with four discrete time delays as follows:

$$\begin{cases} x_1'(t) = -x_1(t) + f(x_2(t-\tau_2)), \\ x_2'(t) = -x_2(t) + f(x_3(t-\tau_4)), \\ x_3'(t) = -x_3(t) + af(x_1(t-\tau_1)) + bf(x_2(t-\tau_3)). \end{cases}$$

By means of the method of multiple time scales, the normal forms associated with Hopf-zero bifurcation, non-resonant and resonant double Hopf bifurcations were derived. Hajihosseini et al., see [5] generalized two-neuron system (1.1) to the following three-node network model with distributed delay:

$$\begin{cases} y_1'(t) = -y_1(t) + \int_0^{+\infty} F(r)\tanh(y_2(t-r))dr, \\ y_2'(t) = -y_2(t) + \int_0^{+\infty} F(r)\tanh(y_3(t-r))dr, \\ y_3'(t) = -y_3(t) + w_1\int_0^{+\infty} F(r)\tanh(y_1(t-r))dr + w_2\int_0^{+\infty} F(r)\tanh(y_2(t-r))dr. \end{cases}$$

where w_1 and w_2 are parameters, $F(r)$ is a strong kernel. The authors have discussed the Hopf bifurcation and stability of the bifurcating periodic solutions by taking μ as a bifurcating parameter. For an inertial four-neuron network model

$$\begin{cases} v_1''(t) = -v_1'(t) - \mu_1 v_1(t) + a_1 f(v_3(t-\tau)) + a_2 f(v_4(t-\tau)), \\ v_2''(t) = -v_2'(t) - \mu_2 v_2(t) + b_1 f(v_3(t-\tau)) + b_2 f(v_4(t-\tau)), \\ v_3''(t) = -v_3'(t) - \mu_3 v_3(t) + c_1 f(v_1(t-\tau)) + c_2 f(v_2(t-\tau)), \\ v_4''(t) = -v_4'(t) - \mu_4 v_4(t) + d_1 f(v_1(t-\tau)) + d_2 f(v_2(t-\tau)). \end{cases} \tag{1.3}$$

Ge and Xu, see [6] have discussed the local stability for the trivial solution. Stability switches and fold-Hopf bifurcations are found to occur in model (1.3). Dynamical behaviors are qualitatively classified in the neighbor of fold-Hopf bifurcation point. Bifurcating periodic solutions are expressed analytically in an approximate form (Xiao et al., see [7]).

Motivated by the above models, in this paper, we will discuss the following four-node neural network model with discrete and distributed delays:

$$\begin{cases} x_1'(t) = -r_1 x_1(t) + \beta_{11} f(x_1(t-\tau_1)) + \beta_{13} f(x_3(t-\tau_3)) + \beta_{14} f(x_4(t-\tau_4)) \\ \qquad\qquad +\gamma_2 \int_0^{+\infty} F(r) f(x_2(t-r))dr, \\ x_2'(t) = -r_2 x_2(t) + \beta_{21} f(x_1(t-\tau_1)) + \beta_{22} f(x_2(t-\tau_2)) + \beta_{24} f(x_4(t-\tau_4)) \\ \qquad\qquad +\gamma_3 \int_0^{+\infty} F(r) f(x_3(t-r))dr, \\ x_3'(t) = -r_3 x_3(t) + \beta_{31} f(x_1(t-\tau_1)) + \beta_{32} f(x_2(t-\tau_2)) + \beta_{33} f(x_3(t-\tau_3)) \\ \qquad\qquad +\gamma_4 \int_0^{+\infty} F(r) f(x_4(t-r))dr, \\ x_4'(t) = -r_4 x_4(t) + \beta_{42} f(x_2(t-\tau_2)) + \beta_{43} f(x_3(t-\tau_3)) + \beta_{44} f(x_4(t-\tau_4)) \\ \qquad\qquad +\gamma_1 \int_0^{+\infty} F(r) f(x_1(t-r))dr. \end{cases} \tag{1.4}$$

where the passive decay rates $r_i (i = 1, 2, 3, 4)$ are positive constants. $f(x_i)$ are activation functions, $\beta_{ij}, \gamma_i (i, j = 1, 2, 3, 4)$ are nonzero constants and $F(r) = \mu^2 r e^{-\mu r} (\mu > 0)$ is a strong kernel. Model (1.4) is a four-node recurrent neural network which includes distributed delays. Reminding that Liao et al., discussed an eight degree equation by using the bifurcating approach to deal with system (1.1) (see [2], equation (42), page 550), we should investigate a sixteen degree algebraic equation if we want to follow their bifurcating method. It is extremely hard to deal with a sixteen degree algebraic equation when the $\beta_{ij}, \gamma_i (i, j = 1, 2, 3, 4)$ are different real numbers. In order to discuss the existence of periodic solutions for system (1.4), we adopt Chafee's criterion [8]. Indeed, system (1.4) is in accordance with Chafee's criterion (we refer to the appendix of Feng and Plamondon [9], for more information) and, in this context, as for the class of time delay system which has a unique unstable equilibrium point, all solutions of the system are bounded, and this particular instability of the unique equilibrium point and the boundedness of the solutions will force the system (1.4) to generate a limit cycle, namely, a periodic solution.

2. Preliminaries

For the activation functions $f(x_i)$, we assume that $f(x_i), (i = 1, 2, 3, 4)$ are continuous bounded differentiable functions, satisfying:

$$f(0) = 0, \quad uf(u) > 0 \quad (u \neq 0)\,. \tag{2.1}$$

The general activation functions such as $\tanh(x), \arctan(x)$ satisfy condition (2.1). According to the linear chain trick [10], for system (1.4) with the strong kernel $F(r) = \mu^2 r e^{-\mu r} (\mu > 0)$, we have:

$$\begin{aligned} \int_0^{+\infty} F(r) f(x_i(t-r))dr &= \int_0^{+\infty} \mu^2 r e^{-\mu r} f(x_i(t-r))dr \\ &= \mu^2 e^{-\mu t} (t \int_{-\infty}^{t} e^{\mu s} f(x_i(s))ds - \int_{-\infty}^{t} s e^{\mu s} f(x_i(s))ds)\,, (i = 1, 2, 3, 4)\,. \end{aligned} \tag{2.2}$$

From (2.2) we get:

$$\begin{aligned} \frac{d}{dt}(\int_0^{+\infty} F(r) f(x_i(t-r))dr) &= \\ &= -\mu \int_0^{+\infty} \mu^2 r e^{-\mu r} f(x_i(t-r))dr + \mu^2 e^{-\mu t} \int_{-\infty}^{t} e^{-\mu s} f(x_i(s))ds \\ &= -\mu \int_0^{+\infty} F(r) f(x_i(t-r))dr + \mu^2 e^{-\mu t} \int_{-\infty}^{t} e^{-\mu s} f(x_i(s))ds\,. \end{aligned}$$

Thus, taking the derivative on both sides of system (1.4), and setting $f(x_i(t-\tau_i)) = f(x_i)$ $(i = 1, 2, 3, 4)$ we obtain:

$$\begin{cases} x_1''(t) = -r_1 x_1'(t) + \beta_{11} f'(x_1) x_1'(t-\tau_1) + \beta_{13} f'(x_3) x_3'(t-\tau_3) + \beta_{14} f'(x_4) x_4'(t-\tau_4) \\ \qquad -\mu\gamma_2 \int_0^{+\infty} F(r) f(x_2(t-r))dr + \gamma_2 \mu^2 e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} f(x_2(s))ds, \\ x_2''(t) = -r_2 x_2'(t) + \beta_{21} f'(x_1) x_1'(t-\tau_1) + \beta_{22} f'(x_2) x_2'(t-\tau_2) + \beta_{24} f'(x_4) x_4'(t-\tau_4) \\ \qquad -\mu\gamma_3 \int_0^{+\infty} F(r) f(x_3(t-r))dr + \gamma_3 \mu^2 e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} f(x_3(s))ds, \\ x_3''(t) = -r_3 x_3'(t) + \beta_{31} f'(x_1) x_1'(t-\tau_1) + \beta_{32} f'(x_2) x_2'(t-\tau_2) + \beta_{33} f'(x_3) x_3'(t-\tau_3) \\ \qquad -\mu\gamma_4 \int_0^{+\infty} F(r) f(x_4(t-r))dr + \gamma_4 \mu^2 e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} f(x_4(s))ds, \\ x_4''(t) = -r_4 x_4'(t) + \beta_{42} f'(x_2) x_2'(t-\tau_2) + \beta_{43} f'(x_3) x_3'(t-\tau_3) + \beta_{44} f'(x_4) x_4'(t-\tau_4) \\ \qquad -\mu\gamma_1 \int_0^{+\infty} F(r) f(x_1(t-r))dr + \gamma_1 \mu^2 e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} f(x_1(s))ds\,. \end{cases} \tag{2.3}$$

From (1.4), the system (2.3) can be written as the follows:

$$
\left\{\begin{array}{l}
x_1''(t)=-r_1x_1'(t)+\beta_{11}f'(x_1)x_1'(t-\tau_1)+\beta_{13}f'(x_3)x_3'(t-\tau_3)+\beta_{14}f'(x_4)x_4'(t-\tau_4)\\
-\mu(x_1'(t)+r_1x_1(t)-\beta_{11}f(x_1)-\beta_{13}f(x_3)-\beta_{14}f(x_4))+\gamma_2\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_2(s))ds,\\
x_2''(t)=-r_2x_2'(t)+\beta_{21}f'(x_1)x_1'(t-\tau_1)+\beta_{22}f'(x_2)x_2'(t-\tau_2)+\beta_{24}f'(x_4)x_4'(t-\tau_4)\\
-\mu(x_2'(t)+r_2x_2(t)-\beta_{21}f(x_1)-\beta_{22}f(x_2)-\beta_{24}f(x_4))+\gamma_3\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_3(s))ds,\\
x_3''(t)=-r_3x_3'(t)+\beta_{31}f'(x_1)x_1'(t-\tau_1)+\beta_{32}f'(x_2)x_2'(t-\tau_2)+\beta_{33}f'(x_3)x_3'(t-\tau_3)\\
-\mu(x_3'(t)+r_3x_3(t)-\beta_{31}f(x_1)-\beta_{32}f(x_2)-\beta_{33}f(x_3))+\gamma_4\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_4(s))ds,\\
x_4''(t)=-r_4x_4'(t)+\beta_{42}f'(x_2)x_2'(t-\tau_2)+\beta_{43}f'(x_3)x_3'(t-\tau_3)+\beta_{44}f'(x_4)x_4'(t-\tau_4)\\
-\mu(x_4'(t)+r_4x_4(t)-\beta_{42}f(x_2)-\beta_{43}f(x_3)-\beta_{44}f(x_4))+\gamma_1\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_1(s))ds.
\end{array}\right. \tag{2.4}
$$

Noting that

$$
\frac{d}{dt}(\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_i(s))ds)=-\mu(\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_i(s))ds)+\mu^2f(x_i(t))\,,\ (i=1,2,3,4)
$$

and taking the derivative again on both sides of system (2.4), we have:

$$
\left\{\begin{array}{l}
x_1'''(t)=-r_1x_1''(t)+\beta_{11}f''(x_1)[x_1'(t-\tau_1)]^2+\beta_{11}f'(x_1)x_1''(t-\tau_1)+\beta_{13}f''(x_3)[x_3'(t-\tau_3)]^2\\
+\beta_{13}f'(x_3)x_3''(t-\tau_3)+\beta_{14}f''(x_4)[x_4'(t-\tau_4)]^2+\beta_{14}f'(x_4)x_4''(t-\tau_4)-\mu[x_1''(t)+r_1x_1'(t)\\
-\beta_{11}f'(x_1)x_1'(t-\tau_1)-\beta_{13}f'(x_3)x_3'(t-\tau_3)-\beta_{14}f'(x_4)x_4'(t-\tau_4)]\\
-\mu\gamma_2\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_2(s))ds+\gamma_2\mu^2f(x_2(t)),\\
x_2'''(t)=-r_1x_2''(t)+\beta_{21}f''(x_1)[x_1'(t-\tau_1)]^2+\beta_{21}f'(x_1)x_1''(t-\tau_1)+\beta_{22}f''(x_2)[x_2'(t-\tau_2)]^2\\
+\beta_{22}f'(x_2)x_2''(t-\tau_2)+\beta_{24}f''(x_4)[x_4'(t-\tau_4)]^2+\beta_{24}f'(x_4)x_4''(t-\tau_4)-\mu[x_2''(t)+r_2x_2'(t)\\
-\beta_{21}f'(x_1)x_1'(t-\tau_1)-\beta_{22}f'(x_2)x_2'(t-\tau_2)-\beta_{24}f'(x_4)x_4'(t-\tau_4)]\\
-\mu\gamma_3\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_3(s))ds+\gamma_3\mu^2f(x_3(t)),\\
x_3'''(t)=-r_3x_3''(t)+\beta_{31}f''(x_1)[x_1'(t-\tau_1)]^2+\beta_{31}f'(x_1)x_1''(t-\tau_1)+\beta_{32}f''(x_2)[x_2'(t-\tau_2)]^2\\
+\beta_{32}f'(x_2)x_2''(t-\tau_2)+\beta_{33}f''(x_3)[x_3'(t-\tau_3)]^2+\beta_{33}f'(x_3)x_3''(t-\tau_3)-\mu[x_3''(t)+r_3x_3'(t)\\
-\beta_{31}f'(x_1)x_1'(t-\tau_1)-\beta_{32}f'(x_2)x_2'(t-\tau_2)-\beta_{33}f'(x_3)x_3'(t-\tau_3)]\\
-\mu\gamma_4\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_4(s))ds+\gamma_4\mu^2f(x_4(t)),\\
x_4'''(t)=-r_4x_4''(t)+\beta_{42}f''(x_2)[x_2'(t-\tau_2)]^2+\beta_{42}f'(x_2)x_2''(t-\tau_2)+\beta_{43}f''(x_3)[x_3'(t-\tau_3)]^2\\
+\beta_{43}f'(x_3)x_3''(t-\tau_3)+\beta_{44}f''(x_4)[x_4'(t-\tau_4)]^2+\beta_{44}f'(x_4)x_4''(t-\tau_4)-\mu[x_4''(t)+r_4x_4'(t)\\
-\beta_{42}f'(x_2)x_2'(t-\tau_2)-\beta_{43}f'(x_3)x_3'(t-\tau_3)-\beta_{44}f'(x_4)x_4'(t-\tau_4)]\\
-\mu\gamma_1\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_1(s))ds+\gamma_1\mu^2f(x_1(t))\,.
\end{array}\right. \tag{2.5}
$$

Also from (2.4), we have that:

$$
\begin{aligned}
\gamma_2\mu^2e^{-\mu t}\int_{-\infty}^{t}e^{\mu s}f(x_2(s))ds\quad &=\\
&=x_1''(t)+r_1x_1'(t)-\beta_{11}f'(x_1)x_1'(t-\tau_1)-\beta_{13}f'(x_3)x_3'(t-\tau_3)\\
&-\beta_{14}f'(x_4)x_4'(t-\tau_4)+\mu(x_1'(t)+r_1x_1(t)-\beta_{11}f(x_1)-\beta_{13}f(x_3)-\beta_{14}f(x_4))\,.
\end{aligned} \tag{2.6}
$$

Next, we combine (2.5) and (2.6). By setting $x_5(t)=x_1'(t),x_6(t)=x_2'(t),x_7(t)=x_3'(t),x_8(t)=x_4'(t),x_9(t)=x_1''(t),x_{10}(t)=x_2''(t),x_{11}(t)=x_3''(t),x_{12}(t)=x_4''(t)$, we have the equivalent version of system (1.4) as follows:

$$
\left\{\begin{array}{l}
x_1'(t) = x_5(t),\\
x_2'(t) = x_6(t),\\
x_3'(t) = x_7(t),\\
x_4'(t) = x_8(t),\\
x_5'(t) = x_9(t),\\
x_6'(t) = x_{10}(t),\\
x_7'(t) = x_{11}(t),\\
x_8'(t) = x_{12}(t),\\
x_9'(t) = -\mu^2 r_1 x_1(t) - (\mu^2 + 2\mu r_1)x_5(t) - (2\mu + r_1)x_9(t) + 2\mu\beta_{11} f'(x_1)x_5(t-\tau_1)\\
+2\mu\beta_{13} f'(x_3)x_7(t-\tau_3) + 2\mu\beta_{14} f'(x_4)x_8(t-\tau_4) + \beta_{11} f'(x_1)x_9(t-\tau_1)\\
+\beta_{13} f'(x_3)x_{11}(t-\tau_3) + \beta_{14} f'(x_4)x_{12}(t-\tau_4) + \beta_{11} f''(x_1)[x_5(t-\tau_1)]^2\\
+\beta_{13} f''(x_3)[x_7(t-\tau_3)]^2 + \beta_{14} f''(x_4)[x_8(t-\tau_4)]^2 + \mu^2\beta_{11} f(x_1)\\
+\mu^2\beta_{13} f(x_3) + \mu^2\beta_{14} f(x_4) + \gamma_2\mu^2 f(x_2(t)),\\
x_{10}'(t) = -\mu^2 r_2 x_2(t) - (\mu^2 + 2\mu r_2)x_6(t) - (2\mu + r_2)x_{10}(t) + 2\mu\beta_{21} f'(x_1)x_5(t-\tau_1)\\
+2\mu\beta_{22} f'(x_2)x_6(t-\tau_2) + 2\mu\beta_{24} f'(x_4)x_8(t-\tau_4) + \beta_{21} f'(x_1)x_9(t-\tau_1)\\
+\beta_{22} f'(x_2)x_{10}(t-\tau_2) + \beta_{24} f'(x_4)x_{12}(t-\tau_4) + \beta_{21} f''(x_1)[x_5(t-\tau_1)]^2\\
+\beta_{22} f''(x_2)[x_6(t-\tau_2)]^2 + \beta_{24} f''(x_4)[x_8(t-\tau_4)]^2 + \mu^2\beta_{21} f(x_1)\\
+\mu^2\beta_{22} f(x_2) + \mu^2\beta_{24} f(x_4) + \gamma_3\mu^2 f(x_3(t)),\\
x_{11}'(t) = -\mu^2 r_3 x_3(t) - (\mu^2 + 2\mu r_3)x_7(t) - (2\mu + r_3)x_{11}(t) + 2\mu\beta_{31} f'(x_1)x_5(t-\tau_1)\\
+2\mu\beta_{32} f'(x_2)x_6(t-\tau_2) + 2\mu\beta_{33} f'(x_3)x_7(t-\tau_3) + \beta_{31} f'(x_1)x_9(t-\tau_1)\\
+\beta_{32} f'(x_2)x_{10}(t-\tau_2) + \beta_{33} f'(x_3)x_{11}(t-\tau_3) + \beta_{31} f''(x_1)[x_5(t-\tau_1)]^2\\
+\beta_{32} f''(x_2)[x_6(t-\tau_2)]^2 + \beta_{33} f''(x_3)[x_7(t-\tau_3)]^2 + \mu^2\beta_{31} f(x_1)\\
+\mu^2\beta_{32} f(x_2) + \mu^2\beta_{33} f(x_3) + \gamma_4\mu^2 f(x_4(t)),\\
x_{12}'(t) = -\mu^2 r_4 x_4(t) - (\mu^2 + 2\mu r_4)x_8(t) - (2\mu + r_4)x_{12}(t) + 2\mu\beta_{42} f'(x_2)x_6(t-\tau_2)\\
+2\mu\beta_{43} f'(x_3)x_7(t-\tau_3) + 2\mu\beta_{44} f'(x_4)x_8(t-\tau_4) + \beta_{42} f'(x_2)x_{10}(t-\tau_2)\\
+\beta_{43} f'(x_3)x_{11}(t-\tau_3) + \beta_{44} f'(x_4)x_{12}(t-\tau_4) + \beta_{42} f''(x_2)[x_6(t-\tau_2)]^2\\
+\beta_{43} f''(x_3)[x_7(t-\tau_3)]^2 + \beta_{44} f''(x_4)[x_8(t-\tau_4)]^2 + \mu^2\beta_{42} f(x_2)\\
+\mu^2\beta_{43} f(x_3) + \mu^2\beta_{44} f(x_4) + \gamma_1\mu^2 f(x_1(t)).
\end{array}\right.
\tag{2.7}
$$

The linearized system of (2.7) can be written in a matrix form:

$$X'(t) = AX(t) + BX(t-\tau) \tag{2.8}$$

where $X(t) = [x_1(t), x_2(t), x_3(t), \cdots, x_{12}(t)]^T$ and
$X(t-\tau) = [x_1(t-\tau_1), x_2(t-\tau_2), x_3(t-\tau_3), \cdots, x_{11}(t-\tau_3), x_{12}(t-\tau_4)]^T$. Both A and B are 12×12 matrices as follows:

$$
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
a_{91} & a_{92} & 0 & 0 & a_{95} & 0 & 0 & 0 & a_{99} & 0 & 0 & 0\\
0 & a_{102} & a_{103} & 0 & 0 & a_{106} & 0 & 0 & 0 & a_{1010} & 0 & 0\\
0 & 0 & a_{113} & a_{114} & 0 & 0 & a_{117} & 0 & 0 & 0 & a_{1111} & 0\\
a_{121} & 0 & 0 & a_{124} & 0 & 0 & 0 & a_{128} & 0 & 0 & 0 & a_{1212}
\end{pmatrix},
$$

$$B=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\b_{91}&0&b_{93}&b_{94}&b_{95}&0&b_{97}&b_{98}&b_{99}&0&b_{911}&b_{912}\\b_{101}&b_{102}&0&b_{104}&b_{105}&b_{106}&0&b_{108}&b_{109}&b_{1010}&0&b_{1012}\\b_{111}&b_{112}&b_{113}&0&b_{115}&b_{116}&b_{117}&0&b_{119}&b_{1110}&b_{1111}&0\\0&b_{122}&b_{123}&b_{124}&0&b_{126}&b_{127}&b_{128}&0&b_{1210}&b_{1211}&b_{1212}\end{pmatrix}.$$

where $a_{91}=-\mu^2 r_1, a_{92}=\gamma_2\mu^2\alpha_2, a_{95}=-(\mu^2+2\mu r_1),\cdots,a_{128}=-(\mu^2+2\mu r_4), a_{1212}=-(2\mu+r_4)$, $b_{91}=\mu^2\beta_{11}\alpha_1, b_{93}=\mu^2\beta_{13}\alpha_3,\cdots,b_{1211}=\beta_{43}\alpha_3, b_{1212}=\beta_{44}\alpha_4$.

Lemma 2.1. *All solutions of system (1.4) are bounded.*

Proof. From the condition (2.1), the activation functions are bounded, and assuming that $|f(x_i)|\le N_i\,,\ \ (i=1,2,3,4)$, then we have:

$$\left|\int_0^{+\infty}F(r)f(x_i(t-r))dr\right|\le N_i\int_0^{+\infty}\mu^2 re^{-\mu r}dr=N_i\,,\ \ (i=1,2,3,4)\,.$$

From (1.4), noting that $r_i(i=1,2,3,4)$ are positive real numbers, we obtain

$$\begin{cases}\frac{d|x_1(t)|}{dt}\le -r_1|x_1(t)|+|\beta_{11}|N_1+|\beta_{13}|N_3+|\beta_{14}|N_4+\gamma_2N_2,\\ \frac{d|x_2(t)|}{dt}\le -r_2|x_2(t)|+|\beta_{21}|N_1+|\beta_{22}|N_2+|\beta_{24}|N_4+\gamma_3N_3,\\ \frac{d|x_3(t)|}{dt}\le -r_3|x_3(t)|+|\beta_{31}|N_1+|\beta_{32}|N_2+|\beta_{33}|N_3+\gamma_4N_4,\\ \frac{d|x_4(t)|}{dt}\le -r_4|x_4(t)|+|\beta_{42}|N_2+|\beta_{43}|N_3+|\beta_{44}|N_4+\gamma_1N_1.\end{cases}$$

Thus,

$$\begin{cases}|x_1(t)|\le\dfrac{|\beta_{11}|N_1+|\beta_{13}|N_3+|\beta_{14}|N_4+\gamma_2N_2}{r_1},\\ |x_2(t)|\le\dfrac{|\beta_{21}|N_1+|\beta_{22}|N_2+|\beta_{24}|N_4+\gamma_3N_3}{r_2},\\ |x_3(t)|\le\dfrac{|\beta_{31}|N_1+|\beta_{32}|N_2+|\beta_{33}|N_3+\gamma_4N_4}{r_3},\\ |x_4(t)|\le\dfrac{|\beta_{42}|N_2+|\beta_{43}|N_3+|\beta_{44}|N_4+\gamma_1N_1}{r_4}.\end{cases}$$

This means that the solutions of system (1.4) are uniformly bounded. □

Lemma 2.2. *If the matrix* $C=\begin{pmatrix}\beta_{11}&\gamma_2&\beta_{13}&\beta_{14}\\ \beta_{21}&\beta_{22}&\gamma_3&\beta_{24}\\ \beta_{31}&\beta_{32}&\beta_{33}&\gamma_4\\ \gamma_1&\beta_{42}&\beta_{43}&\beta_{44}\end{pmatrix}$ *is not a positive definite matrix, then the system (1.4) has a unique equilibrium point.*

Proof. Note that system (2.7) is equivalent to system (1.4). An equilibrium point $x^* = (x_1^*, x_2^*, x_3^*, \cdots, x_{12}^*)^T$ is a solution of the following algebraic equation:

$$
\begin{cases}
x_5^* = 0,\\
x_6^* = 0,\\
x_7^* = 0,\\
x_8^* = 0,\\
x_9^* = 0,\\
x_{10}^* = 0,\\
x_{11}^* = 0,\\
x_{12}^* = 0,\\
-\mu^2 r_1 x_1^* - (\mu^2 + 2\mu r_1) x_5^* - (2\mu + r_1) x_9^* + 2\mu\beta_{11} f'(x_1^*) x_5^* + 2\mu\beta_{13} f'(x_3^*) x_7^*\\
+2\mu\beta_{14} f'(x_4^*) x_8^* + \beta_{11} f'(x_1^*) x_9^* + \beta_{13} f'(x_3^*) x_{11}^* + \beta_{14} f'(x_4^*) x_{12}^*\\
+\beta_{11} f''(x_1^*)[x_5^*]^2 + \beta_{13} f''(x_3^*)[x_7^*]^2 + \beta_{14} f''(x_4^*)[x_8^*]^2 + \mu^2\beta_{11} f(x_1^*)\\
+\mu^2\beta_{13} f(x_3^*) + \mu^2\beta_{14} f(x_4^*) + \gamma_2\mu^2 f(x_2^*) = 0,\\
-\mu^2 r_2 x_2^* - (\mu^2 + 2\mu r_2) x_6^* - (2\mu + r_2) x_{10}^* + 2\mu\beta_{21} f'(x_1^*) x_5^* + 2\mu\beta_{22} f'(x_2^*) x_6^*\\
+2\mu\beta_{24} f'(x_4^*) x_8^* + \beta_{21} f'(x_1^*) x_9^* + \beta_{22} f'(x_2^*) x_{10}^* + \beta_{24} f'(x_4^*) x_{12}^*\\
+\beta_{21} f''(x_1^*)[x_5^*]^2 + \beta_{22} f''(x_2^*)[x_6^*]^2 + \beta_{24} f''(x_4^*)[x_8^*]^2 + \mu^2\beta_{21} f(x_1^*)\\
+\mu^2\beta_{22} f(x_2^*) + \mu^2\beta_{24} f(x_4^*) + \gamma_3\mu^2 f(x_3^*) = 0,\\
-\mu^2 r_3 x_3^* - (\mu^2 + 2\mu r_3) x_7^* - (2\mu + r_3) x_{11}^* + 2\mu\beta_{31} f'(x_1^*) x_5^* + 2\mu\beta_{32} f'(x_2^*) x_6^*\\
+2\mu\beta_{33} f'(x_3^*) x_7^* + \beta_{31} f'(x_1^*) x_9^* + \beta_{32} f'(x_2^*) x_{10}^* + \beta_{33} f'(x_3^*) x_{11}^*\\
+\beta_{31} f''(x_1^*)[x_5^*]^2 + \beta_{32} f''(x_2^*)[x_6^*]^2 + \beta_{33} f''(x_3^*)[x_7^*]^2 + \mu^2\beta_{31} f(x_1^*)\\
+\mu^2\beta_{32} f(x_2^*) + \mu^2\beta_{33} f(x_3^*) + \gamma_4\mu^2 f(x_4^*) = 0,\\
-\mu^2 r_4 x_4^* - (\mu^2 + 2\mu r_4) x_8^* - (2\mu + r_4) x_{12}^* + 2\mu\beta_{42} f'(x_2^*) x_6^* + 2\mu\beta_{43} f'(x_3^*) x_7^*\\
+2\mu\beta_{44} f'(x_4^*) x_8^* + \beta_{42} f'(x_2^*) x_{10}^* + \beta_{43} f'(x_3^*) x_{11}^* + \beta_{44} f'(x_4^*) x_{12}^*\\
+\beta_{42} f''(x_2^*)[x_6^*]^2 + \beta_{43} f''(x_3^*)[x_7^*]^2 + \beta_{44} f''(x_4^*)[x_8^*]^2 + \mu^2\beta_{42} f(x_2^*)\\
+\mu^2\beta_{43} f(x_3^*) + \mu^2\beta_{44} f(x_4^*) + \gamma_1\mu^2 f(x_1^*) = 0.
\end{cases}
\tag{2.9}
$$

Since $x_5^* = 0, x_6^* = 0, \cdots, x_{12}^* = 0$, system (2.9) changes to:

$$
\begin{cases}
-\mu^2 r_1 x_1^* + \mu^2\beta_{11} f(x_1^*) + \mu^2\beta_{13} f(x_3^*) + \mu^2\beta_{14} f(x_4^*) + \gamma_2\mu^2 f(x_2^*) = 0,\\
-\mu^2 r_2 x_2^* + \mu^2\beta_{21} f(x_1^*) + \mu^2\beta_{22} f(x_2^*) + \mu^2\beta_{24} f(x_4^*) + \gamma_3\mu^2 f(x_3^*) = 0,\\
-\mu^2 r_3 x_3^* + \mu^2\beta_{31} f(x_1^*) + \mu^2\beta_{32} f(x_2^*) + \mu^2\beta_{33} f(x_3^*) + \gamma_4\mu^2 f(x_4^*) = 0,\\
-\mu^2 r_4 x_4^* + \mu^2\beta_{42} f(x_2^*) + \mu^2\beta_{43} f(x_3^*) + \mu^2\beta_{44} f(x_4^*) + \gamma_1\mu^2 f(x_1^*) = 0.
\end{cases}
\tag{2.10}
$$

System (2.10) can be written as follows:

$$
CF(X^*) = DX^* \tag{2.11}
$$

where $X^* = (x_1^*, x_2^*, x_3^*, x_4^*)^T, F(X^*) = (f(x_1^*), f(x_2^*), f(x_3^*), f(x_4^*))^T$, and the matrix D is a diagonal matrix, $D = diag(r_1, r_2, r_3, r_4)$. From condition (2.1), when $x_i^* > 0$ then $f(x_i^*) > 0$, when $x_i^* < 0$ then $f(x_i^*) < 0$, $(i = 1, 2, 3, 4)$. Noting that $r_i > 0$, $(i = 1, 2, 3, 4)$, and C is not a positive definite matrix. Therefore, on the one hand, the right hand of (2.11) is greater than zero as $x_i^* > 0$, its left hand is not guaranteed to be greater than zero. On the other hand, while the right hand of (2.11) is less than zero as $x_i^* < 0$, $(i = 1, 2, 3, 4)$, its left hand cannot be proved to be less than zero. In this context, condition $f(0) = 0$ implies that system (2.11), namely (2.9), has a unique zero solution. Thus, system (1.4) has a unique equilibrium point and it is exactly the zero point. Obviously, the zero point is not only the equilibrium point of (2.9) but also the equilibrium point of (2.8). □

We adopt the following norms of vectors and matrices in this paper [11]: $\|x(t)\| = \sum_{i=1}^{12} |x_i(t)|$, $\|A\| = \max_{1\le j\le 12} \sum_{i=1}^{12} |a_{ij}|$, the measure $\mu(A)$ of a matrix A is defined by $\mu(A) = \lim_{\theta\to 0^+} \frac{\|I+\theta A\|-1}{\theta}$, which for the chosen norms reduces to $\mu(A) = \max_{1\le j\le 12}[a_{jj} + \sum_{i=1,i\neq j}^{12} |a_{ij}|]$. $A > 0$ (respectively, $A < 0$) which indicates that A is a positive (negative) definite matrix.

Definition 2.1. *The trivial solution of system (2.9) is unstable, if there exists at least one component of the trivial solution which is unstable.*

3. Existence of periodic solutions

Theorem 3.1. *Assume that the system (1.4) has a unique equilibrium point for a given set of parameters. If the following condition holds*

$$(\| B \|)e\tau_* \exp(-\tau_*|\mu(A)|) > 1 \tag{3.1}$$

where $\tau_* = \min\{\tau_1, \tau_2, ..., \tau_4\}$, then the unique equilibrium point of the system (1.4) is unstable and it generates a limit cycle, namely, a periodic solution.

Proof. According to Chafee's criterion, we must prove that the unique equilibrium point of system (1.4) is unstable. Since system (2.7) is an equivalent system of system (1.4), we shall prove that the unique equilibrium point of system (2.7) which is exactly the zero point is unstable. Noting that system (2.8) is a linearized version of system (2.7). Thus, in order to prove the instability of the equilibrium point of system (2.7), first, we prove that the equilibrium point is unstable in system (2.8).

Consider a special case of system (2.8) as follows:

$$X'(t) = AX(t) + BX(t-\tau_*) \tag{3.2}$$

where $\tau_* = \min\{\tau_1, \tau_2, \tau_3, \tau_4\}$. Noting that $|x_i(t)| = x_i(t)$ as $x_i(t) > 0$ and $|x_i(t)| = -x_i(t)$ as $x_i(t) < 0$, $(i = 1, 2, \cdots, 12)$. From (3.2), when each $x_i(t) > 0$, we have:

$$\frac{d|X(t)|}{dt} = AX(t) + BX(t-\tau_*)$$

and each $x_i(t) < 0$ one can obtain

$$\frac{d|X(t)|}{dt} = A(-X(t)) + B(-X(t-\tau_*)) ,$$

noting that $a_{99} < 0, a_{1010} < 0, a_{1111} < 0, a_{1212} < 0$. Therefore, we have:

$$\frac{d(\sum_{i=1}^{12} |x_i(t)|)}{dt} \le \mu(A) \sum_{i=1}^{12} |x_i(t)| + \| B \| \sum_{i=1}^{12} |x_i(t-\tau_*)| .$$

Specially, for the scalar time delay differential equation:

$$\frac{dy(t)}{dt} = \mu(A)y(t) + \| B \| \, y(t-\tau_*) , \tag{3.3}$$

if the unique equilibrium of system (3.3) is stable, then the characteristic equation associated with (3.3) given by:

$$\lambda = \mu(A) + \| B \| e^{-\lambda\tau_*} \tag{3.4}$$

will have a real negative root say λ_0, and we have from (3.4):

$$|\lambda_0| \geq \| B \| e^{|\lambda_0|\tau_*} - |\mu(A)| \,.$$

Using the formula $e^x \geq ex$ for $x \geq 0$ one can get

$$1 \geq \frac{\| B \| e^{|\lambda_0|\tau_*}}{|\mu(A)| + |\lambda_0|} = \frac{\| B \| \tau_* e^{-|\mu(A)|\tau_*} e^{(|\mu(A)|+|\lambda_0|)\tau_*}}{(|\mu(A)| + |\lambda_0|)\tau_*} \geq (\| B \| e\tau_*) e^{-\tau_*|\mu(A)|} \,.$$

The last inequality contradicts Eq.(3.1). Hence, our claim regarding the instability of the equilibrium point of system (3.3) is valid. Based on the comparison theorem of differential equation, we have $\sum_{i=1}^{12} |x_i(t)| \leq y(t)$. According to the definition of the instability of the trivial solution, for an arbitrary $\varepsilon > 0$, there exists a sequence $\{t_k\}_1^{+\infty}$ such that $|y(t_k)| > \varepsilon$, where $y(t)$ represents the trivial solution of system (3.3). Since $\sum_{i=1}^{12} |x_i(t)| \leq y(t)$, this means that there exists a subsequence $\{t_{k_j}\}$ of the sequence $\{t_k\}$ such that $\sum_{i=1}^{12} |x_i(t_{k_j})| = y(t_{k_j})$. Therefore, there exists at least one $x_i(t)$, and without loss of generality, we assume that $|x_1(t_{k_j})| > \dfrac{\varepsilon}{12}$. Since ε is an arbitrary sufficiently small positive number, $\dfrac{\varepsilon}{12}$ is also an arbitrary sufficiently small positive number. Thus, $x_1(t)$ is unstable. According to the definition 2.1, the instability of the component $x_1(t)$ implies that the trivial solution of (3.2) is unstable.

Now we prove that the trivial solution of system (2.8) is also unstable. System (3.2) is a special case of (2.8). Obviously, $x_i(t - \tau_*)$ is equivalent to $x_i(t - \tau_i)(\tau_* \leq \tau_i)(i = 1, 2, \cdots, 12)$ as $t(> 0)$ is sufficiently large. So, we still have $|x_1(t_{k_j})| > \dfrac{\varepsilon}{12}$ as t_{k_j} is sufficiently large ($t_{k_j} > \tau_* + \tilde{\tau}$, where $\tilde{\tau} = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$). This means that the trivial solution of system (2.8) is unstable. We can then prove that the trivial solution of system (2.7) is unstable. Indeed, noting that system (2.8) is a linearized version of system (2.7), in other words, system (2.7) is a disturbing system of (2.8). However, the disturbing term only affects the final four equations of system (2.7). The solution $x_1(t)$ to $x_8(t)$ are the same, both in system (2.7) and (2.8). From definition 2.1, the instability of component $x_1(t)$ both in system (2.7) and (2.8) suggests that the trivial solution of system (2.7)(thus the equivalent system (1.4)) is unstable. Since all solutions of system (1.4) are bounded, the instability of the unique equilibrium point together with the boundedness of the solutions lead system (1.4) to generate a limit cycle, namely, a periodic solution based on Chafee's criterion. □

Theorem 3.2. *Assume that system (1.4) has a unique equilibrium point for given parameters. If the following condition holds:*

$$\| B \| + \mu(A) > 0 \tag{3.5}$$

then the unique equilibrium point of system (1.4) is unstable. System (1.4) generates a limit cycle, namely, a periodic solution.

Proof. We still prove that the trivial solution of system (3.2) is unstable. The characteristic equation of system (3.4) is the following:

$$\lambda = \mu(A) + \| B \| e^{-\lambda\tau_*} \,,$$

namely

$$\lambda - \mu(A) - \parallel B \parallel e^{-\lambda\tau_*} = 0\,, \tag{3.6}$$

and there exists a positive characteristic root of equation (3.6) under the restrictive condition (3.5). Indeed, let $f(\lambda) = \lambda - \mu(A) - \parallel B \parallel e^{-\lambda\tau_*}$, then $f(\lambda)$ is a continuous function of λ. Obviously, $f(0) = -(\mu(A) + \parallel B \parallel) < 0$, and while as $\lambda > 0$ is sufficiently large, $e^{-\lambda\tau_*}$ will be sufficiently small. Therefore, there exists a $\tilde{\lambda} > 0$ such that $f(\tilde{\lambda}) = \tilde{\lambda} - \mu(A) - \parallel B \parallel e^{-\tilde{\lambda}\tau_*} > 0$. According to the well known Intermediate Value Theorem, there exists a positive value of λ say $\lambda_0, \lambda_0 \in (0, \tilde{\lambda})$ such that $\lambda_0 - \mu(A) - \parallel B \parallel e^{-\lambda_0\tau_*} = 0$. In other words, equation (3.6) has a positive characteristic root. Therefore, the trivial solution of system (3.3) is unstable. Similar to the proof of Theorem 3.1, the unique equilibrium point of system (2.7) (or the equivalent system (1.4)) is unstable. According to the Chafee's criterion, system (1.4) has a limit cycle, namely, a periodic solution. □

Theorem 3.3. *Assume that system (1.4) has a unique equilibrium point for a given set of parameters. Let $\alpha_1, \alpha_2, \cdots, \alpha_{12}$ represent the eigenvalues of matrix A, and $\rho_1, \rho_2, \cdots, \rho_{12}$ the eigenvalues of matrix B, then if there exists one eigenvalue α_j which is a positive real number, then the unique equilibrium of system (1.4) is unstable. System (1.4) generates a limit cycle, namely, a periodic solution.*

Proof. Considering the system (3.2), its characteristic equation is as follows:

$$det(\lambda I_{12} - A - Be^{-\lambda\tau_*}) = 0\,, \tag{3.7}$$

where I_{12} is a 12×12 identity matrix. Since the eigenvalues of matrix A are $\alpha_1, \alpha_2, \cdots, \alpha_{12}$, and the eigenvalues of matrix B are $\rho_1, \rho_2, \cdots, \rho_{12}$, equation (3.7) changes to the following:

$$\prod_{i=1}^{12}(\lambda - \alpha_i - \rho_i e^{-\lambda\tau_*}) = 0\,.$$

Obviously, there is a $\rho_j = 0$. Without loss of generality, assuming that $\rho_1 = 0$, and α_1 is a positive real eigenvalue, then we have

$$\lambda - \alpha_1 = 0\,.$$

This means that there exists a positive eigenvalue of system (3.7), implying that the trivial solution of system (3.2) is unstable. As we did for Theorem 3.1, one can prove that system (1.4) has a limit cycle, namely, a periodic solution. □

4. Simulation result

These simulations were performed by using the equivalent system (2.7) of (1.4). Firstly we selected the activation function as $f(x) = \tanh(x)$. Then $f'(x) = 1 - \tanh^2(x)$, $f''(x) = 2(\tanh^3(x) - \tanh(x))$, so $f'(0) = 1$, and $f''(0) = 0$. The parameters $\mu = 0.58$, $r_1 = 0.15$, $r_2 = 0.25$, $r_3 = 0.36$, $r_4 = 0.24$; $\beta_{11} = 1.65$, $\beta_{13} = 2.75$, $\beta_{14} = 1.35$, $\beta_{21} = 3.45$, $\beta_{22} = 1.85$, $\beta_{24} = 2.35$, $\beta_{31} = 1.32$, $\beta_{32} = 1.42$, $\beta_{33} = 1.28$, $\beta_{42} = 2.42$, $\beta_{43} = 1.16$, $\beta_{44} = 2.95$; $\gamma_1 = 3.95$, $\gamma_2 = 1.25$, $\gamma_3 = 0.75$, $\gamma_4 = 0.64$. The characteristic values of matrix C are $2.1832, 6.2940, -0.3736 \pm 1.7074i$. Therefore, C is not a positive definite matrix. From Lemma 2.1, this system has a unique equilibrium point, namely, the zero point. The characteristic values of matrix A are $0.2993, -1.2335, 0.1950 \pm 0.3623i, -0.0886 \pm$

$0.6297i, -0.4730 \pm 0.7303i, -0.8543 \pm 0.6348i, -1.1320 \pm 0.3673i$. Obviously, there is a positive characteristic value 0.2993 in matrix A. The solutions are oscillatory based on Theorem 3.3 (see Fig.1). In order to see the effect of μ, we changed μ from 0.58

Fig.1 Oscillation behavior of the solutions, delays: [1.5, 1.8, 2.2, 1.4], mu=0.58. activation function: tanh(x).

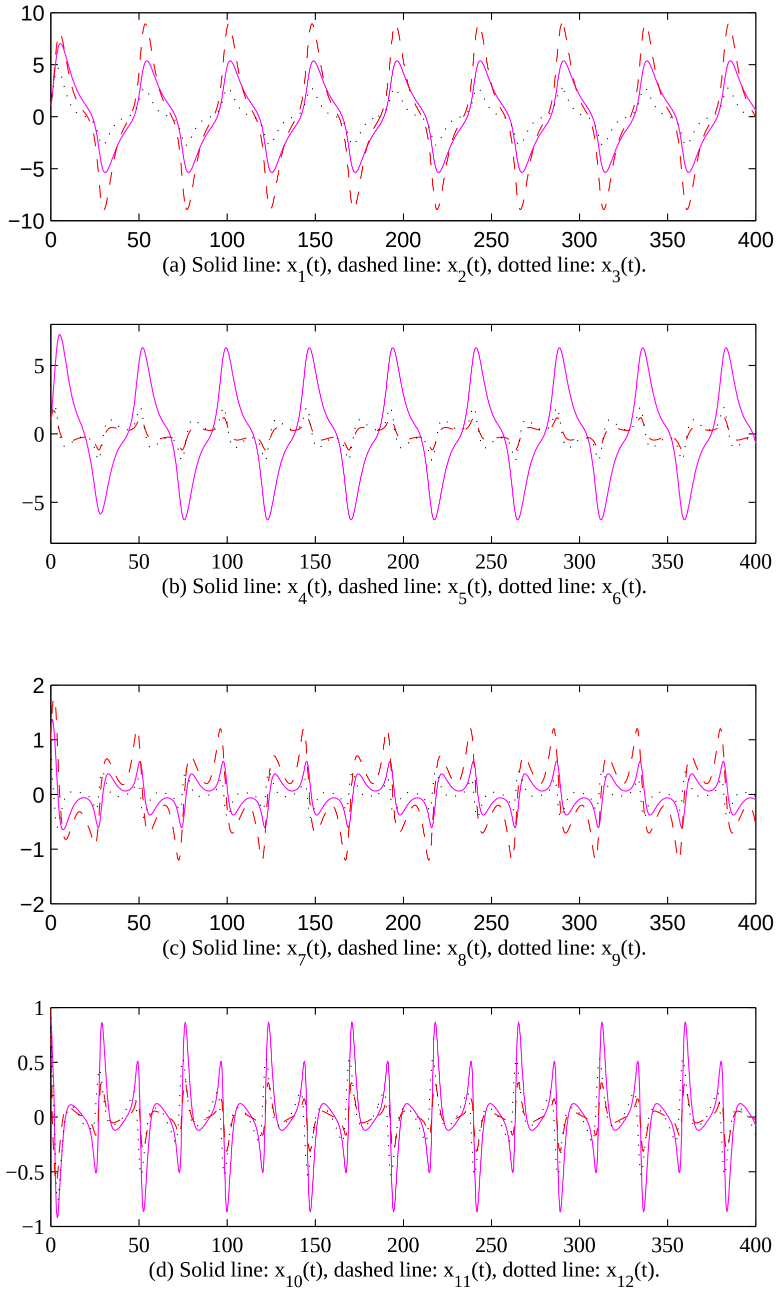
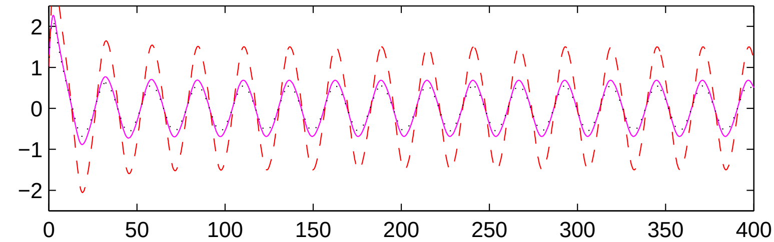
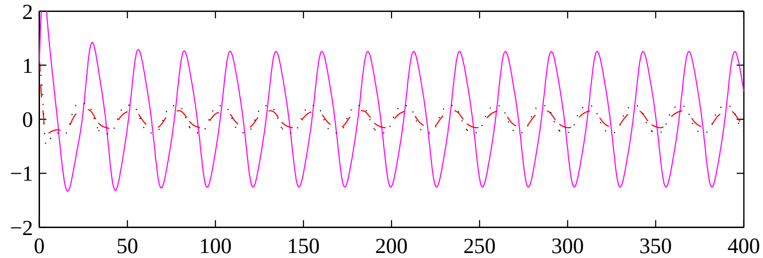


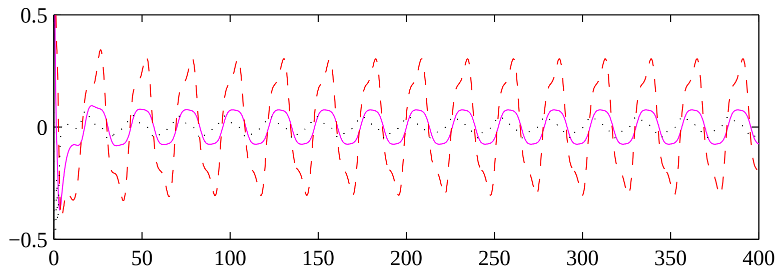
Fig.2 Oscillation behavior of the solutions, delays: [1.5, 1.8, 2.2, 1.4], mu=2.5.
activation function: tanh(x).



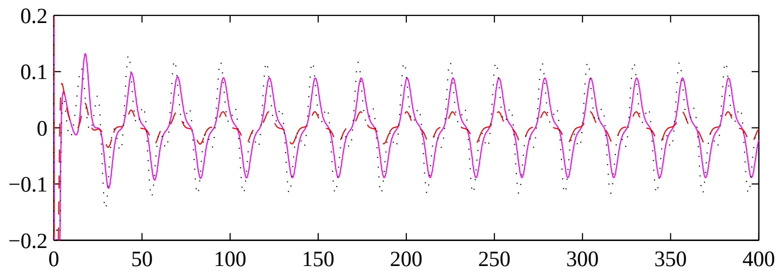
(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$.



(b) Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.



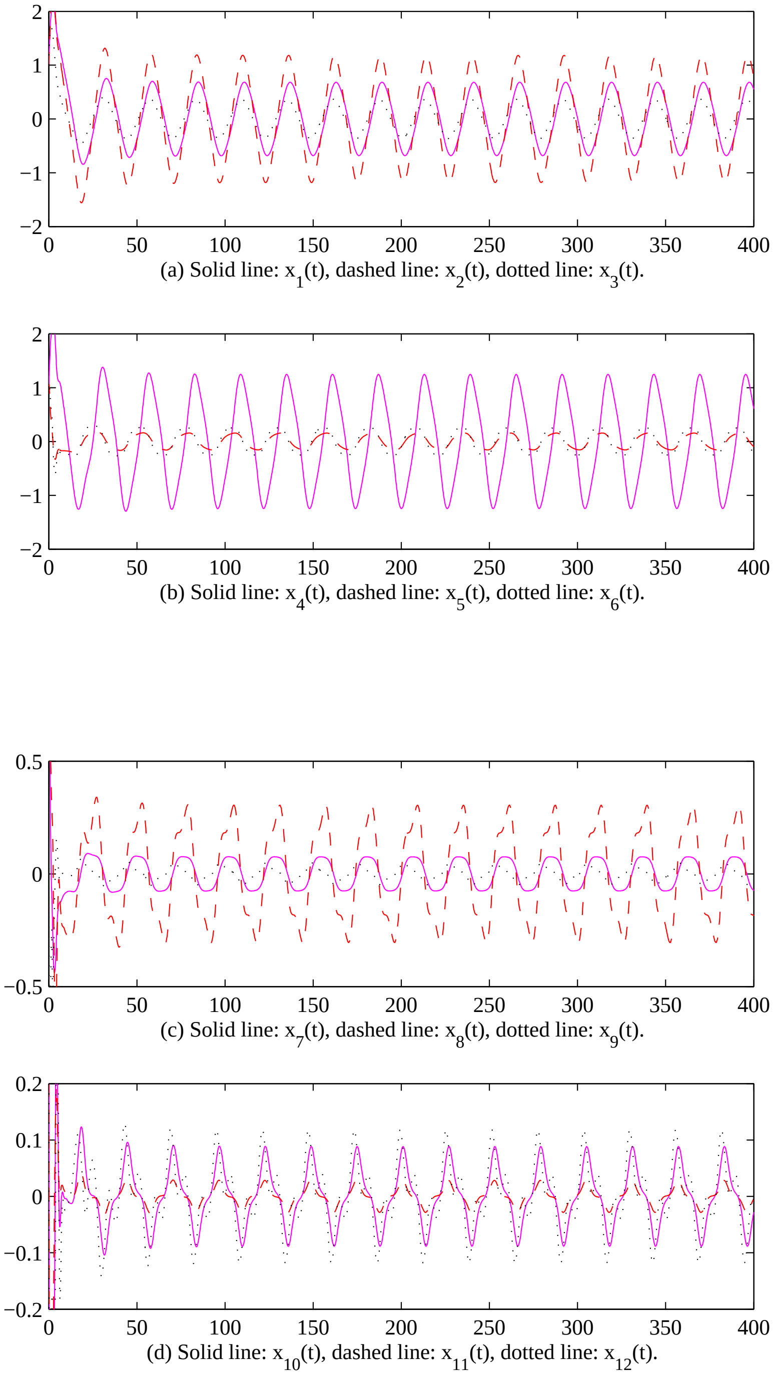
(c) Solid line: $x_7(t)$, dashed line: $x_8(t)$, dotted line: $x_9(t)$.



(d) Solid line: $x_{10}(t)$, dashed line: $x_{11}(t)$, dotted line: $x_{12}(t)$.

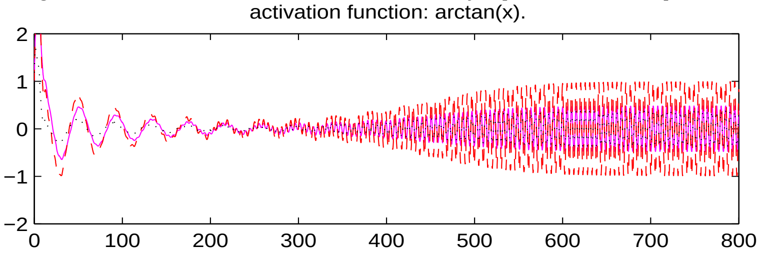
to 2.5, keeping the other parameters as the above, the oscillation was maintained. In this case, $\mu(A) = 9.375$ and $\| B \| + \mu(A) > 0$. The conditions of Theorem 3.2 were satisfied. However, the oscillatory amplitude and frequency changed (see Fig.2). Then we selected activation function as $f(x) = \arctan(x)$, keeping all parameters similar to those used to generate Fig.2, , we also have $f'(0) = 1$ and $f''(0) = 0$. We see that the oscillatory frequency and amplitude remain the same (see Fig.3), implying that the

Fig.3 Oscillation behavior of the solutions, delays: [1.5, 1.8, 2.2, 1.4], mu=2.5. activation function: arctan (x).

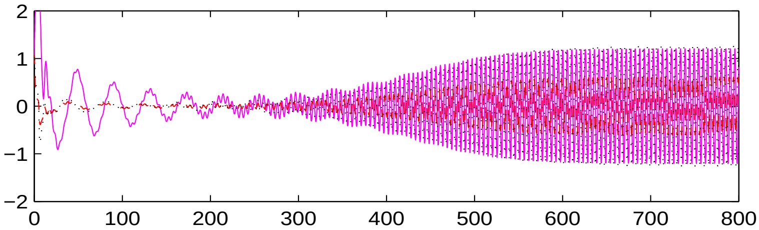


oscillatory behavior is just a little effected by the activation functions. However, when we increase the time delays, the oscillatory frequency changes greatly (see Fig.4).

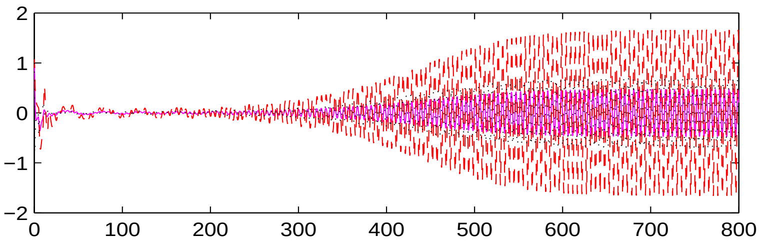
Fig.4 Oscillation behavior of the solutions, delays: [4.5, 4.8, 5.2, 4.4], mu=2.5.
activation function: arctan(x).



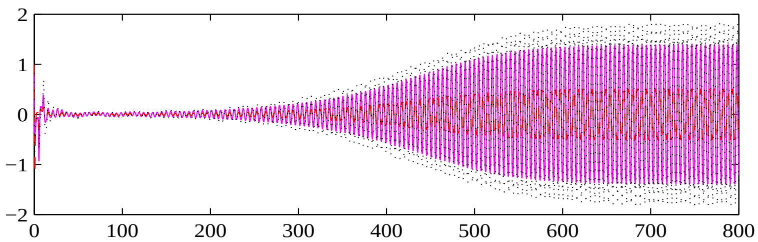
(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$.



(b) Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.



(c) Solid line: $x_7(t)$, dashed line: $x_8(t)$, dotted line: $x_9(t)$.



(d) Solid line: $x_{10}(t)$, dashed line: $x_{11}(t)$, dotted line: $x_{12}(t)$.

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