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PRIME PAIR LABELING OF MORE DIGRAPHS

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Abstract: Let $D(p,q)$ be a digraph. A function $f: V \to \{1,2,3,\dots,p+q\}$ is said to be a prime pair labeling of Dif it is both an in and outdegree prime pair labeling of D. This paper introduces some new digraphs and includes the results of checking the existence of prime pairs labeling of various digraphs.

Keywords: Prime Pair Labeling, Digraphs.

AMS Subject Classification: 05C78.

1. Introduction

A directed graph or digraph D consists of a finite set V of vertices (points) and a collection of ordered pairs of distinct vertices. Any such pair (u,v) is called an arc or directed line and will usually be denoted by $\overrightarrow{uv}$. The arc $\overrightarrow{uv}$ goes from u to v and incident with u and v, we also say u is adjacent to v and v is adjacent from u. A digraph D with p vertices and q arcs is denoted by $D(p,q)$. The indegree $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The outdegree $d^+(v)$of v is the number of arcs having v as its initial vertex [1].A labeling of a graph G is an assignment of integers to either the vertices or the edges or both subject to certain conditions. The notion of prime labeling for graphs originated with Roger Entringer and was introduced in the paper by Tout et.al[7] in the early 1980's and since then it is an active field of research for many scholars. In & Outdegree prime pair labeling in digraphs were introduced by K.Palani et.al[4,5]. Let $D(p,q)$ be a digraph. For $u \in V(D)$,$N^-(u) = \{w \in V(D) | wu \in A(D)\}$. A function $f: V \to \{1,2,3,\dots,p+q\}$ is said to be an indegree prime pair labeling of D if at each $u \in V(D)$,$gcd[f(v),f(w)] = 1,\ \forall\, v,w \in N^-(u)$.For $u \in V(D)$,$N^+(u) = \{w \in V(D) | uw \in A(D)\}$.A function $f: V \to \{1,2,3,\dots,p+q\}$ is said to be an outdegree prime pair labeling of D, if at each $u \in V(D)$, $gcd[f(v),f(w)] = 1$, $\forall\, v,w \in N^+(u)$. Prime pair labeling of directed graphs were introduced by K.Palani et.al[6]. Let $D(p,q)$ be a digraph. A function $f: V \to \{1,2,3,\dots,p+q\}$ is said to be a prime pair labeling of D if it is both an in and outdegree prime pair labeling of D.The comb $P_n \odot K_1$ [3] is obtained by joining a pendent edge to each vertex of P_n.In this paper, we introduce some new digraphs depending upon the orientation of the corresponding simple graphs. Also, investigate the existance of prime pair labeling of those digraphs. The following facts are from[6]:

1.1 Observation: If G is a graph such that $N^+(u)$&$N^-(u)$ are either Φ or a singleton set for every $u \in V(G)$then, G admits prime pair labeling.

1.2 Definition: A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendent edges are oriented towards the end vertices is called a downcomb and is denoted as Down$\overrightarrow{P_n \odot K_1}$.

1.3 Definition: A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendent edges are oriented alternatively is called an alternating comb and is denoted as A$\overrightarrow{P_n \odot K_1}$.

1.4Definition: A comb graph $P_n \odot K_1$ in which the path edges are oriented alternatively and the pendent edges are oriented away from the end vertices is called an alternating upcomb and is denoted as AUp$\overrightarrow{P_n \odot K_1}$.

1.5 Definition: A comb graph $P_n \odot K_1$ in which the path edges are oriented alternatively and the pendent edges are oriented towards the end vertices is called an alternating downcomb and is denoted as ADown$\overrightarrow{P_n \odot K_1}$.

1.6 Definition: A comb graph $P_n \odot K_1$ in which both the path edges and the pendent edges are oriented alternatively is called a double alternating comb and is denoted as DA$\overrightarrow{P_n \odot K_1}$.

The following facts are from[2]:

1.7"Gcd of any two consecutive integers is 1".

1.8"Gcd of any two consecutive odd integers is 1".

2. SOME NEW DIGRAPHS

In this section we define some new digraphs. In[6], we introduced Double alternating comb. Now, we find the existence of two kinds of Double alternating combsdue to the alternative directions for the pendent edges. We call them as sole-Double alternating and Di-Double alternating combs, as in one type the vertices of the path in a comb have either indegree or outdegree as zero and in the other type all the vertices of the path have both indegree and outdegree non zero. Hence we have the following definitions.

*2.1Sole-Double alternating comb(SDA$\overrightarrow{P_n \odot K_1}$):*Consider a comb graph$P_n \odot K_1$. Orient P_n alternatively. Next, orient the pendent edges so that either $d^+(v) = 0$ or$d^-(v) = 0 \forall v \in V(P_n)$. The resulting graph is called Sole-Double alternating comb and is denoted as SDA$\overrightarrow{P_n \odot K_1}$.

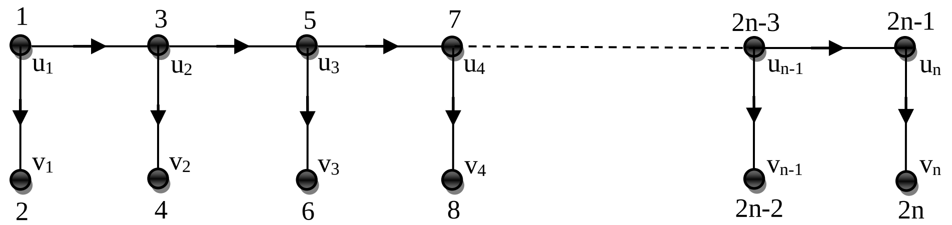
2.2 Di-Double alternating comb(DDA$\overrightarrow{P_n \odot K_1}$): In a comb graph$P_n \odot K_1$, orient P_n alternatively. Now orient the pendent edges so that neither $d^+(v) = 0$ nor $d^-(v) = 0 \forall v \in V(P_n)$. The resulting graph is called Di-Double alternating comb and is denoted as DDA$\overrightarrow{P_n \odot K_1}$.

3. Main Results

3.1 Theorem:Downcomb(Down$\overrightarrow{P_n \odot K_1}$) admits prime pair labeling.

Proof:Let V(Down$\overrightarrow{P_n \odot K_1}$) = $\{u_i, v_i | 1 \le i \le n\}$ be the vertex set where u_i's and v_i's represents the vertices of the path and the i^{th} copy of K_1.

Then A(Down$\overrightarrow{P_n \odot K_1}$) =$\{\overrightarrow{u_\iota u_{\iota+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{u_\iota v_\iota} | 1 \leq i \leq n\}$ is the arc set.



The digraph has 2n vertices and 2n-1 arcs.

Define $f: V \rightarrow \{1,2,3, \ldots, 4n-1\}$ by $f(u_i) = 2i - 1$ for $1 \leq i \leq n$and $f(v_i) = 2i$ for$1 \leq i \leq n$

Now, $N^{-}(u_1) = \Phi$;$N^{-}(u_i) = \{u_{i-1}\}$ for 2 ≤ i ≤ n ; $N^{-}(v_i) = \{u_i\}$ for 1 ≤ i ≤ n

∴ N^{+}(w) contains atmost one element ∀ w ∈Down$\overrightarrow{P_n \odot K_1}$

∴ f is an indegree prime pair labeling −−−−−−−(I)

Now, $N^{+}(u_i) = \{v_i, u_{i+1}\}$ for 1 ≤ i ≤ n−1

Further, $N^{+}(u_n) = \{v_n\}$;$N^{+}(v_i) = \Phi$ for $1 \leq i \leq n$

Hence gcd$[f(v_i), f(u_{i+1})]$ = gcd$[2i, 2i+1]$ = 1 for $1 \leq i \leq n-1$

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{+}(u_i)$ for$1 \leq i \leq n-1$ −−−−−−−(1)

Also, $|N^{+}(u_n)| = 1$ and $|N^{+}(v_i)| = 0$for $1 \leq i \leq n$ −−−−−−−(2)

From (1) & (2) f is an outdegree prime pair labeling −−−−−−−−(II)

By (I) &(II), f is a prime pair labeling of Down$\overrightarrow{P_n \odot K_1}$

Hence Down$\overrightarrow{P_n \odot K_1}$ admits prime pair labeling.

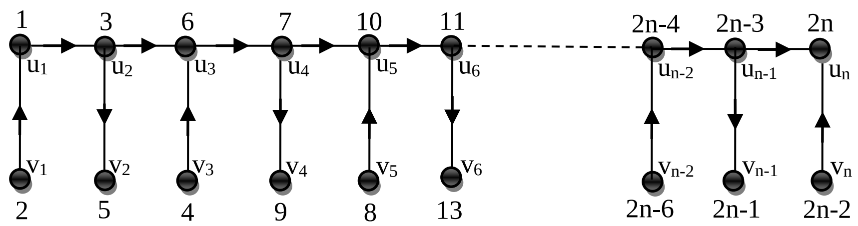
3.2 Theorem:Alternating comb (A$\overrightarrow{P_n \odot K_1}$) admits prime pair labeling.

Proof:Let V(A$\overrightarrow{P_n \odot K_1}$) = $\{u_i, v_i | 1 \leq i \leq n\}$ be the vertex set where u_i's and v_i's represents the vertices of the path and the i^{th} copy of K_1.

Case(i): n is odd

Here, A(A$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_\iota u_{\iota+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{v_{2\iota-1} u_{2\iota-1}} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{u_{2\iota} v_{2\iota}} | 1 \leq i \leq \frac{n-1}{2}\}$is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define a function $f: V \rightarrow \{1,2,3,\dots,4n-1\}$ by $(u_1) = 1$; $f(v_1) = 2$;

For $2 \leq i \leq n, f(u_i) = \begin{cases} 2i if i is odd \\ 2i-1 \quad if i is even \end{cases}$ and $f(v_i) = \begin{cases} 2i-2 \quad if i is odd \\ 2i+1 \quad if i is even \end{cases}$

Now, $N^{-}(u_1)$ = $\{v_1\}$ and $N^{-}(v_1)$ =Φ

For odd i, such that $3 \leq i \leq n, N^{-}(u_i)$ = $\{u_{i-1}, v_i\}$ and $N^{-}(v_i)$ =Φ

Further for even i, such that $2 \leq i \leq n-1$, $N^{-}(u_i)$ = $\{u_{i-1}\}$ and $N^{-}(v_i)$ = $\{u_i\}$

Hence, $|N^{-}(u_1)| = 1$ and $|N^{-}(v_1)\ | = 0$ $------(1)$

Let $3 \leq i \leq n$ and i is odd.

Then gcd[$f(u_{i-1})$,$f(v_i)$] = gcd[$2i-3, 2i-2$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{-}(u_i)$$------(2)$

And $|N^{-}(v_i)\ | = 0$ $------(3)$

Let $2 \leq i \leq n-1$ and i is even.Then $|N^{-}(u_i)| = |N^{-}(v_i)| = 1$ $-------$ (4)

From (1), (2), (3) & (4) f is an indegree prime pair labeling$-------$(I)

For odd i, such that $1 \leq i \leq n-2, N^{+}(u_i)$ = $\{u_{i+1}\}$ and $N^{+}(v_i)$ = $\{u_i\}$

Further for even i, such that $2 \leq i \leq n-1, N^{+}(u_i)$ = $\{v_i\ , u_{i+1}\}$ and $N^{+}(v_i)$ =Φ

Further, $N^{+}(u_n)$ =Φand $N^{+}(v_n)$ = $\{u_n\}$

Let $1 \leq i \leq n-2$ and i is odd.Then $|N^{+}(u_i)| = |N^{+}(v_i)| = 1$ $------(5)$

Let $2 \leq i \leq n-1$ and i is even.

Then gcd[$f(v_i)\ , f(u_{i+1})$] = gcd[$2i+1, 2i+2$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{+}(u_i)$$-------(6)$

And $|N^{+}(v_i)| = 0$ $-------$ (7)

Also, $|N^{+}(u_n)| = 0$ and $|N^{+}(v_n)\ | = 1$ $-------(8)$

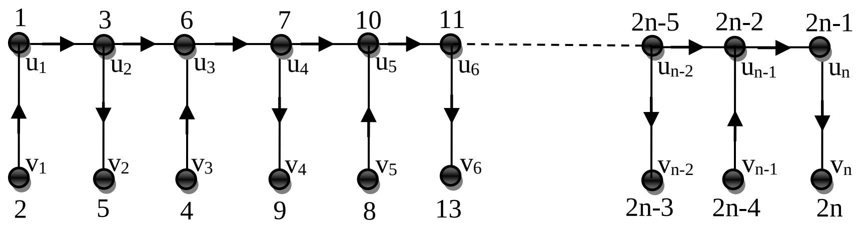
From (5), (6), (7) & (8) f is an outdegree prime pair labeling $-------$(II)

By (I)&(II), f is a prime pair labeling.

Case(ii): n is even

Here, A($\overrightarrow{AP_n \odot K_1}$) = $\{\overrightarrow{u_i u_{i+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{v_{2i-1} u_{2i-1}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{u_{2i} v_{2i}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define $f: V \to \{1,2,3,\dots,4n-1\}$ by $f(u_1) = 1$; $f(v_1) = 2$; $f(u_n) = 2n-1$; $f(v_n) = 2n$

For $2 \le i \le n-1$, $f(u_i) = \begin{cases} 2i \quad if i is odd \\ 2i-1 \quad if i is even \end{cases}$ and $f(v_i) = \begin{cases} 2i-2 \quad if i is odd \\ 2i+1 \quad if i is even \end{cases}$

Now, $N^-(u_1) = \{v_1\}$ and $N^-(v_1) = \Phi$

For odd i, such that $3 \le i \le n-1$, $N^-(u_i) = \{u_{i-1}, v_i\}$ and $N^-(v_i) = \Phi$

Further for even i, such that $2 \le i \le n$, $N^-(u_i) = \{u_{i-1}\}$ and $N^-(v_i) = \{u_i\}$

Hence, $|N^-(u_1)| = 1$ and $|N^-(v_1)| = 0$ ––––––(9)

Let $3 \le i \le n-1$ and i is odd.

Then gcd$[f(u_{i-1}), f(v_i)]$ = gcd$[2i-3, 2i-2]$ = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^-(u_i)$––––––(10)

And $|N^-(v_i)| = 0$ ––––––(11)

Let $2 \le i \le n$ and i is even.Then $|N^-(u_i)| = |N^-(v_i)| = 1$ ––––– (12)

From (9), (10), (11) & (12) f is an indegree prime pair labeling –––––––(III)

For odd i, such that $1 \le i \le n-1$, $N^+(u_i) = \{u_{i+1}\}$ and $N^+(v_i) = \{u_i\}$

Further for even i, such that $2 \le i \le n-2$, $N^+(u_i) = \{v_i, u_{i+1}\}$ and $N^+(v_i) = \Phi$

Further, $N^+(u_n) = \{v_n\}$ and $N^+(v_n) = \Phi$

Let $1 \le i \le n-1$ and i is odd.

Then $|N^+(u_i)| = |N^+(v_i)| = 1$ ––––––(13)

Let $2 \le i \le n-2$ and i is even.

Then gcd$[f(v_i), f(u_{i+1})]$ = gcd$[2i+1, 2i+2]$ = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^+(u_i)$–––––– (14)

And $|N^+(v_i)| = 0$ ––––––– (15)

Also, $|N^+(u_n)| = 1$ and $|N^+(v_n)| = 0$ –––––––(16)

From (13), (14), (15)& (16) f is an outdegree prime pair labeling–––––––(IV)

By (III)&(IV), f is a prime pair labeling.

From case(i) & case(ii), f is a prime pair labeling of $A\overrightarrow{P_n \odot K_1}$

Hence $A\overrightarrow{P_n \odot K_1}$ admits prime pair labeling.

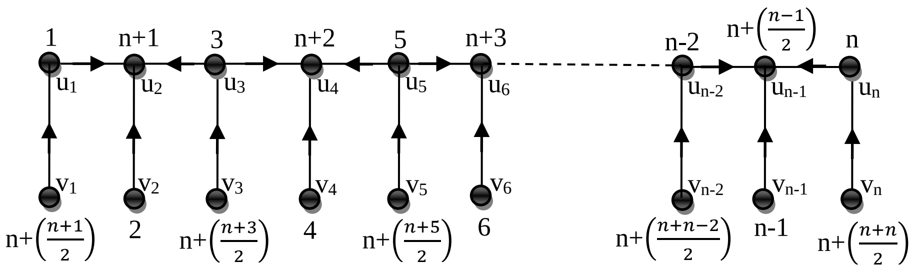
3.3 Theorem:Alternating upcomb ($AUp\overrightarrow{P_n \odot K_1}$) admits prime pair labeling.

Proof:Let V($AUp\overrightarrow{P_n \odot K_1}$) = $\{u_i, v_i | 1 \leq i \leq n\}$ be the vertex set where u_i's and v_i's represents the vertices of the path and the i^{th} copy of K_1.

Case(i): n is odd

Here, A($AUp\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overleftarrow{u_{2i}u_{2i+1}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_i u_i} | 1 \leq i \leq n\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define $f: V \rightarrow \{1,2,3, \dots, 4n-1\}$ by $f(u_i) = \begin{cases} i \text{ if } i \text{ is odd} \\ n + \left(\frac{i}{2}\right) \text{ if } i \text{ is even} \end{cases}$ and

$f(v_i) = \begin{cases} n + \left(\frac{n+i}{2}\right) \text{ if } i \text{ is odd} \\ i \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$.

For odd i, such that $1 \leq i \leq n, N^-(u_i) = \{v_i\}$

Further for even i, such that $2 \leq i \leq n-1, N^-(u_i) = \{u_{i-1}, v_i, u_{i+1}\}$

Further, $N^-(v_i) = \Phi$ for $1 \leq i \leq n$

Let $1 \leq i \leq n$ and i is odd.Then $|N^-(u_i)| = 1$ −−−−−−(1)

Let $2 \leq i \leq n-1$ and i is even.

Then gcd[$f(u_{i-1}), f(v_i), f(u_{i+1})$] = gcd[$i-1, i, i+1$] = 1

∴ gcd[f(u), f(v),f(w)] = 1 ∀u,v,w ∈ $N^-(u_i)$ −−−−−−− (2)

Also$|N^-(v_i)| = 0$ for $1 \leq i \leq n$ −−−−−−(3)

From (1), (2) & (3) f is an indegree prime pair labeling−−−−−−−(I)

Now, $N^+(u_1) = \{u_2\}$; $N^+(v_i) = \{u_i\}$ for $1 \leq i \leq n$

For odd i, such that $3 \leq i \leq n-2, N^+(u_i) = \{u_{i-1}, u_{i+1}\}$

Further for even i, such that $2 \leq i \leq n-1, N^{+}(u_i) = \Phi$

Further, $N^{+}(u_n) = \{u_{n-1}\}$

Hence, $|N^{+}(u_1)| = 1$; $|N^{+}(v_i)| = 1$ for $1 \leq i \leq n$ $--------$(4)

Let $3 \leq i \leq n-2$ and i is odd.

Then gcd$[f(u_{i-1}), f(u_{i+1})]$ = gcd$[n + \left(\frac{i-1}{2}\right), n + \left(\frac{i+1}{2}\right)]$ = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{+}(u_i)$$--------$(5)

Let $2 \leq i \leq n-1$ and i is even.Then $|N^{+}(u_i)| = 0$ $--------$ (6)

Also, $|N^{+}(u_n)| = 1$ $--------$ (7)

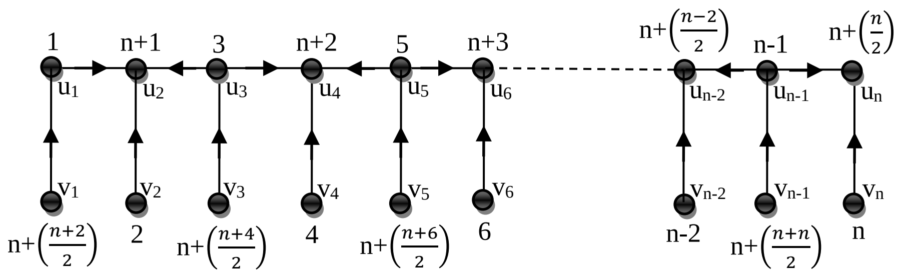
From (4), (5), (6) & (7) f is an outdegree prime pair labeling$--------$(II)

By (I)&(II), f is a prime pair labeling.

Case(ii): n is even

Here, A(AUp$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overleftarrow{u_{2i}u_{2i+1}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_i u_i} | 1 \leq i \leq n\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define $f: V \rightarrow \{1,2,3, \ldots, 4n-1\}$ by $f(u_i) = \begin{cases} i \text{ if } i \text{ is odd} \\ n + \left(\frac{i}{2}\right) \text{ if } i \text{ is even} \end{cases}$ and

$f(v_i) = \begin{cases} n + \left(\frac{n+i+1}{2}\right) \text{ if } i \text{ is odd} \\ i \text{ if } i \text{ is even} \end{cases}$ for $1 \leq i \leq n$.

For odd i, such that $1 \leq i \leq n-1, N^{-}(u_i) = \{v_i\}$

Further for even i, such that $2 \leq i \leq n-2, N^{-}(u_i) = \{u_{i-1}, v_i, u_{i+1}\}$

Further,$N^{-}(u_n) = \{u_{n-1}, v_n\}$;$N^{-}(v_i) = \Phi$ for $1 \leq i \leq n$

Let $1 \leq i \leq n-1$ and i is odd.Then $|N^{-}(u_i)| = 1$ $-------$(8)

Let $2 \leq i \leq n-2$ and i is even.

Then gcd$[f(u_{i-1}), f(v_i), f(u_{i+1})]$ = gcd$[i-1, i, i+1]$ = 1

$\therefore$ gcd[f(u), f(v),f(w)] = 1 $\forall$ u,v,w $\in N^{-}(u_i)$ $-------$ (9)

Also gcd[$f(u_{n-1})$,$f(v_n)$] = gcd[$n-1, n$] = 1$-------$(10)

Also $|N^{-}(v_i)| = 0$ for $1 \le i \le n$ $------$(11)

From (8), (9), (10) & (11) f is an indegree prime pair labeling$-------$(III)

Now, $N^{+}(u_1)$ = $\{u_2\}$;$N^{+}(v_i)$ = $\{u_i\}$ for $1 \le i \le n$

For odd i, such that $3 \le i \le n-1, N^{+}(u_i)$ = $\{u_{i-1}, u_{i+1}\}$

Further for even i, such that $2 \le i \le n, N^{+}(u_i)$ = Φ

Hence, $|N^{+}(u_1)| = 1$; $|N^{+}(v_i)| = 1$ for $1 \le i \le n$ $-------$(12)

Let $3 \le i \le n-1$ and i is odd.

Then gcd[$f(u_{i-1})$,$f(u_{i+1})$] = gcd[$n + \left(\frac{i-1}{2}\right), n + \left(\frac{i+1}{2}\right)$] = 1

$\therefore$ gcd[f(v),f(w)] = 1 $\forall$ v,w $\in N^{+}(u_i)$$-------$(13)

Let $2 \le i \le n$ and i is even. Then $|N^{+}(u_i)| = 0$ $-------$ (14)

From (12), (13) & (14) f is an outdegree prime pair labeling$-------$(IV)

By (III)&(IV), f is a prime pair labeling.

From case(i) & case(ii), f is a prime pair labeling of AUp$\overrightarrow{P_n \odot K_1}$

Hence AUp$\overrightarrow{P_n \odot K_1}$ admits prime pair labeling.

3.4 Theorem:Alternating downcomb (ADown$\overrightarrow{P_n \odot K_1}$) admits prime pair labeling.

Proof:Let V(ADown$\overrightarrow{P_n \odot K_1}$) = $\{u_i, v_i | 1 \le i \le n\}$ be the vertex set where u_i's and v_i's represents the vertices of the path and the i^{th} copy of K_1.

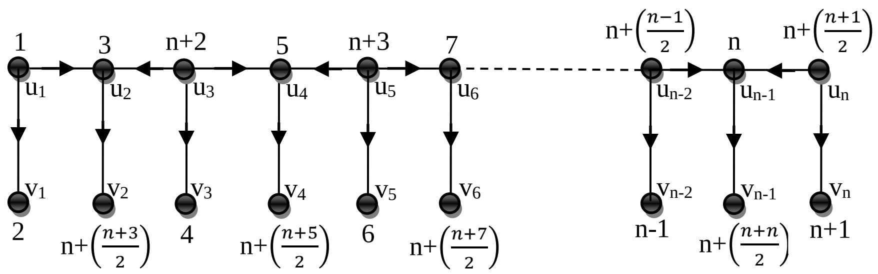
Case(i): n is odd

Here, A(ADown$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overleftarrow{u_{2i}u_{2i+1}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{u_i v_i} | 1 \le i \le n\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.

Define $f: V \to \{1,2,3, \ldots, 4n-1\}$ by $(u_1) = 1$; $f(v_1) = 2$;

For $2 \le i \le n, f(u_i) = \begin{cases} n + \left(\frac{i+1}{2}\right) & if i is odd \\ i+1 & if i is even \end{cases}$ and $f(v_i) = \begin{cases} i+1 & if i is odd \\ n + \left(\frac{n+i+1}{2}\right) & if i is even \end{cases}$



For odd i, such that $1 \leq i \leq n, N^{-}(u_i) = \Phi$

Further for even i, such that $2 \leq i \leq n-1, N^{-}(u_i) = \{u_{i-1}, u_{i+1}\}$

Further, $N^{-}(v_i) = \{u_i\}$ for $1 \leq i \leq n$

Let $1 \leq i \leq n$ and i is odd. Then $|N^{-}(u_i)| = 0$ ‒‒‒‒‒‒(1)

For $i = 2$, $N^{-}(u_2) = \{u_1, u_3\}$. Then gcd[$f(u_1)$,$f(u_3)$] = gcd[1 , $n+2$] = 1 ‒‒‒‒‒‒ (2)

Let $4 \leq i \leq n-1$ and i is even.

Then gcd[$f(u_{i-1})$,$f(u_{i+1})$] = gcd[$n + \left(\frac{i}{2}\right)$, $n + \left(\frac{i+2}{2}\right)$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{-}(u_i)$ ‒‒‒‒‒‒ (3)

Also$|N^{-}(v_i)| = 1$ for $1 \leq i \leq n$ ‒‒‒‒‒‒(4)

From (1), (2), (3) & (4) f is an indegree prime pair labeling ‒‒‒‒‒‒‒(I)

Now, $N^{+}(u_1) = \{v_1, u_2\}$; $N^{+}(v_i) = \Phi$ for $1 \leq i \leq n$

For odd i, such that $3 \leq i \leq n, N^{+}(u_i) = \{u_{i-1}, v_i, u_{i+1}\}$

Further for even i, such that $2 \leq i \leq n-1, N^{+}(u_i) = \{v_i\}$

Hence, gcd[$f(v_1)$,$f(u_2)$] = gcd[2,3] =1; $|N^{+}(v_i)| = 0$ for $1 \leq i \leq n$ ‒‒‒‒‒ (5)

Let $3 \leq i \leq n$ and i is odd.

Then gcd[$f(u_{i-1})$,$f(v_i)$, $f(u_{i+1})$] = gcd[$i, i+1, i+2$] = 1

∴ gcd[f(u), f(v),f(w)] = 1 ∀ u,v,w ∈ $N^{+}(u_i)$‒‒‒‒‒‒‒(6)

Let $2 \leq i \leq n-1$ and i is even. Then $|N^{+}(u_i)| = 1$ ‒‒‒‒‒‒(7)

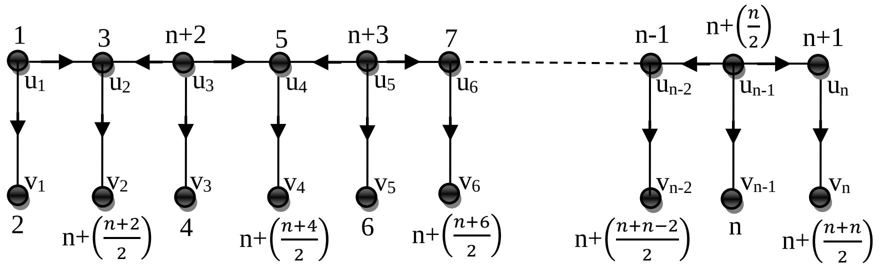
From (5), (6) & (7) f is an outdegree prime pair labeling‒‒‒‒‒‒(II)

By (I) & (II), f is a prime pair labeling.

Case(ii): n is even

Here, A(ADown$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overleftarrow{u_{2i}u_{2i+1}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{u_i v_i} | 1 \leq i \leq n\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define $f: V \rightarrow \{1,2,3, \dots ,4n-1\}$ by $(u_1) = 1$; $f(v_1) = 2$;

For $2 \leq i \leq n, f(u_i) = \begin{cases} n + \left(\frac{i+1}{2}\right) ifiisodd \\ i+1 \quad ifiiseven \end{cases}$ and $f(v_i) = \begin{cases} i+1 \quad ifiisodd \\ n + \left(\frac{n+i}{2}\right) ifiiseven \end{cases}$

For odd i, such that $1 \leq i \leq n-1, N^-(u_i) = \Phi$

Further for even i, such that $2 \leq i \leq n-2, N^-(u_i) = \{u_{i-1}, u_{i+1}\}$

Further, $N^-(u_n) = \{u_{n-1}\}$; $N^-(v_i) = \{u_i\}$ for $1 \leq i \leq n$

Let $1 \leq i \leq n-1$ and i is odd. Then $|N^-(u_i)| = 0$ $- - - - - - -$(8)

For $i = 2$, $N^-(u_2) = \{u_1, u_3\}$. Then gcd[$f(u_1)$,$f(u_3)$] = gcd[1 , $n+2$] = 1$- - - - - -$ (9)

Let $4 \leq i \leq n-2$ and i is even.

Then gcd[$f(u_{i-1})$,$f(u_{i+1})$] = gcd[$n + \left(\frac{i}{2}\right)$, $n + \left(\frac{i+2}{2}\right)$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^-(u_i)$$- - - - - -$ (10)

Also, $|N^-(u_n)| = 1$;$|N^-(v_i)| = 1$ for $1 \leq i \leq n$ $- - - - - - -$(11)

From (8), (9), (10) & (11) f is an indegree prime pair labeling $- - - - - - - -$(III)

Now, $N^+(u_1) = \{v_1, u_2\}$; $N^+(v_i) = \Phi$ for $1 \leq i \leq n$

For odd i, such that $3 \leq i \leq n-1, N^+(u_i) = \{u_{i-1}, v_i, u_{i+1}\}$

Further for even i, such that $2 \leq i \leq n, N^+(u_i) = \{v_i\}$

Hence, gcd[$f(v_1)$,$f(u_2)$] = gcd[2,3] =1; $|N^+(v_i)| = 0$ for $1 \leq i \leq n$ $- - - - - - -$(12)

Let $3 \leq i \leq n-1$ and i is odd.

Then gcd[$f(u_{i-1})$,$f(v_i)$, $f(u_{i+1})$] = gcd[$i, i+1, i+2$] = 1

∴ gcd[f(u), f(v),f(w)] = 1 ∀ u,v,w ∈ $N^+(u_i)$$- - - - - - - -$(13)

Let $2 \leq i \leq n$ and i is even. Then $|N^+(u_i)| = 1$ $- - - - - - -$(14)

From (12), (13) & (14) f is an outdegree prime pair labeling $- - - - - - - -$(IV)

By (III) & (IV), f is a prime pair labeling.

From case(i) & case(ii), f is a prime pair labeling of ADown$\overrightarrow{P_n \odot K_1}$

Hence ADown$\overrightarrow{P_n \odot K_1}$ admits prime pair labeling.

3.5 Theorem:Sole-Double alternating comb(SDA$\overrightarrow{P_n \odot K_1}$) admits prime pair labeling.

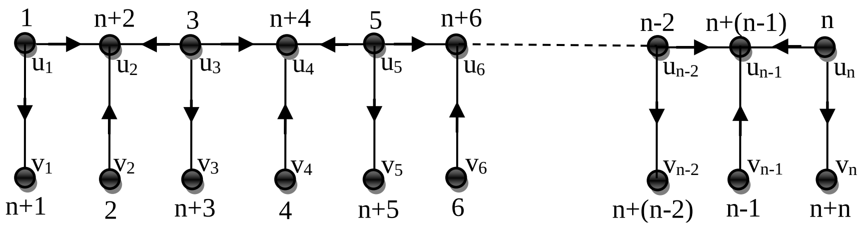
Proof:Let V(SDA$\overrightarrow{P_n \odot K_1}$) = $\{u_i, v_i | 1 \le i \le n\}$ be the vertex set where u_i's and v_i's represents the vertices of the path and the i^{th} copy of K_1.

Case(i): n is odd

Then A(SDA$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overleftarrow{u_{2i}u_{2i+1}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{u_{2i-1}v_{2i-1}} |$

$1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}u_{2i}} | 1 \le i \le \frac{n-1}{2}\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define $f: V \to \{1,2,3,\dots,4n-1\}$ by $f(u_i) = \begin{cases} i \text{ if } i \text{ is odd} \\ n+i \text{ if } i \text{ is even} \end{cases}$ and

$f(v_i) = \begin{cases} n+i \text{ if } i \text{ is odd} \\ i \text{ if } i \text{ is even} \end{cases}$ for $1 \le i \le n$

For odd i, such that $1 \le i \le n, N^-(u_i)$ = Φ and $N^-(v_i)$ =$\{u_i\}$

Further for even i, such that $2 \le i \le n-1, N^-(u_i)$ = $\{u_{i-1}, v_i, u_{i+1}\}$ and $N^-(v_i)$ = Φ

Let $1 \le i \le n$ and i is odd. Then $|N^-(u_i)| = 0$ and $|N^-(v_i)| = 1$ $-------(1)$

Let $2 \le i \le n-1$ and i is even.

Then gcd[$f(u_{i-1})$,$f(v_i)$, $f(u_{i+1})$] = gcd[$i-1, i, i+1$] = 1

∴ gcd[f(u), f(v),f(w)] = 1 ∀ u,v,w ∈ $N^-(u_i)$$------$ (2)

And $|N^-(v_i)| = 0$ $-------(3)$

From (1), (2) & (3) f is an indegree prime pair labeling $--------(I)$

Now, $N^+(u_1)$ = $\{v_1, u_2\}$ and $N^+(v_1)$ = Φ

For odd i, such that $3 \le i \le n-2, N^+(u_i)$ = $\{u_{i-1}, v_i, u_{i+1}\}$ and $N^+(v_i)$ = Φ

Further for even i, such that $2 \le i \le n-1, N^+(u_i)$ = Φ and $N^+(v_i)$ = $\{u_i\}$

Further, $N^+(u_n)$ = $\{u_{n-1}, v_n\}$and $N^+(v_n)$ = Φ

Hence, gcd[$f(v_1)$,$f(u_2)$] = gcd[n+1,n+2] =1 and $|N^+(v_1)| = 0$ $------(4)$

Let $3 \leq i \leq n-2$ and i is odd.

Then gcd[$f(u_{i-1})$,$f(v_i)$, $f(u_{i+1})$] = gcd[$n+i-1, n+i, n+i+1$] = 1

∴ gcd[f(u), f(v),f(w)] = 1 ∀ u,v,w ∈ $N^+(u_i)$− − − − − −(5)

And $|N^+(v_i)| = 0$ − − − − − (6)

Let $2 \leq i \leq n-1$ and i is even. Then $|N^+(u_i)| = 0$ and $|N^+(v_i)| = 1$ − − − − − −(7)

Also gcd[$f(u_{n-1})$,$f(v_n)$] = gcd[n+(n-1), n+n] = gcd[2n-1, 2n] = 1 − − − − − − (8)

Also $|N^+(v_n)| = 0$ − − − − − −(9)

From (4), (5), (6), (7), (8) & (9) f is an outdegree prime pair labeling − − − − − −(II)

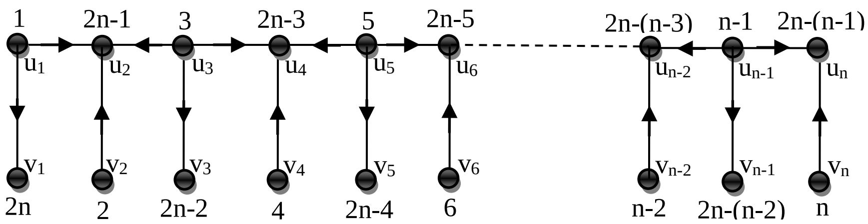
By (I) & (II), f is a prime pair labeling.

Case(ii): n is even

Then A(SDA$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overleftarrow{u_{2i}u_{2i+1}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{u_{2i-1}v_{2i-1}}|$

$1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}u_{2i}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define $f: V \to \{1,2,3,\ldots,4n-1\}$ by $f(u_i) = \begin{cases} i & if\, i\, is\, odd \\ 2n-(i-1) & if\, i\, is\, even \end{cases}$ and

$f(v_i) = \begin{cases} 2n-(i-1) & if\, i\, is\, odd \\ i & if\, i\, is\, even \end{cases}$ for $1 \leq i \leq n$

For odd i, such that $1 \leq i \leq n-1, N^-(u_i)$ = Φ and $N^-(v_i)$ ={u_i}

Further for even i, such that $2 \leq i \leq n-2, N^-(u_i)$ = {u_{i-1} , v_i , u_{i+1}} and $N^-(v_i)$ = Φ

Further, $N^-(u_n)$ = {u_{n-1} , v_n} and $N^-(v_n)$ = Φ

Let $1 \leq i \leq n-1$ and i is odd. Then $|N^-(u_i)| = 0$ and $|N^-(v_i)| = 1$ − − − − − −(10)

Let $2 \leq i \leq n-2$ and i is even.

Then gcd[$f(u_{i-1})$,$f(v_i)$, $f(u_{i+1})$] = gcd[$i-1, i, i+1$] = 1

∴ gcd[f(u), f(v),f(w)] = 1 ∀ u,v,w ∈ $N^-(u_i)$− − − − − − (11)

And $|N^-(v_i)| = 0$ − − − − − −(12)

Also gcd[$f(u_{n-1})$, $f(v_n)$] = gcd[$n-1, n$] = 1 and $|N^-(v_n)| = 0$ $-------$(13)

From (10), (11), (12) & (13) f is an indegree prime pair labeling $-------$(III)

Now, $N^+(u_1)$ = $\{v_1, u_2\}$and $N^+(v_1)$ = Φ

For odd i, such that $3 \le i \le n-1$, $N^+(u_i)$ = $\{u_{i-1}, v_i, u_{i+1}\}$ and $N^+(v_i)$ = Φ

Further for even i, such that $2 \le i \le n$, $N^+(u_i)$ = Φ and $N^+(v_i)$ = $\{u_i\}$

Hence, gcd[$f(v_1)$,$f(u_2)$] = gcd[2n,2n-1] =1 and $|N^+(v_1)| = 0$ $------$(14)

Let $3 \le i \le n-1$ and i is odd.

Then gcd[$f(u_{i-1})$,$f(v_i)$, $f(u_{i+1})$] = gcd[2n-(i-1-1), 2n-(i-1), 2n-(i+1-1)]

= gcd[2n-i+2, 2n-i+1, 2n-i] =1

∴ gcd[f(u), f(v),f(w)] = 1 ∀ u,v,w ∈ $N^+(u_i)$ $------$(15)

And $|N^+(v_i)| = 0$ $------$(16)

Let $2 \le i \le n$ and i is even.

Then $|N^+(u_i)| = 0$ and $|N^+(v_i)| = 1$ $------$(17)

From (14), (15), (16) & (17) f is an outdegree prime pair labeling $-------$(IV)

By (III) & (IV), f is a prime pair labeling.

From case(i) & case(ii), f is a prime pair labeling of SDA$\overrightarrow{P_n \odot K_1}$.

Hence SDA$\overrightarrow{P_n \odot K_1}$ admits prime pair labeling.

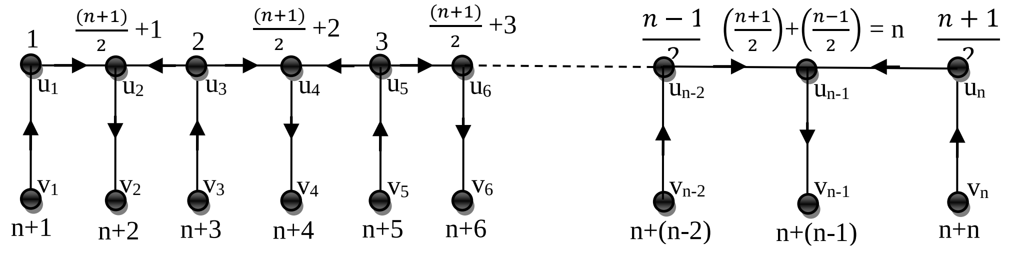
3.6 Theorem:Di-Double alternating comb(DDA$\overrightarrow{P_n \odot K_1}$) admits prime pair labeling.

Proof:Let V(DDA$\overrightarrow{P_n \odot K_1}$) = $\{u_i, v_i | 1 \le i \le n\}$ be the vertex set where u_i's and v_i's represents the vertices of the path and the i^{th} copy of K_1.

Case(i): n is odd

A(DDA$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} | 1 \le i \le \frac{n-1}{2}\}$ ∪ $\{\overleftarrow{u_{2i}u_{2i+1}} | 1 \le i \le \frac{n-1}{2}\}$ ∪ $\{\overrightarrow{v_{2i-1}u_{2i-1}} | 1 \le i \le \frac{n+1}{2}\}$ ∪ $\{\overrightarrow{u_{2i}v_{2i}} | 1 \le i \le \frac{n-1}{2}\}$ is the arc set.

The digraph has 2n vertices and 2n-1 arcs.



Define $f: V \to \{1,2,3,\ldots,4n-1\}$ by $f(u_i) = \begin{cases} \frac{i+1}{2} \text{ if } i \text{ is odd} \\ \left(\frac{n+1}{2}\right) + \frac{i}{2} \text{ if } i \text{ is even} \end{cases}$ and

$f(v_i) = n + i$ for $1 \le i \le n$

For odd i, such that $1 \le i \le n$, $N^-(u_i) = \{v_i\}$ and $N^-(v_i) = \Phi$

Further for even i, such that $2 \le i \le n-1$, $N^-(u_i) = \{u_{i-1}, u_{i+1}\}$ and $N^-(v_i) = \{u_i\}$

Let $1 \le i \le n$ and i is odd.

Then $|N^-(u_i)| = 1$ and $|N^-(v_i)| = 0$ $-------(1)$

Let $2 \le i \le n-1$ and i is even.

Then gcd[$f(u_{i-1})$,$f(u_{i+1})$] = gcd[$\frac{i}{2}$,$\frac{i+2}{2}$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^-(u_i)$$------$ (2)

And $|N^-(v_i)| = 1$ $-------(3)$

From (1), (2) & (3) f is an indegree prime pair labeling $--------$(I)

Now, $N^+(u_1) = \{u_2\}$ and $N^+(v_1) = \{u_1\}$

For odd i, such that $3 \le i \le n-2$, $N^+(u_i) = \{u_{i-1}, u_{i+1}\}$ and $N^+(v_i) = \{u_i\}$

Further for even i, such that $2 \le i \le n-1$, $N^+(u_i) = \{v_i\}$ and $N^+(v_i) = \Phi$

Further, $N^+(u_n) = \{u_{n-1}\}$ and $N^+(v_n) = \{u_n\}$

Hence $|N^+(u_1)| = |N^+(v_1)| = 1$ $------(4)$

Let $3 \le i \le n-2$ and i is odd.

Then gcd[$f(u_{i-1})$,$f(u_{i+1})$] = gcd[$\left(\frac{n+1}{2}\right) + \frac{i-1}{2}$, $\left(\frac{n+1}{2}\right) + \frac{i+1}{2}$]=gcd[$\frac{n+i}{2}$, $\frac{n+i+2}{2}$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^+(u_i)$$------$ (5)

And $|N^+(v_i)| = 1$ $------(6)$

Let $2 \le i \le n-1$ and i is even.Then $|N^+(u_i)| = 1$ and $|N^+(v_i)| = 0$ $------(7)$

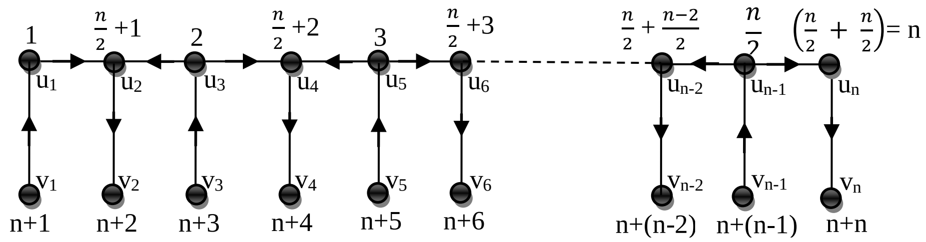
Also, $|N^+(u_n)| = |N^+(v_n)| = 1$ $------(8)$

From (4), (5), (6), (7) & (8) f is an outdegree prime pair labeling $------$(II)

By (I) & (II), f is a prime pair labeling.

Case(ii): n is even

$A(DDA\overrightarrow{P_n \odot K_1}) = \{\overrightarrow{u_{2\iota-1}u_{2\iota}} | 1 \le i \le \frac{n}{2}\} \cup \{\overleftarrow{u_{2\iota}u_{2\iota+1}} | 1 \le i \le \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2\iota-1}u_{2\iota-1}} | 1 \le i \le \frac{n}{2}\} \cup$
$\{\overrightarrow{u_{2\iota}v_{2\iota}} | 1 \le i \le \frac{n}{2}\}$ is the arc set.



The digraph has 2n vertices and 2n-1 arcs.

Define $f: V \to \{1,2,3,\dots,4n-1\}$ by $f(u_i) = \begin{cases} \frac{i+1}{2} if i is odd \\ \left(\frac{n}{2}\right) + \frac{i}{2} if i is even \end{cases}$ and

$f(v_i)$ =n+i for $1 \le i \le n$

For odd i, such that $1 \le i \le n-1, N^-(u_i) = \{v_i\}$ and $N^-(v_i) = \Phi$

Further for even i, such that $2 \le i \le n-2, N^-(u_i) = \{u_{i-1}, u_{i+1}\}$ and $N^-(v_i) = \{u_i\}$

Further, $N^-(u_n) = \{u_{n-1}\}$ and $N^-(v_n) = \{u_n\}$

Let $1 \le i \le n-1$ and i is odd. Then $|N^-(u_i)| = 1$ and $|N^-(v_i)| = 0$ $------(9)$

Let $2 \le i \le n-2$ and i is even.

Then gcd[$f(u_{i-1}), f(u_{i+1})$] = gcd[$\frac{i}{2}, \frac{i+2}{2}$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^-(u_i)$ $------$ (10)

And $|N^-(v_i)| = 1$ $------(11)$

Also, $|N^-(u_n)| = 1$ and $|N^-(v_n)| = 1$ $------(12)$

From (9), (10), (11) & (12) f is an indegree prime pair labeling $-------$(III)

Now, $N^+(u_1) = \{u_2\}$ and $N^+(v_1) = \{u_1\}$

For odd i, such that $3 \le i \le n-1, N^+(u_i) = \{u_{i-1}, u_{i+1}\}$ and $N^+(v_i) = \{u_i\}$

Further for even i, such that $2 \le i \le n, N^+(u_i) = \{v_i\}$ and $N^+(v_i) = \Phi$

Hence $|N^+(u_1)| = |N^+(v_1)| = 1$ $------(13)$

Let $3 \le i \le n-1$ and i is odd.

Then gcd[$f(u_{i-1}), f(u_{i+1})$] = gcd[$\left(\frac{n}{2}\right) + \frac{i-1}{2}$, $\left(\frac{n}{2}\right) + \frac{i+1}{2}$]= gcd[$\frac{n+i-1}{2}, \frac{n+i+1}{2}$] = 1

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^+(u_i)$− − − − − − (14)

And $|N^+(v_i)| = 1$ − − − − − − −(15)

Let $2 \leq i \leq n$ and i is even. Then $|N^+(u_i)| = 1$ and $|N^+(v_i)| = 0$ − − − − − − −(16)

From (13), (14), (15) & (16) f is an outdegree prime pair labeling − − − − − − − −(IV)

By (III) & (IV), f is a prime pair labeling.

From case(i) & case(ii), f is a prime pair labeling of DDA$\overrightarrow{P_n \odot K_1}$.

Hence DDA$\overrightarrow{P_n \odot K_1}$ admits prime pair labeling.

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