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NUMERICAL SOLUTION OF INTUITIONISTIC FUZZY DIFFERENTIAL EQUATION USING EXPONENTIAL RUNGE-KUTTA METHOD

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ABSTRAC:.The aim of this paper is to propose Exponential Runge-Kutta (ERK) methods of order 3 and 4 for solvingintuitionistic fuzzy differential equations (IFDEs). Convergence analysis of ERK methods has been carried out. The applicability of ERK methods is illustrated by solving intuitionistic fuzzy differential equations with triangular intuitionistic fuzzy numbers. Comparison of the numerical solution with exact solution shows good accuracy.

Keywords—Intuitionistic Fuzzy differential equations, IVP, ERK methods, Convergence.

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1. Introduction

Fuzzy set theory is a useful tool to describe the situation in which data are imprecise or vague or uncertain. The concept of fuzzy set theory was introduced by Zadeh [22]. This concept was extended to the intuitionistic fuzzy set (IFS) theory by Atanassov [7]. Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of science and engineering. Chang and Zadeh [10] first introduced the concept of fuzzy derivative. Kaleva [11, 12] and Seikkala [19] extensively studied the fuzzy initial value problems.

Recently many research articles are focused on numerical solution of fuzzy initial value problems. Ming Ma et al. [14] introduced Euler method for solving FDEs numerically, Ahmad and Hasan [3] have discussed a new fuzzy version of Euler's method with fuzzy initial values. The authors [5, 6] have developed Exponential Runge-Kutta (ERK) methods to obtain the numerical solution of FDEs and different means. Bede et al. [8, 9] introduced a new concept of the generalized of fuzzy interval valued functions.

Melliani et al. [13] have discussed differential and partial differential equations under intuitionistic fuzzy environment. Abbasbandy and Allahviranloo [1,2,4] introduced the numerical solution of FDE by Runge-Kutta method with intuitionistic treatment. Mondal and Roy [21] have discussed the first order homogeneous ordinary differential equation with initial value as triangular intuitionistic fuzzy number.

Sneh Lata and Amit Kumar [20] have introduced time dependent intuitionistic fuzzy linear differential equation. Nirmala et al. [15 – 18] have discussed numerical solution of intuitionistic fuzzy differential equation by Euler, Modified Euler, fourth order RK and predictor-corrector methods under generalized differentiability concept.

This paper presents Exponential RK methods of order 3 and 4 for solving intuitionistic fuzzy IVPs. The convergence analysis of Exponential RK methods has been discussed. The efficiency of these methods has been illustrated by numerical examples.

2. Preliminaries

Definition 2.1 Intuitionistic Fuzzy Set (IFS): Let a set X be fixed. An Intuitionistic fuzzy set A in X is an object having the form $A=\{< x,\mu_A(x),\vartheta_A(x)>0 : x\in X\}$ where the $\mu_A(x):\mathrm{X}\to[0,1]$ and $\vartheta_A(x):\mathrm{X}\to[0,1]$ define the degree of membership and degree of non-membership respectively, The element $x\in X$ to the set A which is a subset of X, for every element of $x\in X$, $0<\mu_A(x)+\vartheta_A(x)\le 1.$

Definition2.2 Intuitionistic Fuzzy Number (IFN):An Intuitionistic fuzzy number A is defined as follows:

(i) an intuitionistic fuzzy subset of real line.

(ii) normal.

i.e., there is any $x_0\in R$ such that $\mu_A(x)=1$ (so $\vartheta_{\tilde{A}^i}(x)=0$)

(iii) a convex set for the membership function $\mu_A(x)$

i.e., $\mu_A(\lambda x_1+(1-\lambda)\,\mathrm{x}_2)\ge\min(\mu_A(x_1),\mu_A(x_2))\ \forall x_1,x_2\in R,\lambda\in[0,1]$

(iv)a concave set for the non-membership function $\vartheta_{\tilde{A}^i}(x)$,

i.e., $\vartheta_A(\lambda x_1+(1-\lambda)\,\mathrm{x}_2)\le\min(\vartheta_A(x_1),\vartheta_A(x_2))\ \forall x_1,x_2\in R,\lambda\in[0,1]$.

Definition2.3. An intuitionistic fuzzy set $A=\{x,\mu_A(x),\vartheta_A(x)\setminus x\in X\}$ such that $\mu_A(x)$ and $(1-\vartheta_A)(x)=1-\vartheta_A(x),\forall\,\mathrm{x}\in R$ are fuzzy numbers. Therefore IFS $A=\{x,\mu_A(x),\vartheta_A(x)\setminus x\in X\}$ is a conjecture of two fuzzy numbers, A^+ with a membership function $\mu_{A^+}(x)=\mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x)=1-\vartheta_A(x)$.

Definition 2.4.The α,β -cut of an IFN $A=\{x,\mu_A(x),\vartheta_A(x)\setminus x\in X\}$ is defined as

$A=\{x,\mu_A(x),\vartheta_A(x)\setminus x\in X,\mu_A(x)\ge\alpha\ and\ \vartheta_A(x)\le\beta\}\ \forall\,x\in[0,1]$, where $\beta=1-\alpha$.

The α,β -cut representation of intuitionistic fuzzy number. A generates the following pair of intervals and is denoted by $[\mathrm{A}]_{\alpha,\beta}=\{[\mathrm{A}_L^+(\alpha),\mathrm{A}_U^+(\alpha)],[\mathrm{A}_L^-(\beta),\mathrm{A}_U^-(\beta)]\}$.

Definition 2.5 **Triangular Intuitionistic Fuzzy Number.** A Triangular Intuitionistic Fuzzy Number (TIFN) A is an intuitionistic fuzzy set in R with the following membership function $\mu_A(x)$ and non-membership function $\vartheta_A(x)$ given as follows:

$$\mu_A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2}, & a_2\le x\le a_3\\ 0, & \text{otherwise}\end{cases}\qquad \vartheta_A(x)=\begin{cases}\dfrac{a_2-x}{a_2-a_1'}, & a_1'\le x\le a_2\\ \dfrac{x-a_2}{a_3'-a_2}, & a_2\le x\le a_3'\\ 1, & \text{otherwise}\end{cases}$$

where $a_1' \le a_1 \le a_2 \le a_3 \le a_3'$ and TIFN is denoted by $A=(a_1,a_2,a_3;\ a_1',\ a_2,\ a_3')$.

Definition 2.7. For arbitrary $u,\ v\in E^n$, the quantity $\mathrm{D}(u,\ v)=\sup-0\le\alpha\le 1, d([\mathrm{u}]^\alpha,[\mathrm{v}]^\alpha)$ is the distance between u and v, where d is the Hausdroff metric in E^n.

Definition2.8. Let mapping $f: I\to W^n$ for some interval I be an intuitionistic fuzzy function. The $\alpha,\beta-$cut of f is given by $[f(t)]_\alpha=\left\{[\underline{f}^+(\mathrm{t};\alpha),\overline{f}^+(\mathrm{t};\alpha)],[\underline{f}^-(\mathrm{t};\alpha),\overline{f}^-(\mathrm{t};\alpha)]\right\}$, where

$$\underline{f}^+(\mathrm{t};\alpha)=\min\left\{f^+(\mathrm{t};\alpha)\,|\,\mathrm{t}\in\mathrm{I},\ 0\le\alpha\le 1\right\},\ \overline{f}^+(\mathrm{t};\alpha)=\max\left\{f^+(\mathrm{t};\alpha)\,|\,\mathrm{t}\in\mathrm{I},\ 0\le\alpha\le 1\right\},$$
$$\underline{f}^-(\mathrm{t};\alpha)=\min\left\{f^-(\mathrm{t};\alpha)\,|\,\mathrm{t}\in\mathrm{I},\ 0\le\beta\le 1\right\},\ \overline{f}^-(\mathrm{t};\alpha)=\max\left\{f^-(\mathrm{t};\alpha)\,|\,\mathrm{t}\in\mathrm{I},\ 0\le\beta\le 1\right\}.$$

3. Intuitionistic Fuzzy Cauchy Problem

A first order intuitionistic fuzzy differential equation is of the form

$$y'(t)=f(t,y(t)),\qquad t\in[\mathrm{a,b}]\qquad y(t_0)=y_0 \tag{3.1}$$

where y is an intuitionistic fuzzy function of the crisp variable t, $f(t,y(t))$ is an intuitionistic fuzzy function of the crisp variable *t* and the intuitionistic fuzzy variable y and *y'* is the intuitionistic fuzzy derivative.

If an initial value $y(t_0)=y_0$ {intuitionistic fuzzy number}, we get an intuitionistic fuzzy Cauchy problem of first order $y'(t)=f(t,y(t),\ \ y(t_0)=y_0$

As each intuitionistic fuzzy number is a conjecture two fuzzy numbers of equation (3.1) can be replaced by an equivalent system as follows:

$$y'(t)=\left\{[\underline{y'^+}(\mathrm{t};\alpha),\overline{y'^+}(\mathrm{t};\alpha)],[\underline{y'^-}(\mathrm{t};\beta),\overline{y'^-}(\mathrm{t};\beta)]\right\},$$

where

$$\underline{y}'^+(\mathrm{t};\alpha)=\underline{f}^+(t,y^+)=\min\left\{f^+(t.u)\,|\,u\in[\underline{y}^+,\overline{y}^+]\right\}=F(t,\ \underline{y}^+,\overline{y}^+),\quad \underline{y}^+(t_0)=\underline{y_0}^+ \tag{3.2}$$
$$\overline{y}'^+(\mathrm{t};\alpha)=\overline{f}^+(t,y^+)=\max\left\{f^+(t.u)\,|\,u\in[\underline{y}^+,\overline{y}^+]\right\}=G(t,\ \underline{y}^+,\overline{y}^+),\quad \overline{y}^+(t_0)=\overline{y_0}^+ \tag{3.3}$$
$$\underline{y}'^-(\mathrm{t};\beta)=\underline{f}^-(t,y^-)=\min\left\{f^-(t.u)\,|\,u\in[\underline{y}^-,\overline{y}^-]\right\}=H(t,\ \underline{y}^-,\overline{y}^-),\quad \underline{y}^-(t_0)=\underline{y_0}^- \tag{3.4}$$
$$\overline{y}'^-(\mathrm{t};)\beta=\overline{f}^-(t,y^-)=\max\left\{f^-(t.u)\,|\,u\in[\underline{y}^-,\overline{y}^-]\right\}=I(t,\ \underline{y}^-,\ \overline{y}^-),\quad \overline{y}^-(t_0)=\overline{y_0}^- \tag{3.5}$$

The system of equations given in (3.2) and (3.3) will have unique solution $[\underline{y}^+,\overline{y}^+]\in B=\overline{c}[0,1]\times\overline{c}[0,1]$ and the system of equations given in (3.4) and (3.5) will have unique solution $[\underline{y}^-,\overline{y}^-]\in B=\overline{c}[0,1]\times\overline{c}[0,1]$

Therefore, the system given from equations (3.2) to (3.5) possesses unique solution

$$\mathrm{y(t)} = \left\{ [\underline{y}^{+}(\mathrm{t}), \overline{y^{+}}(\mathrm{t})], [\underline{y}^{-}(\mathrm{t}), \overline{y^{-}}(\mathrm{t})] \right\} \in B \times B$$

which is an intuitionistic fuzzy function.

(i. e) for each t, $\mathrm{y}(\mathrm{t};\alpha;\beta) = \left\{ [\underline{y}^{+}(\mathrm{t};\alpha), \overline{y^{+}}(\mathrm{t};\alpha)], [\underline{y}^{-}(\mathrm{t};\beta), \overline{y^{-}}(\mathrm{t};\beta)] \right\}, \alpha, \beta \in [0,1]$ is an intuitionistic fuzzy number.

The parametric form of the system of equations (3.2) to (3.5) is given by

$$\underline{y}'^{+}(\mathrm{t};\alpha) = F(t, \underline{y}^{+}(\mathrm{t};\alpha), \overline{y^{+}}(\mathrm{t};\alpha)), \quad \underline{y}^{+}(t_0;\alpha) = \underline{y_0}^{+}(\alpha)$$

$$\overline{y'^{+}}(\mathrm{t};\alpha) = G(t, \underline{y}^{+}(\mathrm{t};\alpha), \overline{y^{+}}(\mathrm{t};\alpha)), \quad \overline{y^{+}}(t_0;\alpha) = \overline{y_0}^{+}(\alpha)$$

$$\underline{y}'^{-}(\mathrm{t};\beta) = H(t, \underline{y}^{-}(\mathrm{t};\beta), \overline{y^{-}}(\mathrm{t};\beta)), \quad \underline{y}^{-}(t_0;\beta) = \underline{y_0}^{-}(\beta)$$

$$\overline{y'^{-}}(\mathrm{t};\beta) = I(t, \underline{y}^{-}(\mathrm{t};\beta), \overline{y^{-}}(\mathrm{t};\beta)), \quad \overline{y^{-}}(t_0;\beta) = \overline{y_0}^{-}(\beta) \text{ for } \alpha, \beta \in [0,1].$$

4. Exponential Runge-Kutta methods for solving Intuitionistic Fuzzy differential equations

Consider the intuitionistic fuzzy IVP

$$y'(t) = Ly + f(t, y(t)), \quad y(t_0) = y_0 \quad \text{for } t \in [t_0, T] \tag{4.1}$$

where y is a intuitionistic fuzzy function of t, $f(t, y)$ is a intuitionistic fuzzy function of the crisp variable t and the intuitionistic fuzzy variable y, y' is the intuitionistic fuzzy derivative of y. Here $y(t_0) = y_0$ is a intuitionistic fuzzy number and L is crisp number.

The general s-stage Exponential RK method for intuitionistic fuzzy IVPs is given by

$$\begin{aligned} \underline{y}^{+}_{n+1} &= e^{Lh}\underline{y}^{+}_{n} + h\sum_{i=1}^{s} b_i K_i \quad \overline{y^{+}}_{n+1} = e^{Lh}\overline{y^{+}_{n}} + h\sum_{i=1}^{s} b_i L_i \\ \underline{y}^{-}_{n+1} &= e^{Lh}\underline{y}^{-}_{n} + h\sum_{i=1}^{s} b_i M_i \quad \overline{y^{-}}_{n+1} = e^{Lh}\overline{y^{-}_{n}} + h\sum_{i=1}^{s} b_i N_i \end{aligned} \tag{4.2}$$

where

$$K_i = F(t_n + \mathrm{c}_i\,\mathrm{h}, \; e^{\mathrm{c}_i Lh}\underline{y}^{+}_{n} + \mathrm{h}\sum_{j=1}^{i-1} \mathrm{a}_{ij}\mathrm{K}_{\mathrm{j}}),$$

$$L_i = G(t_n + \mathrm{c}_i\,\mathrm{h}, \; e^{\mathrm{c}_i Lh}\overline{y}^{+}_{n} + \mathrm{h}\sum_{j=1}^{i-1} \mathrm{a}_{ij}\, L_{\mathrm{j}}),$$

$$M_i = H(t_n + \mathrm{c}_i\,\mathrm{h}, \; e^{\mathrm{c}_i Lh}\underline{y}^{-}_{n} + \mathrm{h}\sum_{j=1}^{i-1} \mathrm{a}_{ij}\, M_{\mathrm{j}}),$$

$$N_i = I(t_n + \mathrm{c}_i\,\mathrm{h}, \; e^{\mathrm{c}_i Lh}\overline{y}^{-}_{n} + \mathrm{h}\sum_{j=1}^{i-1} \mathrm{a}_{ij}\, N_{\mathrm{j}}), \quad i = 1, 2, \cdots, s$$

and $\sum_{i=1}^{s} b_i(0) = 1, \quad \sum_{j=1}^{s} a_{ij}(0) = c_i, \quad i = 1, 2, \cdots, s.$

Setting s = 3, we get third order Exponential RK method for Intuitionistic Fuzzy IVP which is given by

$$\underline{y^+}(t_{n+1};\,\alpha)=e^{Lh}\,\underline{y^+}(t_n;\,\alpha)+\frac{h}{6}(\mathrm{K}_1+4K_2+K_3)$$
$$\overline{y^+}(t_{n+1};\,\alpha)=e^{Lh}\,\overline{y^+}(t_n;\,\alpha)+\frac{h}{6}(\mathrm{L}_1+4L_2+L_3)$$
$$\underline{y^-}(t_{n+1};\,\beta)=e^{Lh}\,\underline{y^-}(t_n;\,\beta)+\frac{h}{6}(\mathrm{M}_1+4M_2+M_3)$$
$$\overline{y^-}(t_{n+1};\,\beta)=e^{Lh}\,\overline{y^-}(t_n;\,\beta)+\frac{h}{6}(\mathrm{N}_1+4N_2+N_3)$$

where

$$K_1=F(t_n,\underline{y^+}(\mathrm{t}_n;\alpha),\overline{y^+}(\mathrm{t}_n;\alpha))$$
$$L_1=G(t_n,\underline{y^+}(\mathrm{t}_n;\alpha),\overline{y^+}(\mathrm{t}_n;\alpha))$$
$$M_1=H(t_n,\underline{y^-}(\mathrm{t}_n;\beta),\overline{y^-}(\mathrm{t}_n;\beta))$$
$$N_1=I(t_n,\underline{y^-}(\mathrm{t}_n;\beta),\overline{y^-}(\mathrm{t}_n;\beta))$$
$$K_2=F(t_n+\frac{1}{2}\mathrm{h},\ e^{\frac{1}{2}Lh}\underline{y^+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hK}_1,e^{\frac{1}{2}Lh}\overline{y^+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hL}_1)$$
$$L_2=G(t_n+\frac{1}{2}\mathrm{h},\ e^{\frac{1}{2}Lh}\underline{y^+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hK}_1,e^{\frac{1}{2}Lh}\overline{y^+}(\mathrm{t}_n;\alpha)+\frac{1}{2}\mathrm{hL}_1)$$
$$M_2=H(t_n+\frac{1}{2}\mathrm{h},\ e^{\frac{1}{2}Lh}\underline{y^-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hM}_1,e^{\frac{1}{2}Lh}\overline{y^-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hN}_1)$$
$$N_2=I(t_n+\frac{1}{2}\mathrm{h},\ e^{\frac{1}{2}Lh}\underline{y^-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hM}_1,e^{\frac{1}{2}Lh}\overline{y^-}(\mathrm{t}_n;\beta)+\frac{1}{2}\mathrm{hN}_1)$$
$$K_3=F(t_n+\mathrm{h},\ e^{L\mathrm{h}}\underline{y^+}(\mathrm{t}_n;\alpha)+h(-K_1+2K_2),\ e^{L\mathrm{h}}\,\overline{y^+}(\mathrm{t}_n;\alpha)+h(-L_1+2L_2))$$
$$L_3=G(t_n+\mathrm{h},\ e^{L\mathrm{h}}\underline{y^+}(\mathrm{t}_n;\alpha)+h(-K_1+2K_2),\ e^{L\mathrm{h}}\,\overline{y^+}(\mathrm{t}_n;\alpha)+h(-L_1+2L_2))$$
$$M_3=H(t_n+\mathrm{h},\ e^{L\mathrm{h}}\,\underline{y^-}(\mathrm{t}_n;\beta)+h(-M_1+M_2),\ e^{L\mathrm{h}}\overline{y^-}(\mathrm{t}_n;\beta)+h(-N_1+N_2))$$
$$N_3=I(t_n+\mathrm{h},\ e^{L\mathrm{h}}\,\underline{y^-}(\mathrm{t}_n;\beta)+h(-M_1+M_2),e^{L\mathrm{h}}\,\overline{y^-}(\mathrm{t}_n;\beta)+h(-N_1+N_2))$$

Setting s = 4, we get fourth order Exponential RK method for Intuitionistic Fuzzy IVP which is given by

$$\underline{y^+}(t_{n+1};\,\alpha)=e^{Lh}\,\underline{y^+}(t_n;\,\alpha)+\frac{h}{6}(\mathrm{K}_1+2K_2+2K_3+K_4)$$
$$\overline{y^+}(t_{n+1};\,\alpha)=e^{Lh}\,\overline{y^+}(t_n;\,\alpha)+\frac{h}{6}(\mathrm{L}_1+2L_2+2L_3+L_4)$$
$$\underline{y^-}(t_{n+1};\,\beta)=e^{Lh}\,\underline{y^-}(t_n;\,\beta)+\frac{h}{6}(\mathrm{M}_1+2M_2+2M_3+M_4)\ \text{ and}$$
$$\overline{y^-}(t_{n+1};\,\beta)=e^{Lh}\,\overline{y^-}(t_n;\,\beta)+\frac{h}{6}(\mathrm{N}_1+2N_2+2N_3+N_4)$$
$$K_1=F(t_n,\underline{y^+}(\mathrm{t}_n;\alpha),\overline{y^+}(\mathrm{t}_n;\alpha))\quad L_1=G(t_n,\underline{y^+}(\mathrm{t}_n;\alpha),\overline{y^+}(\mathrm{t}_n;\alpha))$$
$$M_1=H(t_n,\underline{y^-}(\mathrm{t}_n;\beta),\overline{y^-}(\mathrm{t}_n;\beta))\quad N_1=I(t_n,\underline{y^-}(\mathrm{t}_n;\beta),\overline{y^-}(\mathrm{t}_n;\beta))$$

$$K_2 = F(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}Lh}\underline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}\text{hK}_1, e^{\frac{1}{2}Lh}\overline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}\text{hL}_1)$$

$$L_2 = G(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}Lh}\underline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}\text{hK}_1, e^{\frac{1}{2}Lh}\overline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}\text{hL}_1)$$

$$M_2 = H(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}Lh}\underline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}\text{hM}_1, e^{\frac{1}{2}Lh}\overline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}\text{hN}_1)$$

$$N_2 = I(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}Lh}\underline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}\text{hM}_1, e^{\frac{1}{2}Lh}\overline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}\text{hN}_1)$$

$$K_3 = F(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}L\text{h}}\underline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}hK_2,\ e^{\frac{1}{2}L\text{h}}\overline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}hL_2)$$

$$L_3 = G(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}L\text{h}}\underline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}hK_2,\ e^{\frac{1}{2}L\text{h}}\overline{y}^{+}(\text{t}_n;\alpha) + \frac{1}{2}hL_2)$$

$$M_3 = H(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}L\text{h}}\underline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}hM_2,\ e^{\frac{1}{2}L\text{h}}\overline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}hN_2)$$

$$N_3 = I(t_n + \frac{1}{2}\text{h},\ e^{\frac{1}{2}L\text{h}}\underline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}hM_2, e^{\frac{1}{2}L\text{h}}\overline{y}^{-}(\text{t}_n;\beta) + \frac{1}{2}hN_2)$$

$$K_4 = F(t_n + \text{h},\ e^{Lh}\underline{y}^{+}(\text{t}_n;\alpha) + hK_3,\ e^{Lh}\underline{y}^{+}(\text{t}_n;\alpha) + hL_3)$$

$$L_4 = G(t_n + \text{h},\ e^{Lh}\underline{y}^{+}(\text{t}_n;\alpha) + hK_3,\quad e^{Lh}\underline{y}^{+}(\text{t}_n;\alpha) + hL_3,\ )$$

$$M_4 = H(t_n + \text{h},\ e^{Lh}\underline{y}^{-}(\text{t}_n;\beta) + hM_3,\ e^{Lh}\underline{y}^{-}(\text{t}_n;\ \beta) + hN_3)\ .$$

$$N_4 = I(t_n + \text{h},\ e^{Lh}\underline{y}^{-}(\text{t}_n;\ \beta) + hM_3,\ e^{Lh}\underline{y}^{-}(\text{t}_n;\ \beta) + hN_3)$$

5. Convergence of Exponential Runge-Kutta methods for Intuitionistic Fuzzy IVPs

The solution is obtained by grid points at

$$a = t_0 \le t_1 \le \cdots\cdots \le t_N = b \text{ and } h = \frac{b-a}{N} = t_{n+1} - t_n\ . \tag{5.1}$$

We define

$$\begin{aligned} &F[t, y(t;a)] = \sum_{i=1}^{s} b_i K_i(t, y(t;a)) \quad G[\text{t}, y(t;a)] = \sum_{i=1}^{s} b_i L_i(t, y(t;a)) \\ &H[t, y(t;\beta)] = \sum_{i=1}^{s} b_i M_i(t, y(t;\beta))\ I[\text{t}, y(t;\beta)] = \sum_{i=1}^{s} b_i N_i(t, y(t;\beta))\ . \end{aligned} \tag{5.2}$$

The exact and approximate solutions at $t_n,\ 0 \le n \le N$ are denoted respectively by

$$[\text{y}(\text{t}_n)]_{\alpha,\beta} = [\underline{y}^{+}(\text{t}_n;\alpha), \overline{y}^{+}(\text{t}_n;\alpha), \underline{y}^{-}(\text{t}_n;\beta), \overline{y}^{-}(\text{t}_n;\beta)]$$

$$[\text{Y}(\text{t}_n)]_{\alpha,\beta} = [\underline{Y}^{+}(\text{t}_n;\alpha), \overline{Y}^{+}(\text{t}_n;\alpha), \underline{Y}^{-}(\text{t}_n;\beta), \overline{Y}^{-}(\text{t}_n;\beta)]\ .$$

We have:

$$\underline{y}^{+}(t_{n+1};\ \alpha) \approx e^{Lh}\,\underline{y}^{+}(t_n;\ \alpha) + h\,\mathrm{F}[\mathrm{t}_n, \underline{y}^{+}(\mathrm{t}_n;\alpha), \overline{y^{+}}(\mathrm{t}_n;\alpha)],$$
$$\overline{y^{+}}(t_{n+1};\ \alpha) \approx e^{Lh}\,\overline{y^{+}}(t_n;\ \alpha) + h\,\mathrm{G}[\mathrm{t}_n, \underline{y}^{+}(\mathrm{t}_n;\alpha), \overline{y^{+}}(\mathrm{t}_n;\alpha)],$$
$$\underline{y}^{-}(t_{n+1};\ \beta) \approx e^{Lh}\,\underline{y}^{-}(t_n;\ \beta) + hH[\mathrm{t}_n, \underline{y}^{-}(\mathrm{t}_n;\beta), \overline{y^{-}}(\mathrm{t}_n;\beta)],$$
$$\overline{y^{-}}(t_{n+1};\ \beta) \approx e^{Lh}\,\overline{y^{-}}(t_n;\ \beta) + hI[\mathrm{t}_n, \underline{y}^{-}(\mathrm{t}_n;\beta), \overline{y^{-}}(\mathrm{t}_n;\beta)].$$

and

$$\underline{Y}^{+}(t_{n+1};\ \alpha) \approx e^{Lh}\,\underline{Y}^{+}(t_n;\ \alpha) + h\,\mathrm{F}[\mathrm{t}_n, \underline{Y}^{+}(\mathrm{t}_n;\alpha), \overline{Y^{+}}(\mathrm{t}_n;\alpha)],$$
$$\overline{Y^{+}}(t_{n+1};\ \alpha) \approx e^{Lh}\,\overline{Y^{+}}(t_n;\ \alpha) + h\,\mathrm{G}[\mathrm{t}_n, \underline{Y}^{+}(\mathrm{t}_n;\alpha), \overline{Y^{+}}(\mathrm{t}_n;\alpha)],$$
$$\underline{Y}^{-}(t_{n+1};\ \beta) \approx e^{Lh}\,\underline{Y}^{-}(t_n;\ \beta) + hH[\mathrm{t}_n, \underline{Y}^{-}(\mathrm{t}_n;\beta), \overline{Y^{-}}(\mathrm{t}_n;\beta)],$$
$$\overline{Y^{-}}(t_{n+1};\ \beta) \approx e^{Lh}\,\overline{Y^{-}}(t_n;\ \beta) + hI[\mathrm{t}_n, \underline{Y}^{-}(\mathrm{t}_n;\beta), \overline{Y^{-}}(\mathrm{t}_n;\beta)].$$

We need the following lemmas to show the convergence of these approximates, $\underline{y}^{+}(\mathrm{t}_n;\alpha)$, $\overline{y^{+}}(\mathrm{t}_n;\alpha)$, $\underline{y}^{-}(\mathrm{t}_n;\beta)$ and $\overline{y^{-}}(\mathrm{t}_n;\beta)$ converges to $\underline{Y}^{+}(\mathrm{t}_n;\alpha)$, $\overline{Y^{+}}(\mathrm{t}_n;\alpha)$, $\underline{Y}^{-}(\mathrm{t}_n;\beta)$ and $\overline{Y^{-}}(\mathrm{t}_n;\beta)$ respectively whenever $h \to 0$.

Lemma 5.1. [14] Let the sequence of numbers $\{W_n^{+}\}_{n=0}^{N}, \{W_n^{-}\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le A|W_n| + B \quad 0 \le n \le N-1$$

for some given positive constants, A and B. Then $|W_n| \le A^n|W_0| + B\dfrac{A^n-1}{A-1}$.

Lemma 5.2.[14] Let the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le |W_n| + A.\max\{|W_n||V_n|\} + B,$$
$$|V_{n+1}| \le |V_n| + A.\max\{|W_n||V_n|\} + B,$$

for some given positive constants, A and B and denote

$$U_n = |W_n| + |V_n|, \quad 0 \le n \le N.$$

Then $U_n \le \overline{\mathrm{A}}^{\mathrm{n}} U_0 + \overline{\mathrm{B}}\,\dfrac{\overline{\mathrm{A}}^{\mathrm{n}}-1}{\overline{\mathrm{A}}-1}$, $\quad 0 \le n \le N$ where $\overline{\mathrm{A}}=1+2\mathrm{A}$ and $\overline{\mathrm{B}}=2\mathrm{B}$..

Let F (t, u, v), G (t, u, v), H(t, u', v') and I(t, u', v') are obtained by substituting $[y(t)]_{\alpha,\beta} = [u, v; u', v']$ in (5.2).

The domain where F, G, H and I are defined is therefore

$$K = \{(t,u,v;u',v') / 0 \le t \le T, -\infty < u < v < \infty, -\infty < u' < u \le v < v'\}.$$

Theorem 5.1. Let F (t, u, v), G (t, u, v), H (t, u', v') and I (t, u', v') belonging to $c^{P}(K)$ and let the partial derivatives of F, G, H and I be bounded over K. Then for arbitrary fixed α, β, $0 \le \alpha, \beta \le 1$, the approximate solution of (5.1), $\left[\underline{y}^{+}(\mathrm{t}_n;\alpha),\ \overline{y}^{+}(\mathrm{t}_n;\alpha), \underline{y}^{-}(\mathrm{t}_n;\beta),\ \overline{y}^{-}(\mathrm{t}_n;\beta)\right]$ converge to the exact solution $\left[\underline{Y}^{+}(\mathrm{t}_n;\alpha),\ \overline{Y}^{+}(\mathrm{t}_n;\alpha), \underline{Y}^{-}(\mathrm{t}_n;\beta),\ \overline{Y}^{-}(\mathrm{t}_n;\beta)\right]$.

Proof: By using Taylor's theorem:

$$\underline{Y}^+(t_{n+1};\alpha) = e^{Lh}\left\{\underline{Y}^+(t_n;\alpha) + hF(t_n, \underline{Y}^+(t_n;\alpha), \overline{Y^+}(t_n;\alpha)) + \frac{h^{p+1}}{(p+1)!}\underline{Y}^{+(p+1)}(\xi_{n,1})\right\}$$

$$\overline{Y^+}(t_{n+1};\alpha) = e^{Lh}\left\{\overline{Y^+}(t_n;\alpha) + hG(t_n, \underline{Y}^+(t_n;\alpha), \overline{Y^+}(t_n;\alpha)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y^+}^{(p+1)}(\xi_{n,2})\right\}$$

$$\underline{Y}^-(t_{n+1};\beta) = e^{Lh}\left\{\underline{Y}^-(t_n;\beta) + hH(t_n, \underline{Y}^-(t_n;\beta), \overline{Y^-}(t_n;\beta)) + \frac{h^{p+1}}{(p+1)!}\underline{Y}^{-(p+1)}(\xi_{n,3})\right\}$$

$$\overline{Y^-}(t_{n+1};\beta) = e^{Lh}\left\{\overline{Y^-}(t_n;\beta) + hI(t_n, \underline{Y}^-(t_n;\beta), \overline{Y^-}(t_n;\beta)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y^-}^{(p+1)}(\xi_{n,4})\right\}$$

where $\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \xi_{n,4} \in (t_n, t_{n+1})$

Now if we denote

$$W_n^+ = \underline{Y}^+(t_n,\alpha) - \underline{y}^+(t_n,\alpha)\ ,\quad V_n^+ = \overline{Y}^+(t_n,\alpha) - \overline{y}^+(t_n,\alpha)$$

$$W_n^- = \underline{Y}^-(t_n,\beta) - \underline{y}^-(t_n,\beta)\ \text{ and }\ V_n^- = \overline{Y}^-(t_n,\beta) - \overline{y}^-(t_n,\beta)$$

then the above two expressions converted to

$$W_{n+1}^+ = e^{Lh}\left\{W_n^+ + h(F(t_n, \underline{Y}^+(t_{n+1};\alpha), \overline{Y^+}(t_{n+1};\alpha)) - F(t_n, \underline{y}^+(t_n;\alpha), \overline{y^+}(t_n;\alpha))) + \frac{h^{p+1}}{(p+1)!}\underline{Y}^{+(p+1)}(\xi_{n,1})\right\}$$

$$V_{n+1}^+ = e^{Lh}\left\{V_n^+ + h(G(t_n, \underline{Y}^+(t_{n+1};\alpha), \overline{Y^+}(t_{n+1};\alpha)) - G(t_n, \underline{y}^+(t_n;\alpha), \overline{y^+}(t_n;\alpha))) + \frac{h^{p+1}}{(p+1)!}\overline{Y^+}^{(p+1)}(\xi_{n,2})\right\}$$

$$W_{n+1}^- = e^{Lh}\left\{W_n^- + h(H(t_n, \underline{Y}^-(t_{n+1};\beta), \overline{Y^-}(t_{n+1};\beta)) - H(t_n, \underline{y}^-(t_n;\beta), \overline{y^-}(t_n;\beta))) + \frac{h^{p+1}}{(p+1)!}\underline{Y}^{-(p+1)}(\xi_{n,3})\right\}$$

$$V_{n+1}^- = e^{Lh}\left\{V_n^- + h(I(t_n, \underline{Y}^-(t_{n+1};\beta), \overline{Y^-}(t_{n+1};\beta)) - I(t_n, \underline{y}^-(t_n;\beta), \overline{y^-}(t_n;\beta))) + \frac{h^{p+1}}{(p+1)!}\overline{Y^-}^{(p+1)}(\xi_{n,4})\right\}$$

Hence

$$\left|W_{n+1}^+\right| \le e^{Lh}\{\left|W_n^+\right| + 2Lh\max\left(\left|W_n^+\right|, \left|V_n^+\right|\right) + \frac{h^{p+1}}{(p+1)!}M_1\}$$

$$\left|V_{n+1}^+\right| \le e^{Lh}\{\left|V_n^+\right| + 2Lh\max\left(\left|W_n^+\right|, \left|V_n^+\right|\right) + \frac{h^{p+1}}{(p+1)!}M_1\}$$

$$\left|W_{n+1}^-\right| \le e^{Lh}\{\left|W_n^-\right| + 2Lh\max\left(\left|W_n^-\right|, \left|V_n^-\right|\right) + \frac{h^{p+1}}{(p+1)!}M_2\}$$

$$\left|V_{n+1}^-\right| \le e^{Lh}\{\left|V_n^-\right| + 2Lh\max\left(\left|W_n^-\right|, \left|V_n^-\right|\right) + \frac{h^{p+1}}{(p+1)!}M_2\}$$

where

$$M_1 = \max\left\{\max\left|\underline{Y}^{+(p+1)}(t,\alpha)\right|, \max\left|\overline{Y^+}^{(p+1)}(t,\alpha)\right|\right\}$$

$$M_2 = \max\left\{\max\left|\underline{Y}^{-(p+1)}(t,\beta)\right|, \max\left|\overline{Y^-}^{(p+1)}(t,\beta)\right|\right\} \quad for\ t \in [0,T],$$

and L > 0 is a bound from the partial derivative of F and G.
Therefore,

$$\left|W_n^+\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^n \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh}\},$$

$$\left|V_n^+\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^n \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh}\}$$

$$\left|W_n^-\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^n \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh}\},$$

$$\left|V_n^-\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^n \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh}\}$$

where $|U_0| = |W_0| + |V_0|$ and $|U_0| = |W_0| + |V_0|$.
In particular

$$\left|W_n^+\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^N \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}\}$$

$$\left|V_n^+\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^N \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}\}$$

$$\left|W_n^-\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^N \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}\}$$

$$\left|V_n^-\right| \le e^{Lh}\{(1+4\,\mathrm{Lh})^N \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}\}$$

where $M = \max\left\{M_1, M_2\right\} \quad for\, t \in [0,\mathrm{T}]$.

Since $W_0 = V_0 = 0$, and $W_0 = V_0 = 0$, we have

$$\left|W_n^+\right| \le M\,\frac{e^{\mathrm{Lh}}(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p, \qquad \left|V_n^+\right| \le M\,\frac{e^{\mathrm{Lh}}(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p$$

$$\left|W_n^-\right| \le M\,\frac{e^{\mathrm{Lh}}(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p, \qquad \left|V_n^-\right| \le M\,\frac{e^{\mathrm{Lh}}(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p$$

Thus if $h \to 0$ we get $W_n^+ \to 0,\ V_n^+ \to 0, W_n^- \to 0$ and $V_n^- \to 0$ which completes the proof.

6. Numerical Examples

Example 6.1 Weight-Loss problem: **C**onsider initial value weight y_0 as a triangular intuitionistic fuzzy number (120,130,135:115,130,140) lb. Let us find the weight after 30 days (The constant of proportionality is k = 1/3500 lb/cal)

Solution: The time rate of decrease of weight y is given by

$$y' = -ky$$

with the initial condition $y(t_0)$=(120,130,135:115,130,140), an intuitionistic fuzzy number.

The exact solution is given by

$$\underline{y}^+(t;\alpha) = (120+10\alpha)e^{-kt}; \qquad \overline{y}^+(\mathrm{t};\alpha) = (135-5\alpha)e^{-kt}$$

$$\underline{y}^-(t;\beta) = (130-15\alpha)e^{-kt}; \qquad \overline{y}^-(\mathrm{t};\beta) = (130+10\alpha)e^{-kt},\ \beta = 1-\alpha.$$

The error results for IFRK3 and IFRK4 at t =1 by taking h = 0. 01areshown in the Tables 1 and 2. The solution graphs are given in Figures 1 and 2.

Table 1

(α,β)	Absolute Error for IFERK3 at t=1			
	$\underline{y}^{+}(t;\alpha)$	$\overline{y}^{+}(t;\alpha)$	$\underline{y}^{-}(t;\beta)$	$\overline{y}^{-}(t;\beta)$
0.00	1.194e-10	1.336e-10	1.151e-10	1.392e-10
0.20	1.222e-10	1.335e-10	1.179e-10	1.392e-10
0.40	1.236e-10	1.335e-10	1.208e-10	1.364e-10
0.60	1.265e-10	1.308e-10	1.236e-10	1.335e-10
0.80	1.279e-10	1.308e-10	1.265e-10	1.308e-10
1.00	1.308e-10	1.308e-10	1.308e-10	1.308e-10

Table 2

(α,β)	Absolute Error for IFERK4 at t=1			
	$\underline{y}^{+}(t;\alpha)$	$\overline{y}^{+}(t;\alpha)$	$\underline{y}^{-}(t;\beta)$	$\overline{y}^{-}(t;\beta)$
0.00	4.263e-14	2.842e-14	2.842e-14	2.842e-14
0.20	2.842e-14	0.000e+00	1.421e-14	2.842e-14
0.40	2.842e-14	5.684e-14	1.421e-14	0.000e+00
0.60	4.263e-14	8.527e-14	2.842e-14	0.000e+00
0.80	2.842e-14	2.842e-14	0.000e+00	8.527e-14
1.00	2.842e-14	2.842e-14	2.842e-14	2.842e-14

Fig. 1
Solution graph for IFERK3

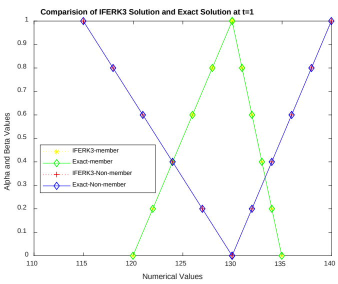
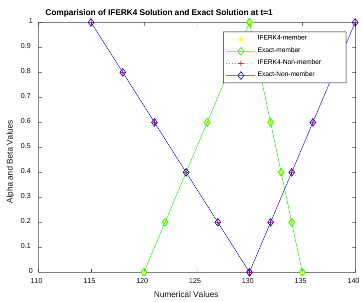


Fig. 2
Solution graph for IFERK3



Example 6.2 Oil production Model: The rate of increase oil production y in million metric tons per year was assumed to be proportional to y itself. Then what is the amount of oil after five years if initially
(1061, 1091, 1131; 1066, 1091, 1111) million metric ton oil is there. (Constant of proportionality is 0.084).

Solution: The rate of increase oil production y is given by $y' = ky$ with k=0.084 and $y_0 = (1061,\ 1091,\ 1131;\ 1066,\ 1091,\ 1111)$ million.

The exact solution is given by :

$$\underline{y}^{+}(t;\alpha) = (1061 + 30\alpha)e^{kt}; \quad \overline{y}^{+}(t;\alpha) = (1131 - 40\alpha)e^{kt}$$

$$\underline{y}^{-}(t;\beta) = (1091 - 25\beta)e^{kt}; \quad \overline{y}^{-}(t;\beta) = (1091 + 20\beta)e^{kt}, \beta = 1 - \alpha.$$

The error results for IFRK3 and IFRK4 at t =5 by taking h = 0. 01are shown in the Tables 3 and 4. The solution graphs are given in Figures 3 and 4.

Table 3

(α,β)	**Absolute Error for IFERK3 at t=5**			
	$\underline{y}^{+}(t;\alpha)$	$\overline{y}^{+}(t;\alpha)$	$\underline{y}^{-}(t;\beta)$	$\overline{y}^{-}(t;\beta)$
0.00	8.672e-09	9.246e-09	8.716e-09	9.083e-09
0.20	8.721e-09	9.184e-09	8.758e-09	9.047e-09
0.40	8.774e-09	9.116e-09	8.793e-09	9.015e-09
0.60	8.818e-09	9.047e-09	8.835e-09	8.986e-09
0.80	8.872e-09	8.986e-09	8.877e-09	8.952e-09
1.00	8.919e-09	8.919e-09	8.919e-09	8.919e-09

Table 4

(α,β)	**Absolute Error for IFERK4 at t=5**			
	$\underline{y}^{+}(t;\alpha)$	$\overline{y}^{+}(t;\alpha)$	$\underline{y}^{-}(t;\beta)$	$\overline{y}^{-}(t;\beta)$
0.00	3.638e-12	2.274e-12	4.547e-12	4.093e-12
0.20	3.183e-12	9.095e-13	1.137e-12	6.821e-13
0.40	3.865e-12	4.547e-12	2.501e-12	2.501e-12
0.60	1.592e-12	6.821e-13	1.137e-12	4.320e-12
0.80	1.137e-12	4.320e-12	3.183e-12	2.046e-12
1.00	0.000e+00	0.000e+00	0.000e+00	0.000e+00

Figure 3
Solution graph for IFERK3

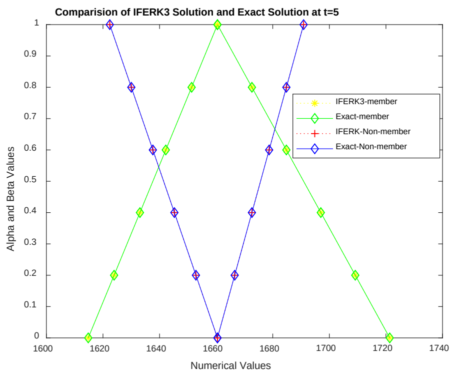
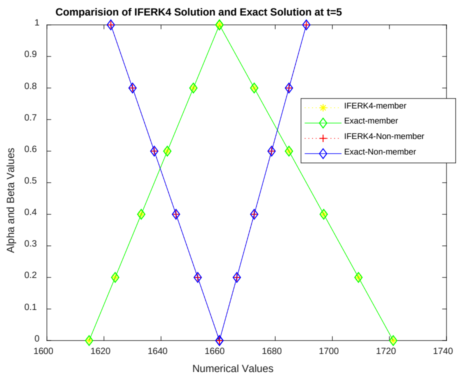


Figure 4
Solution graph for IFERK3



7. Conclusion

In this paper, the Exponential Runge-Kutta methods of order three and four have been proposed to solve the intuitionistic fuzzy IVPs. The convergence analysis of these methods has been discussed. The applicability of these methods has been illustrated through examples of intuitionistic fuzzy IVPs. From the numerical results, it is observed that the absolute error is negligibly small and the accuracy increases as the order of the ERK method increases. Hence, it is seen that the ERK method is suitable for solving intuitionistic fuzzy IVPs.

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