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DECOMPOSING A GRAPH INTO TWO SUBGRAPHS WITH PRESCRIBED PARITIES OF VERTEX DEGREES

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ABSTRACT. Given a finite graph G and a parity 2-dimensional vector-function $\pi = (\pi_1, \pi_2)$: $V(G) \to \{0,1\}\times\{0,1\}$, a parity decomposition of (G, π) is an ordered 2-partition (E_1, E_2) of $E(G)$ such that the degree functions $d_{G[E_i]}$ $(i = 1, 2)$ of the subgraphs induced by the partite sets are in parity accordance with the respective components of π, i.e., $d_{G[E_i]}(v) \equiv_2 \pi_i(v)$ for each vertex v of $G[E_i]$. We show that the decision problem whether (G, π) admits a parity decomposition is solvable in polynomial time. Contrarily, we conjecture that the analogous decision problem involving a parity 3-dimensional vector-function and concerning the existence of an adequate ordered 3-partition is not solvable in polynomial time.

1. INTRODUCTION

All considered graphs are finite, loops and multiple edges are allowed. For general terminology and notation we refer the reader to [1] or to the end of this section. A graph is said to be *odd* (resp. *even*) if all its vertex degrees are odd (resp. even). Recently, Kano et al. [3] characterized the family of graphs that can be decomposed into (at most) two odd subgraphs and gave a polynomial time algorithm for finding such a decomposition or showing its non-existence. The same paper contains a structural characterization of graphs that can be decomposed into an odd subgraph and an even subgraph. The present article acts as a follow-up to [3]. Namely, we generalize the above mentioned results of Kano et al. through the notion 'parity decomposition' defined as follows.

Given a graph G, a *parity 2-dimensional vector-function* π is an arbitrary assignment $\pi = (\pi_1, \pi_2) : V(G) \to \{0,1\}\times\{0,1\}$. A *parity decomposition* of (G, π) is an ordered 2-partition (E_1, E_2) of $E(G)$ such that the degree functions $d_{G[E_i]}$ $(i = 1, 2)$ of the subgraphs induced by the partite sets are in parity accordance with the respective components of π, i.e., are such that the congruence $d_{G[E_i]}(v) \equiv_2 \pi_i(v)$ holds for each vertex v of $G[E_i]$. We prove that the decision problem whether (G, π) admits a parity decomposition is solvable in polynomial time. Contrarily, we conjecture that the analogous decision problem involving a parity 3-dimensional vector-function and dealing with the existence of an adequate ordered 3-partition is not solvable in polynomial time.

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Incidentally, in the following two particular cases regarding $\pi = (\pi_1, \pi_2)$, a parity decomposition of (G, π) amounts to decomposing G into:

(1) two odd subgraphs, if $\pi_1 = \pi_2 \equiv 1$;
(2) an odd subgraph and an even subgraph if $\pi_1 \equiv 1$ and $\pi_2 \equiv 0$ (or vice versa).

Thus, our work here can be seen as a natural generalization of [3]. We postpone proofs to the next section, and end this one with some common notions and facts that are used throughout. The final section of the article conveys some possibilities for further related work.

General terminology and notation. An *ordered k-partition* of a given set S is a family $(S_1, S_2, \dots, S_k)$ of (possibly empty) pairwise disjoint subsets $S_i \subseteq S$ whose union gives S. An ordered k-partition of the edge set $E(G)$ of a graph G can be equivalently interpreted as an *edge-colouring* of G with the *colour set* $[k] = \{1, 2, \dots, k\}$, i.e., a (not necessarily surjective) mapping $E(G) \to [k]$; namely, colour i is assigned to the edge $e \in E(G)$ if and only if $e \in S_i$. For a subset S of $V(G) \cup E(G)$, $G[S]$ denotes the subgraph of G induced on S. The graph with no vertices (and hence no edges) is the *null graph*. For a vertex $v \in V(G)$, $N_G(v)$ is the set of neighbouring vertices of v and $E_G(v)$ is the set of incident edges to v. The size of $E_G(v)$ (with every loop counted twice) is the *degree* $d_G(v)$, and v is said to be an *odd* (resp. *even*) *vertex* of G if $d_G(v)$ is odd (resp. even).

If X and Y are (not necessarily disjoint) subsets of $V(G)$, then $E[X, Y]$ denotes the set of edges with one end in X and the other end in Y, and $e(X, Y)$ is their number. A graph G is *connected* if for every partition of $V(G)$ into two nonempty sets X and Y, it holds that $E[X, Y] \neq \emptyset$. The maximal connected subgraphs of a graph are its *components* (of connectedness). The problem of determining the components of any given graph is solvable in polynomial time. Given a graph G and an even-sized subset T of $V(G)$, a spanning subgraph H (of G) is called a *T-join* of G if $T = V_o(H)$. The symmetric difference of a T-join and an S-join is clearly a $T \oplus S$-join (notation $\oplus$ denotes both the symmetric difference of spanning subgraphs and of sets). Using this simple fact, it can be readily deduced (see e.g. [5]) that every connected G admits a T-join; moreover, a T-join of G can always be found in polynomial time.

2. Results

Given a parity 2-dimensional vector-function $\pi = (\pi_1, \pi_2)$, we distinguish between two types of vertices $v \in V(G)$ depending on the parity compliance (or non-compliance) of the degree $d_G(v)$ with the sum $\pi_1(v) + \pi_2(v)$: namely, the *compliant vertices* are those v's satisfying $d_G(v) \equiv_2 \pi_1(v) + \pi_2(v)$, whereas the *non-compliant vertices* are those v's for which the previous congruence fails to hold. Denote by $\mathcal{C}$ and $\mathcal{N}$ the respective subsets of $V(G)$, and let $\mathcal{X}$ and $\mathcal{Y}$ be the respective sets of components of $G[\mathcal{N}]$ and $G[\mathcal{C}]$. First we single out two necessary conditions for parity decomposability of (G, π).

Proposition 2.1. *If (G, π) admits a parity decomposition, then it holds that*

(NC_1) *Each $v \in \mathcal{N}$ has $\pi(v) \neq (0, 0)$;*
(NC_2) *No $X \in \mathcal{X}$ intersects both $\pi^{-1}(0, 1)$ and $\pi^{-1}(1, 0)$.*

Proof. Interpret any parity decomposition of (G, π) as an edge-colouring $E(G) \to \{1, 2\}$. Assume such an (adequate) edge-colouring is applied to G. The failure of (i) would imply the existence of an odd vertex v of G such that $\pi_1(v) = \pi_2(v) = 0$. But then the local colouring of $E_G(v)$ is surely inadequate (no odd number can be written as a sum of two even numbers). This contradiction establishes (NC_1).

In order to demonstrate (NC_2), start by observing that for every non-compliant vertex v the edge set $E_G(v)$ is not dichromatic, i.e., at most one of the colours 1 and 2 appears on the edges incident to v. Consequently, for every $X \in \mathcal{X}$ the edge set $E[V(X), V(G)]$ is monochromatic. Now, suppose there is such an $X \in \mathcal{X}$ that intersects both $\pi^{-1}(0,1)$ and $\pi^{-1}(1,0)$, say $v' \in V(X) \cap \pi^{-1}(0,1)$ and $v'' \in V(X) \cap \pi^{-1}(1,0)$. Since v', v'' are clearly even vertices of G, it must be that $E_G(v')$ is coloured entirely with 1 whereas $E_G(v'')$ is coloured entirely with 2. However, this contradicts the monochromaticity of the edge set $E(X)$ (as $E(X) \subseteq E[V(X), V(G)]$). □

Whenever (NC_1) and (NC_2) are fulfilled, call (G, π) an *appropriate pair*. For every such (G, π) we devise the following ad-hoc notation. Let $\mathcal{X}'$ and $\mathcal{X}''$, respectively, denote the (mutually disjoint) collections of components $X \in \mathcal{X}$ that intersect $\pi^{-1}(0,1)$ and $\pi^{-1}(1,0)$, and let $\mathcal{X}''' = \mathcal{X} \backslash (\mathcal{X}' \cup \mathcal{X}'')$. Thus, in view of (NC_1), the set $\mathcal{X}'''$ consists of the components of $G[\mathcal{N}]$ which intersect solely $\pi^{-1}(1,1)$, and, in view of (NC_2), $\{\mathcal{X}', \mathcal{X}'', \mathcal{X}'''\}$ constitutes a 3-partition of $\mathcal{X}$. We are ready to characterize parity decomposability for appropriate pairs.

Theorem 2.1. *An appropriate pair (G, π) admits a parity decomposition if and only if there exists a subset $\mathcal{S} \subseteq \mathcal{X}'''$ such that for each $Y \in \mathcal{Y}$ it holds that*

$$e\Big(\bigcup_{X \in \mathcal{S}} V(X), V(Y)\Big) \equiv_2 |V(Y) \cap \pi_1^{-1}(1)| + e\Big(\bigcup_{X \in \mathcal{X}'} V(X), V(Y)\Big). \tag{2.1}$$

Proof. For each subset $S \subseteq V(G)$, define $S^0 = S \cap \pi_1^{-1}(0)$ and $S^1 = S \cap \pi_1^{-1}(1)$. Assuming that (G, π) admits a parity decomposition, consider the accompanying edge-colouring of G with the colour set $\{1, 2\}$. As already observed (in the proof of Proposition 2.1), for every $X \in \mathcal{X}$ the edge set $E[V(X), V(G)]$ is monochromatic; moreover, for every $X \in \mathcal{X}'$ (resp. $X \in \mathcal{X}''$) all edges incident to $V(X)$ are coloured with 1 (resp. 2).

Define $\mathcal{S}$ to be the collection of those X's belonging to $\mathcal{X}'''$ whose incident edges are coloured with 1, and denote $\mathcal{S}^c = \mathcal{X}''' \backslash \mathcal{S}$. Thus, for each $Y \in \mathcal{Y}$ the whole set $E[\bigcup_{X \in \mathcal{S} \cup \mathcal{X}'} V(X), V(Y)]$ is coloured with 1, whereas whole of $E[\bigcup_{X \in \mathcal{S}^c \cup \mathcal{X}''} V(X), V(Y)]$ is coloured with 2. For each $v \in V(G)$, let $d_1(v)$ denote the degree of v in the spanning subgraph of G whose edge set is the colour class 1. Observe that for each $v \in \mathcal{C}$, $d_1(v)$ is even or odd depending on whether $\pi_1(v)$ equals 0 or 1. Therefore

$$\begin{aligned} e\Big(\bigcup_{X \in \mathcal{S} \cup \mathcal{X}'} V(X), V(Y)\Big) &\equiv_2 \sum_{v \in V(Y)} d_1(v) \\ &= \sum_{v \in V(Y)^0} d_1(v) + \sum_{v \in V(Y)^1} d_1(v) \\ &\equiv_2 |V(Y)^1|, \end{aligned}$$

which is equivalent to (2.1) (as $\mathcal{S} \cap \mathcal{X}' = \emptyset$).

Proving the other direction, take $\mathcal{S} \subseteq \mathcal{X}'''$ such that (2.1) is fulfilled for each $Y \in \mathcal{C}$, and again let $\mathcal{S}^c$ be its set complement with respect to $\mathcal{X}'''$. Colour with 1 the edges incident to $\bigcup_{X \in \mathcal{S} \cup \mathcal{X}'} V(X)$, and colour with 2 the edges incident to $\bigcup_{X \in \mathcal{S}^c \cup \mathcal{X}''} V(X)$. Thus, the remaining uncoloured part of $E(G)$ is the edge set of $G[\mathcal{C}]$. Extend the colouring to $E(Y)$

for each $Y \in \mathcal{Y}$ in the following manner. With the same meaning of $d_1(v)$, consider the set T_Y defined by

$$T_Y = \{v : v \in V(Y)^1 \text{ and } d_1(v) \text{ is even}\} \cup \{v : v \in V(Y)^0 \text{ and } d_1(v) \text{ is odd}\},$$

and observe that

$$|V(Y)^1| \equiv_2 |T_Y| + e(\bigcup_{X \in \mathcal{S} \cup \mathcal{X}'} V(X), V(Y)). \tag{2.2}$$

Indeed, the last congruence holds since

$$\begin{aligned} |V(Y)^1| &= |\{v : v \in V(Y)^1 \text{ and } d_1(v) \text{ is even}\}| \\ &\qquad + |\{v : v \in V(Y)^1 \text{ and } d_1(v) \text{ is odd}\}| \\ &\equiv_2 |\{v : v \in V(Y)^1 \text{ and } d_1(v) \text{ is even}\}| + \sum_{v \in V(Y)^1} d_1(v) \\ &\equiv_2 |T_Y| + \sum_{v \in V(Y)^0} d_1(v) + \sum_{v \in V(Y)^1} d_1(v) \\ &= |T_Y| + e(\bigcup_{X \in \mathcal{S} \cup \mathcal{X}'} V(X), V(Y)). \end{aligned}$$

From (2.1) and (2.2) it follows that T_Y is even-sized. Colour with 1 the edges of an arbitrary T_Y-join of Y, and colour with 2 the rest of $E(Y)$. This completes an edge-colouring of G which corresponds to a parity decomposition of (G, π). □

The given characterization of parity decomposability can also be interpreted in terms of solvability of a certain system of linear equations over the field $GF(2)$.

Theorem 2.2. *Given an appropriate pair (G, π), let G^* be the simple bipartite graph constructed as follows: its partite sets are $\mathcal{X}''' = \{X_1, X_2, \ldots, X_r\}$ and $\mathcal{C} = \{Y_1, Y_2, \ldots, Y_s\}$, and vertex X_i is joined by an edge with vertex Y_j if and only if $e(V(X_i), V(Y_j))$ is odd. Relate to each X_i a variable x_i and consider the system (S) consisting of s linear equations over the field $GF(2)$ whose j-th equation $(j = 1, 2, \ldots, s)$ reads*

$$\sum_{X_i \in N_{G^*}(Y_j)} x_i \equiv_2 |V(Y_j) \cap \pi_1^{-1}(1)| + e(\bigcup_{X \in \mathcal{X}'} V(X), V(Y_j)).$$

Then (G, π) admits a parity decomposition if and only if the system (S) is solvable.

Proof. Consider an arbitrary $\mathcal{S} \subseteq \mathcal{X}'''$, and define $x(\mathcal{S}) = (x_i)_{i=1}^r$ by setting $x_i = 1$ if $X_i \in \mathcal{S}$, and $x_i = 0$ otherwise. It suffices to observe that for any $Y \in \mathcal{C}$, the subset $\mathcal{S}$ satisfies (2.1) if and only if the r-tuple $x(\mathcal{S})$ satisfies

$$\sum_{X_i \in N_{G^*}(Y)} x_i \equiv_2 |V(Y) \cap \pi_1^{-1}(1)| + e(\bigcup_{X \in \mathcal{X}'} V(X), V(Y)).$$

□

Since row reduction (also known as Gaussian elimination) is an efficient algorithm for solving a system of linear equations over any given field (see e.g. [4]), Theorem 2.2 and its proof imply the following:

Corollary 2.1. *The decision problem whether (G, π) admits a parity decomposition, where π is a parity 2-dimensional vector-function, is solvable in polynomial time. Moreover, in the affirmative case, such a decomposition can be found in polynomial time.*

Let us end this section with a remark on the complexity of the analogous decision problem involving a 2-covering instead of a 2-partition. Namely, given a parity 2-dimensional vector-function π for a graph G, a *parity covering* of (G, π) is an ordered 2-covering (E_1, E_2) of $E(G)$, i.e., a family of (possibly empty) sets E_1, E_2 satisfying $E_1 \cup E_2 = E(G)$, such that $d_{G[E_i]}(v) \equiv_2 \pi_i(v)$ for each vertex v of $G[E_i]$ $(i = 1, 2)$. Then, the decision problem whether (G, π) admits a parity covering is $\mathcal{NP}$-hard. Indeed, it is a well-known fact (see e.g. [2]) that for an arbitrary cubic graph G the following three properties are equivalent:

(i) G is a class 1 graph;
(ii) G admits a nowhere-zero 4-flow;
(iii) G can be covered by 2 even subgraphs.

Moreover, the decision problem whether G satisfies (i) is known to be $\mathcal{NP}$-hard, whereas (iii) is a particular instance of parity covering.

3. FURTHER WORK

The notion 'parity decomposition' clearly does not rely on the parity vector-function being 2-dimensional. Thus, given a graph G and an integral $k \geq 2$, one may equally well consider a parity k-dimensional vector-function $\pi = (\pi_1, \ldots, \pi_k) : V(G) \to \{0, 1\}^k$ and study the existence of a parity decomposition of (G, π), i.e., an ordered k-partition $(E_1, \ldots, E_k)$ of $E(G)$ such that the degree functions $d_{G[E_i]}$ $(i = 1, \ldots, k)$ of the subgraphs induced by the partite sets are in parity accordance with the respective components of π. (Note that we allow for some of the E_i's to be empty.) It is our belief that for $k = 3$ this decision problem is no longer solvable in polynomial time.

Conjecture 3.1. *The decision problem whether (G, π) admits a parity decomposition, where π is a parity 3-dimensional vector-function, is not solvable in polynomial time.*

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