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# OSCILLATORY SOLUTIONS ON A SIX-NEURON INERTIAL NEURAL SYSTEM WITH MULTIPLE DELAYS

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ABSTRACT. In this paper, a six-neuron inertia neural system with multiple delays is investigated. By means of mathematical analysis method, some sufficient conditions to guarantee the existence of oscillatory solution for the model are obtained. Computer simulations are provided to demonstrate the proposed results.

#### 1. INTRODUCTION

In the last few years, bifurcation analysis on various neural networks has been investigated and many excellent and interesting results have been obtained [1],[5]. For example, Wang *et al.* have discussed a simplified six-neuron of two-layer neural network model with delays [1]:

 $\begin{cases} x_1'(t) = -k_1 x_1(t) + c_{14} f_{14}(x_4(t-\tau_1) + c_{15} f_{15}(x_5(t-\tau_1)), \\ x_2'(t) = -k_2 x_2(t) + c_{24} f_{24}(x_4(t-\tau_1) + c_{25} f_{25}(x_5(t-\tau_1) + c_{26} f_{26}(x_6(t-\tau_1)), \\ x_3'(t) = -k_3 x_3(t) + c_{35} f_{35}(x_5(t-\tau_1) + c_{36} f_{36}(x_6(t-\tau_1)), \\ x_4'(t) = -k_4 x_4(t) + c_{41} f_{41}(x_1(t-\tau_2) + c_{42} f_{42}(x_2(t-\tau_2)), \\ x_5'(t) = -k_5 x_5(t) + c_{51} f_{51}(x_1(t-\tau_2) + c_{52} f_{52}(x_2(t-\tau_2) + c_{53} f_{53}(x_3(t-\tau_2)), \\ x_6'(t) = -k_6 x_6(t) + c_{62} f_{62}(x_2(t-\tau_2) + c_{63} f_{63}(x_3(t-\tau_2)). \end{cases}$ (1.1)

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Set  $\tau = \tau_1 + \tau_2$ ,  $u_i(t) = x_i(t - \tau_2)(1 \le i \le 3)$ ,  $u_j(t) = x_j(t)$ ,  $4 \le j \le 6$ ,, and  $k_i = k(1 \le i \le 6)$ , then the linearized system of model (1.1) is the following:

$$\begin{cases} u_1'(t) = -ku_1(t) + a_1u_4(t-\tau) + b_1u_5(t-\tau), \\ u_2'(t) = -ku_2(t) + c_1u_4(t-\tau) + a_2u_5(t-\tau) + b_2u_6(t-\tau), \\ u_3'(t) = -ku_3(t) + c_2u_5(t-\tau) + a_3u_6(t-\tau), \\ u_4'(t) = -ku_4(t) + a_1u_1(t) + b_1u_2(t), \\ u_5'(t) = -ku_5(t) + c_1u_1(t) + a_2u_2(t) + b_2u_3(t), \\ u_6'(t) = -ku_6(t) + c_2u_2(t) + a_3u_3(t). \end{cases}$$

By means of matrix decomposition method, the Hopf bifurcation analysis has been investigated. Based on the normal form method and the center manifold theorem, the explicit formulas about the stability of the bifurcating periodic solution and the direction of the Hopf bifurcation are established. Chen *et al.* have studied a simplified three layer-neural network model described by the following system [2]:

$$\begin{cases} x_1'(t) = -kx_1(t) + c_{13}f_{13}(y_1(t-\tau_3) + c_{14}f_{14}(y_2(t-\tau_3), x_2'(t))) + c_{13}f_{13}(y_1(t-\tau_3) + c_{14}f_{14}(y_2(t-\tau_3), x_2'(t))) + c_{13}f_{23}(y_1(t-\tau_3) + c_{24}f_{24}(y_2(t-\tau_3), y_1'(t))) + c_{31}f_{31}(x_1(t-\tau_1) + c_{32}f_{32}(x_2(t-\tau_1))) + c_{35}f_{35}(z_1(t-\tau_4) + c_{36}f_{36}(z_2(t-\tau_4), y_2'(t))) + c_{41}f_{41}(x_1(t-\tau_1) + c_{42}f_{42}(x_2(t-\tau_1))) + c_{45}f_{45}(z_1(t-\tau_4) + c_{46}f_{46}(z_2(t-\tau_4)))) + c_{45}f_{45}(z_1(t-\tau_4) + c_{46}f_{46}(z_2(t-\tau_4)))) + c_{42}f_{45}(z_1(t-\tau_2) + c_{54}f_{54}(y_2(t-\tau_2)))) + c_{42}f_{42}(t) + c_{43}f_{43}(y_1(t-\tau_2) + c_{44}f_{46}(y_2(t-\tau_2)))) + c_{42}f_{42}(t) + c_{43}f_{43}(y_2(t-\tau_2))) + c_{44}f_{44}(y_2(t-\tau_2))) + c_{44}f_{44}(y_2(t-\tau_2)) + c_{44}f_{44}(y_2(t-\tau_2))) + c_{44}f_{44}(y_2(t-\tau_2))) + c_{44}f_{44$$

By analyzing its associated characteristic equation, local stability and the existence of Hopf bifurcation of the system are investigated. By using the normal form method and center manifold theorem, formulas to determine the direction of the Hopf bifurcation and the stability of bifurcating periodic solution are obtained. Xu *et al.* have studied the following six-neuron

BAM neural network model with delays [3]:

$$\begin{cases} x_1'(t) = -\mu_1 x_1(t) + c_{11} f_{11}(y_1(t-\tau_4)) + c_{12} f_{12}(y_2(t-\tau_4)) + c_{13} f_{13}(y_3(t-\tau_4)), \\ x_2'(t) = -\mu_2 x_2(t) + c_{21} f_{21}(y_1(t-\tau_5)) + c_{22} f_{22}(y_2(t-\tau_5)) + c_{23} f_{23}(y_3(t-\tau_5)), \\ x_3'(t) = -\mu_3 x_3(t) + c_{31} f_{31}(y_1(t-\tau_6)) + c_{32} f_{32}(y_2(t-\tau_6)) + c_{33} f_{33}(y_3(t-\tau_6)), \\ y_1'(t) = -\mu_4 y_1(t) + c_{41} f_{41}(x_1(t-\tau_1)) + c_{42} f_{42}(x_2(t-\tau_2)) + c_{43} f_{43}(x_3(t-\tau_3)), \\ y_2'(t) = -\mu_5 y_2(t) + c_{51} f_{51}(x_1(t-\tau_1)) + c_{52} f_{52}(x_2(t-\tau_2)) + c_{53} f_{53}(x_3(t-\tau_3)), \\ y_3'(t) = -\mu_6 y_3(t) + c_{61} f_{61}(x_1(t-\tau_1)) + c_{62} f_{62}(x_2(t-\tau_2)) + c_{63} f_{63}(x_3(t-\tau_3)). \end{cases}$$

$$(1.2)$$

In order to analyze the existence of bifurcating periodic solution, the authors assume that  $\tau_1 + \tau_4 = \tau_2 + \tau_5 = \tau_3 + \tau_6 = \tau$  in model (1.2). By analyzing the associated characteristic transcendental equation, the linear stability of the model and Hopf bifurcation are demonstrated. Recently, Ge and Xu [4], have investigated the following four-neuron inertial neural system with multiple delays:

$$\begin{cases} v_1''(t) = -v_1'(t) - \mu_1 v_1(t) + a_1 f(z_1(t-\tau_1)) + a_2 f(z_2(t-\tau_1)), \\ v_2''(t) = -v_2'(t) - \mu_2 v_2(t) + b_1 f(z_1(t-\tau_2)) + b_2 f(z_2(t-\tau_2)), \\ z_1''(t) = -z_1'(t) - \mu_3 z_1(t) + c_1 f(v_1(t-\tau_3)) + c_2 f(v_2(t-\tau_4)), \\ z_2''(t) = -z_2'(t) - \mu_4 z_2(t) + d_1 f(v_1(t-\tau_3)) + d_2 f(v_2(t-\tau_4)). \end{cases}$$
(1.3)

Under the condition  $\tau_1 + \tau_3 = \tau_2 + \tau_4 = \tau$  in model (1.3), and set the new state variables  $w_1(t) = v_1(t - \tau_3), w_2(t) = v_2(t - \tau_4), w_3(t) = z_1(t), w_4(t) = z_2(t)$ , model (1.3) is equivalent to only one delay system. The linear stability of the model is investigated and Hopf bifurcation of the trivial equilibrium point is demonstrated. Periodic solutions bifurcating from the trivial equilibrium point are obtained analytically by using the perturbation scheme without the normal form method and center manifold theory.

Motivated by the above models, in this paper we will study the following six-neuron inertial neural system with multiple delays:

$$\begin{cases} v_1''(t) = -r_1v_1'(t) - \mu_1v_1(t) + c_{11}f_{11}(z_1(t - \sigma_{21})) + c_{12}f_{12}(z_2(t - \sigma_{22})) \\ + c_{13}f_{13}(w_1(t - \sigma_{31})) + c_{14}f_{14}(w_2(t - \sigma_{32})), \\ v_2''(t) = -r_2v_2'(t) - \mu_2v_2(t) + c_{21}f_{21}(z_1(t - \sigma_{21})) + c_{22}f_{22}(z_2(t - \sigma_{22})) \\ + c_{23}f_{23}(w_1(t - \sigma_{31})) + c_{24}f_{24}(w_2(t - \sigma_{32})), \\ z_1''(t) = -r_3z_1'(t) - \mu_3z_1(t) + c_{31}f_{31}(v_1(t - \sigma_{11})) + c_{32}f_{32}(v_2(t - \sigma_{12})) \\ + c_{33}f_{33}(w_1(t - \sigma_{31})) + c_{34}f_{34}(w_2(t - \sigma_{32})), \\ z_2''(t) = -r_4z_2'(t) - \mu_4z_2(t) + c_{41}f_{41}(v_1(t - \sigma_{11})) + c_{42}f_{32}(v_2(t - \sigma_{12})) \\ + c_{43}f_{43}(w_1(t - \sigma_{31})) + c_{44}f_{44}(w_2(t - \sigma_{32})), \\ w_1''(t) = -r_5w_1'(t) - \mu_5w_1(t) + c_{51}f_{51}(v_1(t - \sigma_{11})) + c_{52}f_{52}(v_2(t - \sigma_{12})) \\ + c_{53}f_{53}(z_1(t - \sigma_{21})) + c_{54}f_{54}(z_2(t - \sigma_{22})), \\ w_2''(t) = -r_6w_2'(t) - \mu_6w_2(t) + c_{61}f_{61}(v_1(t - \sigma_{11})) + c_{62}f_{62}(v_2(t - \sigma_{12})) \\ + c_{63}f_{63}(z_1(t - \sigma_{21})) + c_{64}f_{64}(z_2(t - \sigma_{22})). \end{cases}$$
(1.4)

We pointed out that both  $\tau_1 + \tau_4 = \tau_2 + \tau_5 = \tau_3 + \tau_6 = \tau$  in model (1.2) and  $\tau_1 + \tau_3 = \tau_2 + \tau_4 = \tau$  in model (1.3) are special cases. If time delays  $\sigma_{ij}, i = 1, 2, 3, j = 1, 2$ , are different values in model (1.4), one can hard to use bifurcation method because the existence of bifurcating points in a complex transcendental equation is extremely not easy to find. In this paper we use mathematical analysis method to discuss the existence of oscillatory solution for model (1.4).

For convenience, let  $\sigma_{11} = \tau_1, \sigma_{12} = \tau_3, \sigma_{21} = \tau_5, \sigma_{22} = \tau_7, \sigma_{31} = \tau_9, \sigma_{32} = \tau_{11}$ , model (1.4) can be rewritten as the following equivalent system:

$$\begin{cases} x_1'(t) &= x_2(t), \\ x_2'(t) &= -r_1 x_2(t) - \mu_1 x_1(t) + c_{11} f_{11}(x_5(t-\tau_5)) + c_{12} f_{12}(x_7(t-\tau_7))) \\ &+ c_{13} f_{13}(x_9(t-\tau_9)) + c_{14} f_{14}(x_{11}(t-\tau_{11})), \\ x_3'(t) &= x_4(t), \\ x_4'(t) &= -r_2 x_4(t) - \mu_2 x_3(t) + c_{21} f_{21}(x_5(t-\tau_5)) + c_{22} f_{22}(x_7(t-\tau_7))) \\ &+ c_{23} f_{23}(x_9(t-\tau_9)) + c_{24} f_{24}(x_{11}(t-\tau_{11})), \\ x_5'(t) &= x_6(t), \end{cases}$$

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$$\begin{cases} x_{6}'(t) = -r_{3}x_{6}(t) - \mu_{3}x_{5}(t) + c_{31}f_{31}(x_{1}(t-\tau_{1})) + c_{32}f_{32}(x_{3}(t-\tau_{3})) \\ + c_{33}f_{33}(x_{9}(t-\tau_{9})) + c_{34}f_{34}(x_{11}(t-\tau_{11})), \\ x_{7}'(t) = x_{8}(t), \\ x_{8}'(t) = -r_{4}x_{8}(t) - \mu_{4}x_{7}(t) + c_{41}f_{41}(x_{1}(t-\tau_{1})) + c_{42}f_{32}(x_{3}(t-\tau_{3})) \\ + c_{43}f_{43}(x_{9}(t-\tau_{9})) + c_{44}f_{44}(x_{11}(t-\tau_{11})), \\ x_{9}'(t) = x_{10}(t), \\ x_{10}'(t) = -r_{5}x_{10}(t) - \mu_{5}x_{9}(t) + c_{51}f_{51}(x_{1}(t-\tau_{1})) + c_{52}f_{52}(x_{3}(t-\tau_{3})) \\ + c_{53}f_{53}(x_{5}(t-\tau_{5})) + c_{54}f_{54}(x_{7}(t-\tau_{7})), \\ x_{11}'(t) = x_{12}(t), \\ x_{12}'(t) = -r_{6}x_{12}(t) - \mu_{6}x_{11}(t) + c_{61}f_{61}(x_{1}(t-\tau_{1})) + c_{62}f_{62}(x_{3}(t-\tau_{3})) \\ + c_{63}f_{63}(x_{5}(t-\tau_{5})) + c_{64}f_{64}(x_{7}(t-\tau_{7})). \end{cases}$$
(1.5)

where the activation functions  $f_{ij}(x_k)(i = 1, 2, \dots, 6, j = 1, \dots, 4, k = 1, 3, \dots, 11)$  are continuous bounded differentiable functions, satisfying:

(1.6) 
$$f_{ij}(0) = 0, \ uf_{ij}(u) > 0 (u \neq 0).$$

The general activation functions such as tanh(x), arctan(x) satisfy condition (1.6). The linearized system of (1.5) at origin is the following:

$$\begin{cases} x_1'(t) = x_2(t), \\ x_2'(t) = -r_1x_2(t) - \mu_1x_1(t) + b_{11}x_5(t - \tau_5) + b_{12}x_7(t - \tau_7) \\ + b_{13}x_9(t - \tau_9) + b_{14}x_{11}(t - \tau_{11}), \\ x_3'(t) = x_4(t), \\ x_4'(t) = -r_2x_4(t) - \mu_2x_3(t) + b_{21}x_5(t - \tau_5) + b_{22}x_7(t - \tau_7) \\ + b_{23}x_9(t - \tau_9) + b_{24}x_{11}(t - \tau_{11}), \\ x_5'(t) = x_6(t), \\ x_6'(t) = -r_3x_6(t) - \mu_3x_5(t) + b_{31}x_1(t - \tau_1) + b_{32}x_3(t - \tau_3) \\ + b_{33}x_9(t - \tau_9) + b_{34}x_{11}(t - \tau_{11}), \\ x_7'(t) = x_8(t), \\ x_8'(t) = -r_4x_8(t) - \mu_4x_7(t) + b_{41}x_1(t - \tau_1) + b_{42}x_3(t - \tau_3) \\ + b_{43}x_9(t - \tau_9) + b_{44}x_{11}(t - \tau_{11}), \end{cases}$$

$$\begin{cases} x'_{9}(t) = x_{10}(t), \\ x'_{10}(t) = -r_{5}x_{10}(t) - \mu_{5}x_{9}(t) + b_{51}x_{1}(t - \tau_{1}) + b_{52}x_{3}(t - \tau_{3}) \\ + b_{53}x_{5}(t - \tau_{5}) + b_{54}x_{7}(t - \tau_{7}), \\ x'_{11}(t) = x_{12}(t), \\ x'_{12}(t) = -r_{6}x_{12}(t) - \mu_{6}x_{11}(t) + b_{61}x_{1}(t - \tau_{1}) + b_{62}x_{3}(t - \tau_{3}) \\ + b_{63}x_{5}(t - \tau_{5}) + b_{64}x_{7}(t - \tau_{7}). \end{cases}$$
(1.7)

where  $b_{ij} = c_{ij} f'_{ij}(0)$ ,  $v i = 1, \dots, 6, j = 1, \dots, 4$ . The matrix form of system (1.7) can be written as:

$$X'(t) = AX(t) + BX(t - \tau),$$

where  $X(t) = [x_1(t), x_2(t), \dots, x_{12}(t)]^T$ ,  $X(t - \tau) = [x_1(t - \tau_1), 0, x_3(t - \tau_3), 0, \dots, 0, x_{11}(t - \tau_{11}), 0]^T$ . Both  $A = (a_{ij})_{12 \times 12}$  and  $B = (b_{ij})_{12 \times 12}$  are  $12 \times 12$  matrices as follows:

$$A = (a_{ij})_{12 \times 12} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\mu_1 & -r_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -\mu_2 & -r_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\mu_6 & -r_6 \end{pmatrix},$$

$$B = (b_{ij})_{12 \times 12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11} & 0 & \cdots & b_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{21} & 0 & \cdots & b_{24} & 0 \\ \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{61} & 0 & b_{62} & 0 & b_{63} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

**Definition 1.1.** The trivial solution of system (1.4) is unstable, if there exists at least one component of the trivial solution which is unstable.

**Lemma 1.1.** Assume that  $\mu_i > 0 (i = 1, 2, \dots, 6)$ , matrix C is not a positive definite matrix, then the system (1.5) has a unique equilibrium point, where:

$$C = (c_{ij})_{6\times 6} = \begin{pmatrix} 0 & 0 & c_{11} & c_{12} & c_{13} & c_{14} \\ 0 & 0 & c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & 0 & 0 & c_{33} & c_{34} \\ c_{41} & c_{42} & 0 & 0 & c_{43} & c_{44} \\ c_{51} & c_{52} & c_{53} & c_{54} & 0 & 0 \\ c_{61} & c_{62} & c_{63} & c_{64} & 0 & 0 \end{pmatrix}$$

*Proof.* An equilibrium point  $x^* = [x_1^*, x_2^*, \cdots, x_{12}^*]^T$  of system (1.5) is a constant solution of the following algebraic equation:

$$\begin{aligned} x_{2}^{*} &= 0, \\ &-r_{1}x_{2}^{*} - \mu_{1}x_{1}^{*} + c_{11}f_{11}(x_{5}^{*}) + c_{12}f_{12}(x_{7}^{*}) + c_{13}f_{13}(x_{9}^{*}) + c_{14}f_{14}(x_{11}^{*}) = 0, \\ x_{4}^{*} &= 0, \\ &-r_{2}x_{4}^{*} - \mu_{2}x_{3}^{*} + c_{21}f_{21}(x_{5}^{*}) + c_{22}f_{22}(x_{7}^{*}) + c_{23}f_{23}(x_{9}^{*}) + c_{24}f_{24}(x_{11}^{*}) = 0, \\ x_{6}^{*} &= 0, \\ &-r_{3}x_{6}^{*} - \mu_{3}x_{5}^{*} + c_{31}f_{31}(x_{1}^{*}) + c_{32}f_{32}(x_{3}^{*}) + c_{33}f_{33}(x_{9}^{*}) + c_{34}f_{34}(x_{11}^{*}) = 0, \\ x_{8}^{*} &= 0, \\ &-r_{4}x_{8}^{*} - \mu_{4}x_{7}^{*} + c_{41}f_{41}(x_{1}^{*}) + c_{42}f_{32}(x_{3}^{*}) + c_{43}f_{43}(x_{9}^{*}) + c_{44}f_{44}(x_{11}^{*}) = 0, \\ x_{10}^{*} &= 0, \\ &-r_{5}x_{10}^{*} - \mu_{5}x_{9}^{*} + c_{51}f_{51}(x_{1}^{*}) + c_{52}f_{52}(x_{3}^{*}) + c_{53}f_{53}(x_{5}^{*}) + c_{54}f_{54}(x_{7}^{*}) = 0, \\ x_{12}^{*} &= 0, \\ &-r_{6}x_{12}^{*} - \mu_{6}x_{11}^{*} + c_{61}f_{61}(x_{1}^{*}) + c_{62}f_{62}(x_{3}^{*}) + c_{63}f_{63}(x_{5}^{*}) + c_{64}f_{64}(x_{7}^{*}) = 0. \end{aligned}$$
(1.8)

From condition (1.6) we have  $f_{ij}(0) = 0$ . This means that zero is an equilibrium point of system (1.5). From system (1.8),  $x_{2i}^* = 0(i = 1, 2, \dots, 6)$ . System (1.8) reduced to the following:

$$\mu_{1}x_{1}^{*} = c_{11}f_{11}(x_{5}^{*}) + c_{12}f_{12}(x_{7}^{*}) + c_{13}f_{13}(x_{9}^{*}) + c_{14}f_{14}(x_{11}^{*}), \mu_{2}x_{3}^{*} = c_{21}f_{21}(x_{5}^{*}) + c_{22}f_{22}(x_{7}^{*}) + c_{23}f_{23}(x_{9}^{*}) + c_{24}f_{24}(x_{11}^{*}), \mu_{3}x_{5}^{*} = c_{31}f_{31}(x_{1}^{*}) + c_{32}f_{32}(x_{3}^{*}) + c_{33}f_{33}(x_{9}^{*}) + c_{34}f_{34}(x_{11}^{*}), \mu_{4}x_{7}^{*} = c_{41}f_{41}(x_{1}^{*}) + c_{42}f_{32}(x_{3}^{*}) + c_{43}f_{43}(x_{9}^{*}) + c_{44}f_{44}(x_{11}^{*}), \mu_{5}x_{9}^{*} = c_{51}f_{51}(x_{1}^{*}) + c_{52}f_{52}(x_{3}^{*}) + c_{53}f_{53}(x_{5}^{*}) + c_{54}f_{54}(x_{7}^{*}), \mu_{6}x_{11}^{*} = c_{61}f_{61}(x_{1}^{*}) + c_{62}f_{62}(x_{3}^{*}) + c_{63}f_{63}(x_{5}^{*}) + c_{64}f_{64}(x_{7}^{*}).$$

$$(1.9)$$

Noting that condition (1.6) implies that  $f_{ij}(x_k^*) > 0$  when  $x_k^* > 0$ , and  $f_{ij}(x_k^*) < 0$  when  $x_k^* < 0$ . Since  $\mu_i > 0(i = 1, 2, \dots, 6)$ , when we select  $x_k^* > 0$ , one can not guarantee that the right hand of system (1.9) is a positive number since C is not a positive definite matrix. The same as when we select  $x_k^* < 0$ , one can not ensure that the right hand of system (1.9) is a negative number. Therefore, system (1.5), namely system (1.4) has a unique equilibrium point, it is exactly the zero point.

**Lemma 1.2.** Assume that  $r_i > 0, \mu_i > 0 (i = 1, 2, \dots, 6)$ , then all solutions of system (1.4) are bounded.

*Proof.* From condition (1.6), the activation functions  $f_{ij}(x_k)$  are bounded. Let  $N_i = |c_{i2}f_{i2} + c_{i3}f_{i3} + c_{i4}f_{i4}|, i = 1, 2, \cdots, 6$ , then we have:

Since  $r_i > 0, \mu_i > 0$   $(i = 1, 2, \dots, 6)$ , the eigenvalues of the equations  $\lambda^2 + r_i \lambda + \mu_i = 0$  will be  $\frac{-r_i \pm \sqrt{r_i^2 - 4\mu_i}}{2}$   $(i = 1, 2, \dots, 6)$ . Therefore, we get  $|v_i(t)| \le e^{-r_i t} + N_i(i = 1, 2), |z_i(t)| \le e^{-r_i t} + N_i(i = 3, 4)$ , and  $|w_i(t)| \le e^{-r_i t} + N_i(i = 5, 6)$  if  $r_i^2 - 4\mu_i < 0$ , or  $|v_i(t)| \le e^{-(r_i - \sqrt{r_i^2 - 4\mu_i})t} + N_i(i = 1, 2), |z_i(t)| \le e^{-(r_i - \sqrt{r_i^2 - 4\mu_i})t} + N_i(i = 3, 4)$ , and  $|w_i(t)| \le e^{-(r_i - \sqrt{r_i^2 - 4\mu_i})t} + N_i(i = 5, 6)$  if  $r_i^2 - 4\mu_i > 0$ . This means that the solutions of system (1.4) (or system (1.5) ) are bounded.

In this paper we adopt the following norms of vectors and matrices [6]:  $\|x(t)\| = \sum_{i=1}^{12} |x_i(t)|, \|D\| = \max_{1 \le j \le 12} \sum_{i=1}^{12} |d_{ij}|, \text{ the measure } \mu(D) \text{ of a matrix}$   $D \text{ is defined by } \mu(D) = \lim_{\theta \to 0^+} \frac{\|I + \theta D\| - 1}{\theta}, \text{ which for the chosen norms}$ reduces to  $\mu(D) = \max_{1 \le j \le 12} [d_{jj} + \sum_{i=1, i \ne j}^{12} |d_{ij}|]. D > 0 \text{ (respectively, } D < 0)$ which indicates that D is a positive (negative) definite matrix.

**Lemma 1.3.** For each eigenvalue  $\lambda$  of matrix  $A \in \mathbb{R}^{n \times n}$ , the inequality holds:

(1.10)

$$Re\lambda_i(A) \le \mu(A), i \in \{1, 2, \cdots, n.\}$$

Proof. See [7].

## 2. EXISTENCE OF OSCILLATORY SOLUTIONS

**Theorem 2.1.** Assume that the conditions of Lemma 1.1 and Lemma 1.2 hold. Let  $\alpha_1, \alpha_2, \dots, \alpha_{12}$  represent the eigenvalues of matrix A, and  $\beta_1, \beta_2, \dots, \beta_{12}$ the eigenvalues of matrix B. If there exists one eigenvalue, say  $\alpha_1$  which is a positive real number or is a positive real part of a complex number, then the unique equilibrium of system (1.5) is unstable, implying that the equilibrium of system (1.4) is unstable, and system (1.4) generates a limit cycle, namely, a periodic solution.

*Proof.* It is known that the trivial solution of the linearized system (1.7) is unstable, then the trivial solution of original system (1.5) is unstable. Therefore, for proving the instability of the trivial solution of system (1.5) we only need to prove the instability of the trivial solution of system (1.7). Considering an auxiliary equation of the system (1.7) as follows:

(2.1) 
$$X'(t) = AX(t) + BX(t - \tau_*)$$

where  $0 < \tau_* \ll 1(\tau_* < \min\{\tau_1, \tau_3, \cdots, \tau_9, \tau_{11}\}), X(t - \tau_*) = [x_1(t - \tau_*), 0, x_3(t - \tau_*), 0, \cdots, 0, x_{11}(t - \tau_*), 0]^T$ . Based on the property of delayed differential equation, if the trivial solution of (2.1) is unstable then the trivial solution of system (1.7) is unstable, see [8]. Thus in the following we discuss the instability of the trivial solution of system (2.1). Since the eigenvalues of matrix A are  $\alpha_1, \alpha_2, \cdots, \alpha_{12}$ , and the eigenvalues of matrix B are  $\beta_1, \beta_2, \cdots, \beta_{12}$ , system (2.1) has the following characteristic equation:

$$\prod_{i=1}^{12} (\lambda - \alpha_i - \beta_i e^{-\lambda \tau_*}) = 0.$$

Since there exist six row entries of matrix B are zeros, so there is a characteristic value, say  $\beta_1 = 0$ . then we have

$$\lambda - \alpha_1 - \beta_1 e^{-\lambda \tau_*} = \lambda - \alpha_1 = 0$$

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This means that there exists a eigenvalue which is positive number or a positive real part of a complex number of system (2.1), implying that the trivial solution of system (2.1) is unstable. This suggests that the trivial solution of system (1.7) (or system (1.4)) is unstable. The instability of trivial solution with the boundedness of the solution will force system (1.4) to generate a limit cycle, namely, a periodic solution.

**Theorem 2.2.** Assume that the conditions of Lemma 1.1 and Lemma 1.2 hold. Let  $\gamma_1, \gamma_2, \dots, \gamma_{12}$  represent the eigenvalues of matrix (A + B), and  $\beta_1, \beta_2, \dots, \beta_{12}$  the eigenvalues of matrix B. If there exists one eigenvalue, say  $\gamma_1$  which is a positive real number or is a positive real part of a complex number, then the unique equilibrium of system (1.7) is unstable, implying that the equilibrium of system (1.5) is unstable, and system (1.5) (or (1.4)) generates a limit cycle, namely, a periodic solution.

*Proof.* Similar to the proof of Theorem 2.1, we can rewrite the system (1.10) as follows:

(2.2) 
$$X'(t) = (A+B)X(t) + B[X(t-\tau_*) - X(t)].$$

Since the eigenvalues of matrix (A + B) are  $\gamma_1, \gamma_2, \dots, \gamma_{12}$ , and the eigenvalues of matrix B are  $\beta_1, \beta_2, \dots, \beta_{12}$ , system (2.2) has the following characteristic equation:

$$\prod_{i=1}^{12} (\lambda - \gamma_i + \beta_i - \beta_i e^{-\lambda \tau_*}) = 0$$

Now from  $\beta_1 = 0$ . we have

(2.3) 
$$\lambda - \gamma_1 + \beta_1 - \beta_1 e^{-\lambda \tau_*} = \lambda - \gamma_1 = 0$$

This means that there exists a eigenvalue which is positive number or a positive real part of a complex number of system (2.3), implying that the trivial solution of system (2.3) is unstable. This suggests that the trivial solution of system (1.5) (or system (1.4)) is unstable. The instability of trivial solution with the boundedness of the solution will force system (1.4) to generate a limit cycle, namely, a periodic solution.

**Theorem 2.3.** Assume that the conditions of Lemma 1.1 and Lemma 1.2 hold. If the following condition holds:

$$(2.4) 0 < \mu(A) + || B ||,$$

then the unique equilibrium of system (1.5) is unstable, implying that the equilibrium of system (1.4) is unstable, and system (1.4) generates a limit cycle, namely, a periodic solution.

*Proof.* We must prove that the unique equilibrium point of auxiliary system (2.1) is unstable. The characteristic equation associated with system (2.1) is the following:

$$\det(\lambda I_{12} - A - Be^{-\lambda \tau_*}) = 0,$$

where  $I_{12}$  is a  $12 \times 12$  identity matrix. Set:

$$\Phi(\lambda) = \det(\lambda I_{12} - A - Be^{-\lambda \tau_*}).$$

If the trivial solution of auxiliary system (2.1) is unstable, based on Theorem 2.1 there exists a root of  $\Phi(\lambda)$  satisfying  $Re(\lambda) > 0$ . From lemma 1.3, we get:

$$0 < Re(\lambda) \le \mu(A + Be^{-\lambda\tau_*})$$

$$= \lim_{\theta \to 0^+} \frac{\|I + \theta(A + Be^{-\lambda\tau_*})\| - 1}{\theta}$$

$$\le \mu(A) + \|B\| \max_{1 \le k \le 12} |e^{(-\lambda_k\tau_*)}|$$

$$\le \mu(A) + \|B\|.$$

Thus, condition (2.4) holds. The trivial solution of auxiliary system (2.1) is unstable, implying that the trivial solution of system (1.5) (or system (1.4)) is unstable. The instability of trivial solution with the boundedness of the solution will force system (1.4) to generate a limit cycle, namely, a periodic solution. The proof is completed.  $\Box$ 

## 3. SIMULATION RESULT

These simulations were performed by using the equivalent system (1.5) of (1.4). Firstly we selected the activation function as  $f(x) = \arctan(x)$ .



Then  $f'(x) = \frac{1}{1+x^2}$ , so f'(0) = 1. The parameters  $r_1 = 0.25$ ,  $r_2 = 0.23$ ,  $r_3 = 0.24$ ,  $r_4 = 0.28$ ,  $r_5 = 0.22$ ,  $r_6 = 0.26$ ;  $\mu_1 = 0.24$ ,  $\mu_2 = 0.28$ ,  $\mu_3 = 0.24$ ,  $\mu_4 = 0.25$ ,  $\mu_5 = 0.27$ ,  $\mu_6 = 0.22$ ,  $c_{11} = -0.35$ ,  $c_{12} = 0.45$ ,  $c_{13} = -0.36$ ,  $c_{14} = 0.48$ ,  $c_{21} = -0.25$ ,  $c_{22} = 0.75$ ,  $c_{23} = -0.32$ ,  $c_{24} = 0.78$ ,  $c_{31} = 0.65$ ,  $c_{32} = -0.25$ ,  $c_{33} = 0.55$ ,  $c_{34} = -0.35$ ,  $c_{41} = 0.15$ ,  $c_{42} = -0.38$ ,  $c_{43} = 0.18$ ,  $c_{44} = 0.76$ ,  $c_{51} = 0.25$ ,  $c_{52} = -0.26$ ,  $c_{53} = 0.35$ ,  $c_{54} = -0.35$ ,  $c_{61} = 0.45$ ,  $c_{62} = -0.36$ ,  $c_{63} = 0.48$ ,  $c_{64} = -0.46$ . The characteristic values of matrix C are 0.2863, -0.5555,  $-0.0928 \pm 0.9337i$ ,  $0.2274 \pm 0.3164i$ . Therefore, C is not a

positive definite matrix. From Lemma 1.1, the system has a unique equilibrium point, namely, the zero point. Obviously,  $\mu(A) = 0.78 > 0$ . Therefore,  $0 < \mu(A) + ||B||$ . When time delays are selected as 1.8, 1.2, 1.6, 1.4, 1.5, 1.4, based on Theorem 2.3, the system has an oscillatory solution (see Fig 1).



In order to see the effect of time delays, we increase time delays as 3.8, 3.2, 3.6, 3.4, 3.5, 3.4, we see the oscillatory behavior still maintained (see Fig 2), but the oscillatory frequency and amplitude are changed.

When we change  $r_i$  as  $r_1 = 0.75, r_2 = 0.33, r_3 = 0.64, r_4 = 0.78, r_5 = 0.32, r_6 = 0.96$ , the other parameters are the same as in Fig 2, then eigenvalues of A + B are:  $0.2620 \pm 0.6526i, 0.0431 \pm 0.5335i, -0.1299 \pm 0.6980i$ ,

 $-0.8776, -0.0896, -1.0147 \pm 0.7412i, -0.5669 \pm 0.4729i$ , based on Theorem 2.2, the system has an oscillatory solution. Only the oscillatory frequency and amplitude are slightly different from Fig 2, see Fig 3.



In Fig 4, we use tanh(x) as the activation function, we still have  $f'(0) = 1 - tanh^2(0) = 1$ . The other parameters are the same as in Fig 3, we see the oscillatory behavior almost the same as the activation function arctan(x). This means that the activation functions do not affect the oscillatory behavior too much (see Fig 4).



Fig 4. Oscillation of the solutions,  $r_1$ =0.75,  $r_2$ =0.33,  $r_3$ =0.64,  $r_4$ =0.78,  $r_5$ =0.32,  $r_6$ =0.96. Activatifuhnction: tanh(x), delays: 2.8, 2.2, 2.6, 2.4, 2.5, 2.4.

## 4. CONCLUSION

In this paper, we have discussed the dynamical behavior of a six-neuron inertia neural model with time delays. The existence of a limit cycle which is easy to check, as compared to the general bifurcating method. Some simulations are provided to indicate the effectness of the criterion. Time delays only affect the oscillatory frequency when there exists a limit cycle of the system.

### REFERENCES

- T.S. WANG, Z.S. CHENG, R. BU, R.S. MA: Stability and Hopf bifurcation analysis of a simplified six-neuron tridiagonal two-layer neural network model with delays, Neurocomputing, 332 (2019), 203–214.
- [2] Z.H. CHENG, K.H. XIE, T.S WANG, J.D. CAO : Stability and Hopf bifurcation of three-triangle neural networks with delays, Neurocomputing, **322** (2018), 206–215.
- [3] C.J. XU, X.H. TANG, M.X. LIAO: Stability and bifurcation analysis of a six-neuron BAM neural network model with discrete delays, Neurocomputing, 74 (2011), 689– 707.
- [4] J.H. GE, J. XU: Stability and Hopf bifurcation on four-neuron neural networks with inertia and multiple delays, Neurocomputing, **287** (2018), 34–44.
- [5] P. BI, Z.X. HU: *Hopf bifurcation and stability for a neural network model with mixed delays*, Appl. Math. Comput., **218** (2012), 6748–6761.
- [6] K. GOPALSAMY: Stability and oscillations in delay differential equations of populations dynamics, Kluwer, Dordrecht, 1992.
- [7] C.A. DESOER, M. VIDYASAGAR: Feedback System: Input-Output properties, Academic Press, New York 1977.
- [8] J.K. HALE: Theory of functional differential equations, Springer Verlag, Berlin, 1997.

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