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ANALYTICAL EXERTIONS OF J-CLOSED SETS

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Abstract: We initiate J-Closure,J-interior of a subset,J-Derived set, J-Border, J-Frontier, J-Exterior of a subsetand J-saturated set using the concept of J-closed sets &J-open sets here. We discuss some exciting features of J-Closure,J-interior &J-Derived set, J-Saturated sets. Moreover, we analyze interrelations of J-Border, J-Frontier, J-Exterior sets.
Keywords:J-Closure,J-interior of a subset,J-Derived set, J-Border, J-Frontier, J-Exterior.

1. Introduction

In 1937, Stone [6] introduced regular open sets and used it to define the semi-regularization of a topological space. In 1968, Velicko [7]proposed the class of δ-open sets contained in the category of open sets. Levine [2]has brought generalized closed sets in 1970. Dunham [1] has established an operator calledgeneralized closure using Levine's generalized closed sets as Cl*.In 2016, Pious Missier [5] had instituted regular*-open sets using Cl*.In 2019,Meenakshi etal initiated the study of η*-open sets [3] between the classes of δ-Open sets and open sets. Using η*-open sets, J- closed sets are introduced[4], and their features are explored in 2019.In this article, J-Closure,J-interiorof a subset,J-Derived set, J-Border, J-Frontier, J-Exteriorof a subset and J-saturated sets are defined using the concept of J-open sets here. Some exciting features of J-Closure,J-interior &J-Derived set, J-Saturated sets are obtained. Moreover, the interrelations between J-Border, J-Frontier, J-Exterior sets are analysed.

2. Preliminaries

Throughout this article, (Y,ζ) will always denote topological space on which no separation axioms are assumed, unless explicitly stated. If D is a subset of the space (Y,ζ), Cl(D) and int(D) denote the closure and interior of D respectively.

Definition 2.1 If D is a subset of a space (Y,ζ),
(i) The generalized closure of D [1] is defined as the intersection of all g-closed sets in Y containing D and is denoted by Cl*(D).

(ii) The generalized interior of D [1] is defined as the union of all g-open sets in Y contained in D and is denoted by int*(D).

Definition 2.2 Let (Y,ζ) be a topological space. A subset D of (Y,ζ) is called regular*-open (or r*-open) [5] if D = int(Cl*(D)).The complement of a regular*-open set is called a regular*-closed set. The union of all regular*-open sets of Y contained in D is called the regular*-interior of D and is denoted by r*int(D). The intersection of all regular*-closed sets of Y containing D is called the regular*-closure of Dis denoted by r*Cl(D).

Definition 2.3 A subset D of a topological space (Y,ζ) is called a η*-open set [3] if it is a union of regular*-open sets (r*-open sets).The complement of a η*-open set is called a η*-closed set. A subset D of a topological space (Y,ζ) is called η*-Interior of D is the union of all η*-open sets of Y contained in D andis denoted by η*-Int(D).The intersection of all η*-closed sets of Y containing D is called η*-closure of D and is denoted by η*-Cl(D).

Definition 2.4 A subset D of a topological space (Y,ζ)is said to be aJ-closed set [4]if
Cl(D)$\subseteq$M whenever D$\subseteq$M, M is η*-open in (Y,ζ).
The class of all J-closed sets of (Y,ζ) is denoted by JC(Y,ζ).

Definition 2.5 A subset D of a topological space (Y,ζ) is called J-open if itscomplement D^c is J-closed in (Y,ζ). The collection of allJ-open sets in (Y,ζ)is denoted byJO(Y,ζ).

Remark 2.6 *(i)* Every open set is a J-open set [4].

(ii) Finite intersection of J-open sets is J-open [4].

Definition2.7 A subset D of a topological space (Y,ζ) is said to be **Saturatedset** if Cl({x}) ⊆D for each x ∈D.

Definition2.8 Let D ⊆ Y. The **Frontier of D** is defined as Cl(D)−int(D).It is denoted by **Fr(D).**

Definition2.9 Let D be a subset of a topological space (Y,ζ). The **Border of D** is defined as D−int(D). It is denoted by **Br(D).**

Definition2.10 Let D be a subset of a topological space (Y,ζ). The **Exterior** of D is defined as Y−Cl(D) and is denoted by **Er(D).**

3. J-closure operator

In this section, the notion of J-closure of a set is introduced and some of its properties are studied.

Definition 3.1 The **J-closure of D** (briefly JCl(D)) of a topological space (Y,ζ)is defined as follows.

$$\boldsymbol{JCl(D)} = \cap\, \{\boldsymbol{F \subseteq Y : D \subseteq F and\, F \in JC(Y,\zeta)}\}$$

Proposition 3.2 Let D be any subset of (Y,ζ). If D is J -closed in (Y,ζ), thenJCl(D) = D.
Proof: Let D be J-closed in (Y,ζ). By definition, $JCl(D) = \cap\, \{F \subseteq Y: D \subseteq FandF \in JC(\mathrm{Y},\zeta)\}$. Since D is J -closed, the smallest F in the above collection is D itself and hence $\mathrm{J}Cl(D) = D$.

Example 3.3 (counter example) Let Y={p,q,r,s},ζ={ϕ,Y,{p},{p,q}}.HereJCl(Y,ζ)= P(Y)−**{** {p},{q},{p,q}}.Let D = {p,q}.Then JCl(D) ={p,q}=D≠ a J-closedset.

Remark 3.4 For a subset D of (Y,ζ),D ⊆ JCl(D) ⊆ Cl(D)[4].

Proposition 3.5 Let D and B be subsets of (Y, ζ). Then the following statements are true:

(a) JCl(∅) = ∅ and JCl(Y) = Y;

(b) If D⊆ B, thenJCl(D) ⊆ JCl(B);

(c) $D \subseteq JCl(D)$;

(d) $JCl(D) \cup JCl(B) = JCl(D \cup B)$;

(e) $JCl(D \cap B) \subseteq JCl(D) \cap JCl(B)$;

(f) $JCl(JCl(D)) = JCl(D)$.

Proof: (a), (b), (c) follow from Definition 3.1.

(d) Since $D \subseteq D \cup B$ and $B \subseteq D \cup B$. By (b), $JCl(D) \subseteq JCl(D \cup B)$ and $JCl(B) \subseteq JCl(D \cup B)$. Hence $JCl(D) \cup JCl(B) \subseteq JCl(D \cup B)$. To prove the reverse inequality, let $x \notin JCl(D) \cup JCl(B)$, then $x \notin JCl(D)$ and $x \notin JCl(B)$. Therefore, there exist J-closed sets U and V in Y such that $D \subseteq U$ and $B \subseteq V$ and $x \notin U$ and $x \notin V$. Hence we have $D \cup B \subseteq U \cup V$ and $x \notin U \cup V$(By Theorem 4.1from [4]), $U \cup V$ is J -closed and hence $x \notin JCl(D \cup B)$.
(e) Since $D \cap B \subseteq D$ and $D \cap B \subseteq B$. By *(b)*, $JCl(D \cap B) \subseteq JCl(D)$ and $JCl(D \cap B) \subseteq JCl(B)$. Hence $JCl(D \cap B) \subseteq JCl(D) \cap JCl(B)$.

(f) Follows from the definition of J-closure.

The converse of the above Proposition 3.5(e) is not true from the following counter example.

Example3.6 (counter example) Let $Y=\{p,q,r\}, \zeta=\{\phi, Y, \{p\}, \{q\}, \{p,q\}\}$.Then $JC(Y,\zeta)=\{\phi, Y, \{r\}, \{q,r\}, \{p,r\}\}$.Take $D = \{p\}$ and $B = \{q\}$, $D \cap B = \phi$, $JCl(D) = \{p,r\}$, $JCl(B) = \{q,r\}$, $JCl(D) \cap JCl(B) = \{r\}$ but $JCl(D \cap B) = \phi$.Hence $JCl(D) \cap JCl(B) \nsubseteq JCl(D \cap B)$.

From Proposition 3.5 *(a),(b),(c) &(d),(f)* we have that J-closure operator is a Kurtowski's closure operator.

Theorem 3.7 For each $y \in Y$, $y \in JCl(D)$ if and only if $U \cap D \neq \phi$ for every J-open set U in (Y, ζ)containing y.

Proof: Let $y \in JCl(D)$. Suppose that there exists a J-open set U in (Y,ζ)containing y such that $U \cap D = \phi$. Hence $D \subseteq Y - U$ is J-closed in (Y,ζ)which implies that $JCl(D) \subseteq Y - U$. Hence $y \notin JCl(D)$ which is a contradiction. Hence $U \cap D \neq \phi$.

Let us assume that $U \cap D \neq \phi$ for every J-open set U in (Y,ζ)containing y. Suppose that $y \notin JCl(D)$. By definition of J-closure, there exists a J-closed set U in (Y,ζ) containing D such that $y \notin U$. Hence $Y - U$ is J-open in (Y,ζ)containing y.Therefore $(Y-U) \cap D = \phi$, which is a contradiction.Hence $y \in JCl(D)$.

4. J-neighbourhood

Definition4.1 A subset M of a topological space (Y,ζ) is said to be a **J-Neighbourhood of** $x \in Y$ if there exists a J-open set D such that $x \in D \subseteq M$.
The set of all J-Neighbourhoods of x is denoted by **JNr(x)**.

Example4.2 Let $Y=\{p,q,r,s\}$, $\zeta=\{\phi, Y, \{p,q\}\}$.Then $JO(Y,\zeta) = \{\phi, Y, \{p\}, \{q\}, \{r\}, \{s\}, \{p,q\} \{q,r\}, \{p,r\}, \{p,s\}, \{q,s\}, \{p,q,r\}, \{p,q,s\}\}$.Here $\{q,r,s\}$ is a J-Neighbourhood of q as $q \in \{q,r\} \subseteq \{q,r,s\}$.

Theorem4.3 AJ-open set N is a J-Neighbourhood of each of its points.
Proof: Let N be a J-open set and $x \in N$. Then $x \in N \subseteq N$ satisfying the condition of N being a J-Neighbourhood.Since x is an arbitrary point of N, N is a J-Neighbourhood of each of its points.

Corollary 4.4 Every J-open set containing a point x is belongs to JNr(x).

Remark 4.5 A J-Neighbourhood of some point in Yneed not be a J-open set as observed from theupcoming example.

Example4.6 Let Y={p,q,r,s}, ζ={ϕ,Y,{p},{q,r},{p,q,r}}.Then JO(Y,ζ) = {ϕ, Y,{p},{q},{r},{p,q}{q,r},{p,r},{p,q,r}}.A subset {p,q,s} is J-Neighbourhood of p as p∈{p,q}⊆{p,q,s}.But it is not a J-open set.

Definition4.7 A subset M is said to be a **J-Neighbourhood of N**⊆ Y if there exists a J-open set A such that N ⊆ A⊆ M.
The set of all J-Neighbourhoods of N is denoted by **JNr(N)**.

Example4.8 In the above **Example 4.2**, {p,q,s} is J-Neighbourhood of {p} as {p}∈{p,q}⊆{p,q,s}.

Theorem 4.9 If M is a J-closed subset of a topological space(Y,ζ) and x ∈Y−M,then there exists a J-Neighbourhood N of x such that N ∩M =∅.
Proof: LetM be a J-closed subset of Y then Y −M is J-open in Y. By Theorem 4.3,Y−M is a J-Neighbourhood of each of its points. Therefore there exists a J-open set Nof x such that N⊆ Y −M which in turn implies that N ∩ M =∅.

Theorem4.10 In a topological space (Y,ζ) with x ∈ Y, the following results are true.
(i) JNr(x)≠ ∅;
(ii) If M ∈JNr(x),then x ∈ M;
(iii) If M ∈JNr(x) and M⊆ N,then N ∈JNr(x);
(iv) If M ∈JNr(x) and N ∈JNr(x),then M ∩ N∈JNr(x);
(v) If M ∈JNr(x) then there exists a N ∈JNr(x) such that N⊆ M and N∈ JNr(y),for every y ∈ N.
Proof: *(i)* SinceYitself is a J-open set by Theorem 4.3,it is a J-Neighbourhood for every x ∈ Y. That is Y∈JNr(x),for all x ∈Y.HenceJNr(x)≠ ∅ for all x ∈ Y.
(ii)Follows from the Definition 4.1.
(iii) Let M ∈JNr(x) and M⊆ N. Since M is a J-Neighbourhood of x then there exists a J-open set A such that x∈ A ⊆M.SinceM⊆ N,we get x∈ A ⊆ N.Hence N is a J-Neighbourhood of x.
(iv) LetM ∈JNr(x) and N ∈JNr(x) then there exists J-open sets A and B such
Thatx∈ A ⊆ M and x∈ B ⊆ N. This implies x∈ A∩B ⊆ M∩N. Now in order to prove M∩Nis a J-Neighbourhood of x, it is enough to prove that A∩B is J-open. Since finite intersection of J-open sets is J-open(by Remark 2.6 (ii)), A∩Bis J-open and hence M ∩ N is a J-Neighbourhood of x.Therefore M ∩ N∈JNr(x).
(v) Let M ∈JNr(x) then there exists a J-open set N such that x∈ N ⊆ M. Since N is J-open, it is a J-Neighbourhood of all its points (by Theorem 4.3).ThusN∈ JNr(y),for every y ∈ N.

Lemma 4.11 Nr(x) ⊆ JNr(x).
Proof: Let A ∈ Nr(x).Then ∃ B ∈ ζ such that x∈ B⊆A.Since every open set is a J-open set (by Remark 2.6 (ii)), A∈JNr(x).

Lemma 4.12 A collection $\mathcal{C}_x$satisfies:
(i) M ∈ $\mathcal{C}_x$ such that x ∈ M;
(ii) N,M∈ $\mathcal{C}_x$ implies N ∩M ∈ $\mathcal{C}_x$then $\mathcal{B}$ forms a basis for a topology where $\mathcal{B}$ = {∅} ∪{G ⊆ Y / x ∈ G implies there exists N ∈ $\mathcal{C}_x$ such that x∈ N ⊆ G} when (Y,ζ) is a topological spaceand x ∈ Y.

Corollary 4.13 If $\mathcal{C}_x$ =JNr(x) in Lemma 4.12 (ii),t hen JNr(x) forms a basis for a topology.

5. J-derived set

Definition5.1 Let D⊆Yand apoint $y \in Y$ is known as a **J-limit point** of D if every J-Neighbourhood of y intersects D in some point other than y itself.

Example 5.2 Let Y={p,q,r}, ζ={ϕ,Y,{p},{q},{p,q}}.Then JO(Y,ζ) ={ϕ,Y,{p},{q},{p,q}}.Here r is the limit point of {p,r}.

Definition5.3 The set of all J-limit points of D⊆Y is called J-Derived set of D and is denoted by JDr(D).

Theorem5.4 Let A,B⊆Y.Then the following statements are valid in (Y,ζ).

(i) JDr(∅) = ∅;
(ii) JDr(A)⊆Dr(A);
(iii) If A ⊆ B,then JDr(A)⊆JDr(B);
(iv) JDr(A∪B) = JDr(A) ∪ JDr(B);
(v) JDr(A∩B)⊆JDr(A)∩JDr(B);
(vi) [JDr(JDr(A))]– A ⊆JDr(A);
(vii) JDr(A∪JDr(A))⊆A∪JDr(A).

Proof: (i) Follows from the Definition 5.3.
(ii)Let x∈JDr(A).Then every J-Neighbourhood N of x is such thatN∩(A−{x})≠ ∅---(1).Consider N′ is a neighbourhood of x.By the above Lemma 4.11, N′isJ-Neighbourhood of x.Hence by (1), N′ ∩(A−{x})≠ ∅.Hence x ∈Dr(A).
(iii)Let x∈JDr(A).Thenforevery J-Neighbourhood N of x is such thatN∩(A−{x})≠ ∅.Since A ⊆ B, N∩(B−{x})≠ ∅. Therefore x∈JDr(B).
(iv)Since A⊆A∪B,JDr(A)⊆JDr(A∪B).Similarly B⊆A∪B,JDr(B)⊆JDr(A∪B).Therefore JDr(A)∪JDr(B)⊆JDr(A∪B).Suppose x∉ JDr(A)∪JDr(B).Then x∉JDr(A) or x∉JDr(B),that is x is neither a limit point of A nor of B.Therefore there exist J-Neighbourhoods N_1 and N_2,then N_1∩(A−{x}) = ∅ and N_2∩(B−{x}) = ∅.Here to prove N_1∩ N_2 is a J-Neighbourhood containing x.It is enough to prove A∩B is J-open.Sincefiniteintersection of J-open sets is J-open (by Remark 2.6(ii)),A∩B is J-open.Hence N_1∩ N_2 is a J-Neighbourhood containing x is such that(N_1∩ N_2)∩((A∪B)-{x}) = ∅ implies x∉ JDr(A∪B) gives JDr(A∪B)⊆ JDr(A)∪JDr(B).
(v) Since A∩B⊆A,B, the proof follows.
(vi) Let x∈JDr(JDr(A)) – A.Then N∩(JDr(A)−{x})≠ ∅ for each J-Neighbourhood N of x.Now let y∈ N ∩(JDr(A)-{x}) implies y∈N and y∈JDr(A). Here y∈JDr(A)gives N∩(A−{y})≠ ∅ for each J-NeighbourhoodN of y. So that take z∈ N∩(A−{y}).Then z≠x as z∈A and x∉A.Therefore N∩(A−{x})≠ ∅ for each J-Neighbourhood N of x. Hence x∈JDr(A).
(vii)Let x∈JDr(A∪JDr(A)).If x∈A,then the result is obvious. Suppose x∈ JDr(A∪JDr(A)) – A. Then N∩(A∪JDr(A)) −{x})≠ ∅ for each J-Neighbourhood N of x implies N∩(A−{x})≠ ∅ and N∩(JDr(A)−{x})≠ ∅. Now let y∈ N ∩ (JDr(A) − {x}) implies y∈N and y∈JDr(A). So N∩(A−{y})≠ ∅ for each J-Neighbourhood N of y. So that take z∈ N∩(A−{y}).Then z≠x as z∈A and x∉A.Therefore N∩(A−{x})≠ ∅ for each J-Neighbourhood N of x. Hence x∈JDr(A)and thus x∈ A∪JDr(A).

Theorem5.5 Let A⊆ Y.If A is J-closed then JDr(A) ⊆A.
Proof: Let x∈ JDr(A) then N∩(A−{x})≠ ∅ whenever N is a J-Neighbourhood of x implies N∩A≠ ∅ whenever N is a J-Neighbourhood of x-----(1).Consider N'∩A where N' is J-open,then by Theorem 4.3,every J-open is J-Neighbourhood of x and from (1), N'∩A≠ ∅.Therefore x∈JCl(A) = A, since A is J-closed. Hence the proof.

Corollary5.6 **JDr(A) ⊆ JCl(A).**

Theorem5.7 For any subset A of a topological space (Y,ζ),JCl(A) = A ∪ JDr(A).

Proof: By the above Corollary5.6,JDr(A) ⊆ JCl(A),A∪ JDr(A)⊆ JCl(A).On the other hand,let x ∈ JCl(A).If x ∈ A,then the proof is complete. If x ∉ A,each J-open set U containing x intersects A at a point distinct from x ; So x ∈ JDr(A).Thus JDr(A) ⊆ A ∪ JDr(A) which completes the proof.

Lemma5.8 **JCl(A) – A ⊆JDr(A).**
Proof: Let x ∈JCl(A)– A. Then x∈ JCl(A)& x ∉A. For every J-open set N containing x intersects A and x ∉A. By Theorem 4.3, for every J-Neighbourhood N containing x intersects A and x ∉A. Hence N∩(A−{x})≠ ∅ and x ∈ JDr(A). Hence the proof.

6. J-interior operator

In this section,the notion of J-Interior of a set is introduced and some of its properties are studied.

Definition 6.1 Let D be a subset of Y.A point y∈ D is said to be **J-interior point**of D if D is a J-neighbourhood of y.The set of all J-interior points of D is called the J-interior of D and is denoted by Jint(D).

Lemma 6.2 If D is a subset of Y,thenJint(D) = $\cup\{G: G \subseteq D \text{and } G \in JO((Y,\zeta))\}$.

Proof: Let D be a subset of Y. Let y ∈ Jint(D) ⟺ y is a J-interior point of D ⟺ D is a J-neighbourhood of y ⟺ ∃ a J-open set G such that y ∈ G⊆ D⟺y∈$\cup\{G: G \subseteq D \text{ and } G \in JO((Y,\zeta))\}$.Hence Jint (D) =$\cup\{G \subseteq Y: G \subseteq D \text{ and } G \in JO((Y,\zeta))\}$.

Lemma 6.3 If D is a subset of Y,thenint(D) ⊆Jint(D).

Proof*:* Let D be a subset of Y. Let y ∈int(D) ⇒ y∈$\cup\{G: G \subseteq D \text{ and } G \in \zeta$ ⇒ ∃an open set G such that y∈ G⊆ D.Since every open set is a J-open set (by Remark 2.6*(i)*) in Y⇒ y∈ $\cup\{G: G \subseteq D\text{and } G \in JO((Y,\zeta))\}$ ⇒ y∈ Jint(D).Hence int(D)⊆Jint(D).

The converse of the above Lemma6.3 is not true from the following counter example.

Example 6.4 (counter example) LetY={p,q,r},ζ ={ϕ, Y, {p,q}}.Here JO((Y,ζ))= P(Y). Let D={r},Jint(D) = {r},int(D) = ϕ.Hence Jint(D)⊈int(D).

In general intD is an open set. But JintD need not be a J-open set.It can be proved by the following counter example.

Example 6.5 (counter example) Let Y={p,q,r,s},ζ ={ϕ, Y, {p}}.Here JO((Y,ζ))= P(Y)−{q,r,s}.Let D = {q,r,s}, Jint(D) = {q,r,s} but it is not a J-open set.

Theorem 6.6 For any two subsets D and B of (Y, ζ), the following statements are true:

(a) Jint(Y) = Y andJint(ϕ) = ϕ;

(b) Jint(D) ⊆ D;

(c) If B is any J-open set contained in D, then B ⊆Jint(D);

(d) If D⊆B, then Jint(D)⊆Jint(B);

Proof*: (a)*It is obvious.

*(b)*Let y$\in$Jint(D).Then y is a J-interior point of D$\Rightarrow$ D is a J-neighbourhood of y$\Rightarrow$y$\in$ D.Hence Jint(D) $\subseteq$ D.

*(c)*Let B be any J-open set such that B $\subseteq$ D. Let y$\in$ B. Since B is a J-open set contained in D, y is a J-interior point of D. That is y$\in$Jint(D). Hence B $\subseteq$Jint(D).

*(d)*Let D and B be subsets of Y such that D$\subseteq$ B. Let y$\in$Jint(D). Then y is a J-interior point of D and so D is a J-neighbourhood of y.(ie) there exists a J-open set G such that y$\in$ G$\subseteq$ D.Now D$\subseteq$ B implies y$\in$ G$\subseteq$ B.Hence B is also J-neighbourhood of y then y$\in$Jint(B). Hence Jint(D)$\subseteq$Jint(B).

Remark 6.7 For a subset D of (Y,ζ), int(D)$\subseteq$Jint(D) $\subseteq$ D.

Lemma 6.8 Jint(D) = D − JDr(Y −D) .

Proof: Let x $\in$D − JDr(Y −D),then x$\notin$ JDr(Y −D).Therefore $\exists$ a J-open set G containing x such that G $\cap$(Y −D) = $\emptyset$ implies x$\in$G $\subseteq$D and hence x $\in$Jint(D).Now to prove Jint(D)$\subseteq$ D − JDr(Y −D).Let x $\in$Jint(D)then JintD is a J-Neighbourhood of x.Since JintD $\cap$(Y−D)=$\emptyset$,there is a J-Neighbourhood of x doesnot intersect Y −D implies x$\notin$ JDr(Y −D). Therefore x $\in$D − JDr(Y −D). Hence Jint(D) = D − JDr(Y −D) .

Proposition 6.9 Let D be any subset of (Y,ζ). If D is J -open in (Y,ζ)then Jint(D) = D.

Proof: Let D be J-open in (Y,ζ).We know that Jint(D) $\subseteq$ D. Also D is a J-open set contained in D. From above Theorem 6.7 *(c)*, D$\subseteq$Jint(D). Hence Jint(D) = D.

The Converse of Proposition6.10 need not be true.

Example 6.10 (counter example) Same as in Counter Example 6.5. Here Jint(D) = D but D is not a J-open set.

Corollary 6.11 Jint(Jint(D)) =Jint(D).

Theorem 6.12 If D and B are subsets of Y, then Jint(D) $\cup$Jint(B) $\subseteq$Jint(D$\cup$B).

Proof: Since D$\subseteq$D$\cup$B and B $\subseteq$D$\cup$B, by Theorem6.6 *(d)*, Jint(D) $\subseteq$Jint(D$\cup$B) and Jint(B) $\subseteq$Jint(D$\cup$B). Hence Jint(D) $\cup$Jint(B) $\subseteq$Jint(D$\cup$B).

The converse of Theorem 6.12 need not be true as seen from the following counter example.

Example 6.13 (counter example) Let Y = {p,q,r,s}, ζ= {Y,ϕ, {r}, {p,q}, {p,q,r}. Here JO(Y,ζ) = { Y,ϕ,{p},{q},{r},{p,r},{q,r},{p,q}, {p,q,r}}.Let D = {p,q,r}, B = {p,q,s} and D$\cup$B = {p,q,r,s } =Y then Jint{D} = {p,q,r}, Jint{B} = {p,q} and Jint{D$\cup$B} = Y. Hence Jint(D$\cup$B) = Y$\nsubseteq$Jint{D} $\cup$Jint{B} = {p,q,r}.

Theorem 6.14 If D and B are subsets of Y, then Jint(D$\cap$B) = Jint(D) $\cap$Jint(B).

Proof: Since D$\cap$B $\subseteq$D and D$\cap$B $\subseteq$B, by Theorem 6.7*(d)*, Jint(D$\cap$B)$\subseteq$Jint(D) and Jint(D$\cap$B)$\subseteq$Jint(B). Hence Jint(D$\cap$B)$\subseteq$Jint(D)$\cap$Jint(B). In other way,t o prove Jint(D) $\cap$Jint(B) $\subseteq$Jint(D$\cap$B).Let y $\in$ Jint(D)&y $\in$ Jint(B).Then y is a J-interior point of D &y is a J-interior point of B. That implies D is a J-neighbourhood of y& B is a J-neighbourhood of y. Therefore $\exists$ a J-open set G such that y $\in$ G$\subseteq$ D&$\exists$ a J-open set M such that y$\in$M $\subseteq$ D. By Remark 2.6*(ii)*,$\exists$ a J-open set such that y $\in$ G$\cap$ M $\subseteq$D$\cap$B. Hence y $\in$Jint(D$\cap$B).

Remark: From Theorem 6.7 (a),(b) & Corollary 6.12, Theorem 6.15, J-interior operator is a Kuratowski's interior operator.

Theorem 6.15 Let D be any subset of (Y,ζ), then

- *(a)* $(\mathrm{Jint(D)})^c = \mathrm{JCl}(D^c)$.
- *(b)* $(\mathrm{JCl(D)})^c = \mathrm{Jint}(D^c)$.

Proof*:* *(a)*Let $x \in (\mathrm{Jint(D)})^c$.Then $x \notin Jint(D)$.That is every J-open set U containing x is such that $U \nsubseteq$ D.That is every J-open set U containing x is such that $U \cap D^c \neq \emptyset$.By Theorem 3.8,$x \in JCl(D^c)$ and therefore $(\mathrm{Jint(D)})^c \subseteq \mathrm{JCl}(D^c)$.

Conversely, let$x \in \mathrm{JCl}(D^c)$.Then by Theorem 3.8,every J-open set U containing x is such that $U \cap D^c \neq \emptyset$. Then $x \notin Jint(D)$. Hence $x \notin \left(Jint(D)\right)^C$ and $JCl(D^c) \subseteq \left(\mathrm{Jint(D)}\right)^c$.Thus $(\mathrm{Jint(D)})^c = \mathrm{JCl}(D^c)$.

(b) Follows by replacing int by Cl & Cl by int in *(a).*

Remark*:* In other notation, the above Theorem 6.15 can be stated as follows :

- *(a)* Y−Jint(D) = JCl(Y −D).
- *(b)* Y−JCl(D) = Jint(Y −D).

7. J-saturated set

Definition7.1 *A* subset D of a topological space (Y,ζ) is said to be **J-Saturated set** if JCl({x}) ⊆D for each x ∈D.The set of all J-saturated sets in (Y,ζ) is denoted by **JSr(Y)**.

Example7.2 Consider Y={m,n,o},ζ ={ϕ, Y,{m},{n},{m,n}}.Then JC(Y,ζ) = {ϕ, Y,{n,o},{m,o},{o}}. Let D = {n,o}.Then JCl({x}) ⊆D for each x∈D,it is a J-saturated set.

Theorem 7.3 Every J-closed set is a J-Saturated set but not conversely.

Proof: Let D be a J-closed set and x ∈D.Then {x} ⊆ D.Take J-closure on both sides of {x} ⊆D,we get JCl({x}) ⊆ JCl(D) = D,as D is J-closed.Therefore D is J-Saturated.

Example 7.4 (counter example) Let Y={p,q,r,s}, ζ={ϕ, Y,{p},{q},{p,q},{p,q,s},{p,q,r}}. Then JC(Y,ζ) = {ϕ, Y,{r},{s},{q,r},{p,r},{p,s},{r,s},{q,s},{p,q,r},{p,r,s},{p,q,s},{q,r,s}}.Let D = {p,q}.Then JCl({x}) ⊆D for each x∈D,it is J-saturated but D is not J-closed.

8. J-frontier

Definition8.1 Let D ⊆ Y. A subset D of (Y,ζ) is known as the **J-Frontier of D** is defined as JCl(D)−Jint(D) and is denoted by **JFr(D)**.

Example 8.2 In the above Example 4.6, JC(Y,ζ) = {ϕ, Y, {s},{p,s},{r,s},{q,s},{p,r,s},{p,q,s}{q,r,s}}.Let D = {r},JCl(D) = {r,s}.Here Jint(D) = {r}.Therefore JFr(D) = JCl(D) −Jint(D) = {r,s} −{r} = {s}.

Proposition 8.3 Let D ⊆ Y. Then the upcoming results are perfect.

(a) JFr(D) ⊆Fr(D);

(b) JCl(D) = Jint(D) ∪ JFr(D);

(c) Jint(D) ∩ JFr(D) = ∅;

(d) If D is a J-open set then JFr(D) ⊆ JDr(D);

(e) JFr(D) = JCl(D) ∩ JCl(Y− D);

(f) JFr(D) is J-closed;

(g) JFr(D) = JFr(Y−D);

(h) JFr(JFr(D)) ⊆ JFr(D);

(i) JFr(Jint(D)) ⊆ JFr(D);

(j) JFr(JCl(D)) ⊆ JFr(D);

Proof :

(a) Since JCl(D) ⊆ Cl(D) (by Remark 3.4)& int(D) ⊆ Jint(D) (by Lemma 6.3).It givesJCl(D) – Jint(D) ⊆ Cl(D) –int(D) implies JFr(D) ⊆ Fr(D).

(b) RHS = Jint(D) ∪ JFr(D) = Jint(D) ∪ [JCl(D)−Jint(D)] =Jint(D) ∪ [JCl(D)∩ (Y −Jint(D))] =(Jint(D) ∪ JCl(D))∩(Jint(D) ∪ (Y −Jint(D))) = JCl(D)∩ Y = JCl(D) [By Jint(D) ⊆ D ⊆ JCl(D)] =LHS.

(c) Jint(D) ∩ JFr(D) = Jint(D) ∩[JCl(D) –Jint(D)] = Jint(D) ∩[JCl(D)∩(Y −Jint(D))] = Jint(D) ∩(Y −Jint(D))∩JCl(D) =JCl(D)∩ ∅ = ∅.

(d) Given D is J-open implies that Jint(D) = D.JFr(D) = JCl(D) – Jint(D) = JCl(D) – D ⊆ JDr(D).(By Lemma 5.8)

(e) RHS = JCl(D) ∩ JCl(Y− D) =JCl(D) − Jint(D) = JFr(D) = LHS.

(f) JCl(JFr(D)) = JCl(JCl(D) –Jint(D)) = JCl(JCl(D)∩JCl(Y−D)) ⊆JCl(JCl(D)) ∩JCl(JCl(Y−D)) =JCl(D) ∩JCl(Y−D) (by Proposition 3.5(g)) =JCl(D) −Jint(D) =JFr(D).Hence JFr(D) is J-closed.

(g) RHS = JFr(Y−D) = JCl(Y −D) –Jint(Y−D) = (Y –Jint(D)) –(Y – JCl(D)) = (Y –Jint(D)) ∩JCl(D) = JCl(D) –Jint(D) = JFr(D) = LHS.

(h) JFr(JFr(D)) = JCl(JFr(D)) ∩JCl(Y−JFr(D))⊆JCl(JFr(D)) = JFr(D) by (e) & (f).

(i) JFr(Jint(D)) = JCl(Jint(D))–Jint(Jint(D)) = JCl(Jint(D))–Jint(D)⊆ JCl (D)–Jint(D) = JFr(D) (Since JCl(Jint(D))⊆JCl (D)).

(j) JFr(JCl(D)) = JCl(JCl(D)) –Jint(JCl(D))⊆JCl (D)–Jint (D) =JFr(D) (Since Jint(JCl (D))⊇Jint (D)).

Proposition 8.4 Let A⊆B and Jint(B) = ∅ then JFr(A)⊆JFr(B).

Proof: Let A⊆B and Jint(B) = ∅. Let x ∈JFr(A) = JCl(A) –Jint(B) as A⊆B and since Jint(B) = ∅,we get x ∈ JCl(A)⊆JCl(B) = JCl(B)− Jint(B) = JFr(B).Hence x∈JFr(B).

9. J-border

Definition9.1 Let D be a subset of a topological space (Y,ζ). The **J-Border of D** is defined as D −Jint(D) and is denoted by **JBr(D)**.

Example9.2 In the above Example 4.6,*l*et D = {q,r,s},then Jint(D) = {q,r}.Therefore JBr(D) = {s}.

Theorem 9.3 Let A be a subset of a topological space (Y,ζ).Then the following results hold:

(a) JBr(A) ⊆Br(A) ;

(b) JBr(∅) = ∅;

(c) JBr(Y) = ∅;

(d) JBr(A) ⊆ A;

(e) A = Jint(A) ∪JBr(A) ;

(f) If A is J-open,thenJBr(A) = ∅;

(g) Jint(A) ∩JBr(A) = ∅;

(h) JBr(Jint(A)) = ∅;

(i) Jint(JBr(A)) = ∅;

(j) JBr(JBr(A)) = JBr(A) ;

(k) JBr(A) = A ∩ JCl(Y−A);

(l) JBr(A) = JDr(Y –A);

(m) Jint(A) = A –JBr(A);

(n) JBr(A) ⊆ JFr(A);

Proof:

(a) JBr(A) = A – JintA ⊆ A –int A =Br(A) (By Lemma 6.3).
(b) It is a trivial one.
(c) By definition JBr(Y) = Y−Jint(Y) = Y – Y = ∅.
(d) It follows from the Definition of Border of A.
(e) RHS = Jint(A) ∪JBr(A) =Jint(A) ∪[A−Jint(A)] = Jint(A) ∪ (A∩(Y−JintA)) = (Jint(A) ∪ A) ∩ (Jint(A) ∪ (Y−JintA)) = A∩ Y = A=LHS.
(f) Let A be J-open.ThenJint(A) = A.By definition of JBr(A) = A –Jint(A) = A− A = ∅.
(g) LHS = Jint(A) ∩JBr(A) = Jint(A) ∩[A−Jint(A)] = Jint(A) ∩ [A∩(Y –Jint A)] =∅ =RHS.
(h) JBr(Jint(A)) = Jint(A) – Jint(Jint(A)) = Jint(A) – Jint(A) = ∅.
(i) Let x ∈ Jint(JBr(A)).Then x ∈JBr(A) .In other way JBr(A) ⊆ A by (d).Hence x ∈Jint(JBr(A)) ⊆ Jint(A).This implies that x ∈ Jint(A) ∩JBr(A) which contradicts (g).Therefore Jint(JBr(A)) = ∅.
(j) LHS = JBr(JBr(A)) = JBr(A −Jint(A)) = A –Jint(Jint(A)) = A –Jint(A) =JBr(A) = RHS.
(k) JBr(A) = A – Jint(A) = A – (Y – JCl(Y –A)) = A ∩ JCl(Y−A).
(l) JBr(A) = A –Jint(A) = A –(A –JDr(Y−A))(by Lemma 6.9)=JDr(Y –A).
(m) RHS = A –JBr(A) = A – (A –Jint(A)) = Jint(A) = LHS.
(n) Direct Proof.

Remark 9.4 The converse of the above Theorem 9.3 *(f)* is not true as seen from the following counter example.

Example 9.5 (counter example) In Counter Example 6.5, JBr(D) = ∅.But D is not J-open.

10. J-exterior

Definition 10.1 Let D be a subset of a topological space (Y,ζ). The **J-Exterior** of D is defined as Y−JCl(D) and is denoted by **JEr(D).**

Example 10.2 In the above Example 8.2,take D = {p}.Then JCl(D) = {p,s}.Therefore JEr(D) = Y−JCl(D) = Y−{p,s} = {q,r}.

Theorem 10.3 Let A be a subset of a topological space (Y,ζ).Then the following results hold:

(a) JEr(A) ⊆Er(A);

(b) JEr(Y) = ∅;
(c) JEr(∅) = Y;
(d) JEr(A) = Jint(Y −A) = Y −JCl(A);
(e) If A ⊆B then JEr(A) ⊇ JEr(B);
(f) JEr(A∪B) = JEr(A) ∩JEr(B);
(g) JEr(JEr(A)) = Jint(JCl(A));
(h) JEr(A) = JEr(Y − JEr(A));
(i) Y = Jint(A) ∪ JEr(A) ∪ JFr(A);
(j) Jint(A) ⊆ JEr(JEr(A));
(k) JEr(A) ∪ JEr(B) ⊆JEr(A ∩ B);

Proof :

(a) JEr(A) = Y −JCl(A) ⊆Y−Cl(A) = Er(A).(By Remark 3.4).
(b) JEr(Y) = Y −JCl(Y) = Jint(Y −Y) =Jint(∅) = ∅.
(c) JEr(∅) = Y − JCl(∅) = Y.
(d) From this JEr(A) = Y − JCl(A) = Jint(Y − A)(by Remark 6.18(b)),we got the proof.
(e) JEr(A) = Y− JCl(A) = Jint(Y−A) ⊇ Jint(Y−B) as A ⊆B .Hence JEr(A) ⊇ JEr(B).
(f) LHS = JEr(A∪B) = Y −JCl(A ∪ B) = Y − (JCl(A) ∪ JCl(B))(by Proposition 3.5(d)) = (Y − JCl(A)) ∩ (Y − JCl(B)) = JEr(A) ∩ JEr(B) = RHS.
(g) JEr(JEr(A)) =JEr(Y −JCl(A)) = Jint(Y −(Y −JCl(A)))(by Remark 5.9(b)) =Jint(JCl(A)).
(h) JEr(Y − JEr(A)) = JEr(Y −(Y−JCl(A))) = JEr(JCl(A)) = (Y −JCl(JCl(A)))(by (d))= Y− JCl(A) = JEr(A).
(i) Jint(A) ∪ JEr(A) ∪ JFr(A) = (Jint(A) ∪ JFr(A)) ∪JEr(A) = JCl(A) ∪ (Y −JCl(A)) = Y.(By Proposition 8.3 (b)).
(j) Jint(A) ⊆ Jint(JCl(A)) = JEr(JEr(A)),by (h).
(k) JEr(A) ∪ JEr(B) = Jint(Y−A) ∪ Jint(Y−B) ⊆ Jint((Y−A)∪ (Y-B))(by Theorem 6.13) =Jint(Y −(A∩B)) = JEr(A∩B), by (d).

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