Advances in Mathematics: Scientific Journal 8 (2019), no.3, 189–199
(Special issue on ICRAPAM)
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

NEW COUPLED COINCIDENCE POINT IN GENERALIZED INTUITIONISTIC FUZZY METRIC SPACES WITH PARTIAL ORDER

M. Suganthi[1] and M Jeyaraman[2]

[1]*Department of Mathematics, Government Arts College, Melur.*

Part-time Ph.D., Research Scholar, PG & Research Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai. Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India

[2]*PG & Research Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai. Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India . E-mail:* [1]*vimalsugan@gmail.com,* [2]*jeya.math@gmail.com*

Abstract: The aim of this paper is to establish a new coupled fixed point theorem in generalized intuitionistic fuzzy metric spaces having partial ordering.For this purpose, we prove a lemma which simultaneously establishes the Cauchy criterion for the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ stated here.The Hadžić type t-norm and t-conorm mentioned here are characterized by the equicontinuity of iterates. These concepts are used to obtain a new coupled fixed point result in the generalized intuitionistic fuzzy metric spaces having partial ordering.

1. Introduction

Zadeh[20] introduced fuzzy sets which made easy to handle uncertain situations that cannot be approached by non-probabilistic techniques.These sets are used to fuzzify the mathematical structures. The flexible structure of fuzzy ideas leads to various definitions of fuzzy metric spaces which are not equivalent to each other. Here we work on the definition given by George et al. [8] and the topology is taken to be Hausdorff. The purpose of this paper is to generalize an existing coupled fixed point result[6] to generalized intuitionistic fuzzy metric spaces having partial ordering. We also extended the Cauchy criterion for the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ to support the main results.

2. Preliminaries

Definition 2.1.

A 5-tuple $(X, E, F, *, \lozenge)$ is said to be a generalized intuitionistic fuzzy metric space, if X is an arbitrary nonempty set, $*$is a continuous t-norm, $\lozenge$is a continuous t-conorm, G and H are fuzzy sets on $X^3 \times (0,\infty)$ satisfying the following conditions: For every x, y, z, a$\in$ X and t,s>0,

(i) $E(x, y, z, t) + F(x, y, z, t) \leq 1$,
(ii) $E(x, x, y, t) > 0$ for $x \neq y$,
(iii) $E(x, x, y, t) \geq E(x, y, z, t)$ for $y \neq z$,
(iv) $E(x, y, z, t) = 1$ if and only if $x = y = z$,
(v) $E(x, y, z, t) = E(p(x,y,z),t)$, where p is a permutation function,
(vi) $E(x,a,a,t) * E(a,y,z,s) \leq E(x, y, z, t+ s)$,
(vii) $E(x, y, z, .) : (0,\infty) \to [0,1]$ is continuous,
(viii) E is a non-decreasing function on R^+, $\lim_{t\to 0} E(x, y, z, t) = 0$ and $\lim_{t\to\infty} E(x, y, z, t) = 1$,
(ix) (ix) $F(x, x, y, t) < 1$ for $x \neq y$,
(x) $F(x, x, y, t) \leq F(x, y, z, t)$ for $y \neq z$,
(xi) $F(x, y, z, t) = 0$ if and only if $x = y = z$,
(xii) $F(x, y, z, t) = F(p(x,y,z),t)$, where p is a permutation function,
(xiii) $F(x, a, a, t) \lozenge F(a, y, z, s) \geq F(x, y, z, t+s)$,
(xiv) $F(x, y, z, .) : (0,\infty) \to [0,1]$ is continuous,
(xv) F is a non-increasing function on R^+, $\lim_{t\to 0} F(x, y, z, t) = 1$ and $\lim_{t\to\infty} F(x, y, z, t) = 0$.

In this case, the pair (E,F) is called a generalized intuitionistic fuzzy metric on X.

Example 2.2.

Let (X,E) be a G-metric space, and let $u * v = \min\{u,v\}$ and $u \diamond v = \min\{u + v,1\}$ for all u,v$\in$ [0,1]. Define E and F by

$$E(x, y, z, t) = \frac{t}{t+E(x,y,z)} \text{ and } E(x, y, z, t) = \frac{E(x,y,z)}{t+E(x,y,z)}$$

for all x,y,z$\in$ X and for t >0. Then$(X, E, F, *, \lozenge)$ is a generalized intuitionistic fuzzy metric space.

Definition 2.3.

Let X be a nonempty set. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $\Delta: X \times X \to X$ if $x = \Delta(\lambda,\mu)$ and $y = \Delta(\mu,\lambda)$.

Definition 2.4.

Let X be a nonempty set.An element$(x,y) \in X \times X$ is called a coupled coincidence point of $\Delta:X \times X \to X$ and $\Gamma:X \to X$ if $\Gamma x= \Delta(x,y)$ and $\Gamma y= \Delta(\mu,\lambda)$, a common coupled fixed point of $\Delta:X \times X \to X$ and $\Gamma:X \to X$ if $x = \Gamma x= \Delta(x,y)$ and $y = \Gamma y= \Delta(\mu,\lambda)$.

Lemma 2.5.

Let $(X, E, F, *, \lozenge)$ be a generalized intuitionistic fuzzy metric space.
Then$E(x, y, z, .)$ is non-decreasing and $F(x, y, z, .)$ is non-increasing with respect to t for all x,y,z$\in$ X.

Lemma 2.6.

Let $(X, E, F,*,\Diamond)$ be a generalized intuitionistic fuzzy metric space.Then E and F are continuous functions on $X^3 \times (0,\infty)$.

Definition 2.7.

Let $(X, E, F,*,\Diamond)$be a generalized intuitionistic fuzzy metric space. The pair (G,h), where $\Delta:X \times X \to X$ and $\Gamma:X \to X$, are said to be compatible if for all $t > 0$

$$\lim_{n\to\infty} E(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(\Delta(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n),t)) = 1,$$

$$\lim_{n\to\infty} E(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n),t)) = 1 \text{ and}$$

$$\lim_{n\to\infty} F(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(\Delta(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n),t)) = 0,$$

$$\lim_{n\to\infty} F(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n),t)) = 0$$

whenever$\{\lambda_n\}$ and $\{\mu_n\}$ are sequences in X such that $\lim_{n\to\infty} \Delta(\lambda_n,\mu_n) = \lim_{n\to\infty} \Gamma(\lambda_n) = \lambda$ and $\lim_{n\to\infty} \Delta(\mu_n,\lambda_n) = \lim_{n\to\infty} \Gamma(\mu_n) = \mu$, for some $\lambda,\mu \in X$.

Lemma 2.8.

If the pair (Δ,Γ), where $\Delta:X\times X \to X$ and $\Gamma:X \to X$, is compatible in (X,E), then the pair (Δ,Γ) is compatible in the corresponding space $(X, E, F,*,\Diamond)$as in the example stated above.

Proof.Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences in X such that $\lim_{n\to\infty} \Delta(\lambda_n,\mu_n) = \lim_{n\to\infty} \Gamma(\lambda_n) = \lambda$ and $\lim_{n\to\infty} \Delta(\mu_n,\lambda_n) = \lim_{n\to\infty} \Gamma(\mu_n) = \mu$ for some $\lambda,\mu \in X$. Then the same limits also hold in $(X, E, F,*,\Diamond)$. Since Δ and Γ are compatible in (X,E), we have

$$\lim_{n\to\infty} E(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(\Delta(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n)) = 0,$$
$$\lim_{n\to\infty} E(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n)) = 0.$$

For all $t > 0$,

$$E(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(\Delta(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n)),t) = \frac{t}{t+E(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(\Delta(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n))};$$
$$E(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n)),t) = \frac{t}{t+E(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n))}$$
$$F(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(\Delta(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n)),t) = \frac{E(\Gamma(F(\lambda_n,\mu_n)),\Gamma(F(\lambda_n,\mu_n)),F(\Gamma(\lambda_n),\Gamma(\mu_n))}{t+E(\Gamma(F(\lambda_n,\mu_n)),\Gamma(F(\lambda_n,\mu_n)),F(\Gamma(\lambda_n),\Gamma(\mu_n))}$$
$$F(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n)),t) = \frac{E(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n))}{t+E(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(\Delta(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n))}.$$

,

Taking $n \to \infty$ in the above equations, we have, for all $t > 0$

$$\lim_{n\to\infty} E(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(F(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n),t)) = 1,$$
$$\lim_{n\to\infty} E(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(F(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n),t)) = 1,$$
$$\lim_{n\to\infty} F(\Gamma(\Delta(\lambda_n,\mu_n)),\Gamma(F(\lambda_n,\mu_n)),\Delta(\Gamma(\lambda_n),\Gamma(\mu_n),t)) = 0,$$
$$\lim_{n\to\infty} F(\Gamma(\Delta(\mu_n,\lambda_n)),\Gamma(F(\mu_n,\lambda_n)),\Delta(\Gamma(\mu_n),\Gamma(\lambda_n),t)) = 0.$$

Therefore (Δ,Γ) is compatible in $(X, E, F,*,\Diamond)$. □

Lemma 2.9.

Let $(X, E, F,*,\Diamond)$ be a generalized intuitionistic fuzzy metric space having t-norm and t-conorm of Hadžićtype for which $E(x, x, y, u) \to 1$ and $F(x, x, y, u) \to 0$ as $u \to \infty$, $\{\lambda_n\}$ and $\{\mu_n\}$ in X satisfy, for all $n \geq 1$, $t > 0$,

$$E(\lambda_n,\lambda_n,\lambda_{n+1},t)*E(\mu_n,\mu_n,\mu_{n+1},t)\geq E(\lambda_{n-1},\lambda_{n-1},\lambda_n,\frac{t}{k})*E(\mu_{n-1},\mu_{n-1},\mu_n,\frac{t}{k}) \quad (2.9.1)$$

$$F(\lambda_n,\lambda_n,\lambda_{n+1},t)\diamond F(\mu_n,\mu_n,\mu_{n+1},t)\leq F(\lambda_{n-1},\lambda_{n-1},\lambda_n,\frac{t}{k})\diamond F(\mu_{n-1},\mu_{n-1},\mu_n,\frac{t}{k}) \quad (2.9.2)$$

with some $0 < k < 1$, then $\{\lambda_n\}$ and $\{\mu_n\}$ are Cauchy sequences.

Proof. We successively apply (2.9.1) and (2.9.2) to obtain for all integers $i \geq 1$,$t>0$ and $q \geq 0$,

$$E(\lambda_{q+i},\lambda_{q+i},\lambda_{q+i+1},t)*E(\mu_{q+i},\mu_{q+i},\mu_{q+i+1},t)\quad\geq E(\lambda_q,\lambda_q,\lambda_{q+1},\frac{t}{k^i})*E(\mu_q,\mu_q,\mu_{q+1},\frac{t}{k^i}), \quad (2.9.3)$$

$$F(\lambda_{q+i},\lambda_{q+i},\lambda_{q+i+1},t)\diamond F(\mu_{q+i},\mu_{q+i},\mu_{q+i+1},t)\quad\geq F(\lambda_q,\lambda_q,\lambda_{q+1},\frac{t}{k^i})\diamond F(\mu_q,\mu_q,\mu_{q+1},\frac{t}{k^i}). \quad (2.9.4)$$

Let $\epsilon > 0$ and $0 < \lambda < 1$ be given. Let p be an integer such that $p > q$. Then

$$\epsilon > \epsilon\frac{1-k}{1-k} > \epsilon(1-k)(1+k+\ldots+k^{p-q-1}).$$

By Lemma(2.5), for all $p > q$,

$$E(\lambda_q,\lambda_q,\lambda_p,\epsilon)*E(\mu_q,\mu_q,\mu_p,\epsilon)$$
$$\geq\{E(\lambda_q,\lambda_q,\lambda_p,\epsilon(1-k)(1+k+\cdots+k^{p-q-1}))*E(\mu_q,\mu_q,\mu_p,\epsilon(1-k)(1+k+\cdots+k^{p-q-1}))\}$$

or

$$\begin{aligned}
&E(\lambda_q,\lambda_q,\lambda_p,\epsilon)*E(\mu_q,\mu_q,\mu_p,\epsilon)\\
&\geq\{E(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))*E(\lambda_{q+1},\lambda_{q+1},\lambda_{q+2},\epsilon k(1-k))*\cdots*E(\lambda_{p-1},\lambda_{p-1},\lambda_p,\epsilon k^{p-q-1}(1-k))\\
&*E(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))*E(\mu_{q+1},\mu_{q+1},\mu_{q+2},\epsilon k(1-k))*\cdots*E(\mu_{p-1},\mu_{p-1},\mu_p,\epsilon k^{p-q-1}(1-k))\}\\
&=\{E(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))*E(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))\}\\
&*\{E(\lambda_{q+1},\lambda_{q+1},\lambda_{q+2},\epsilon k(1-k))*E(\mu_{q+1},\mu_{q+1},\mu_{q+2},\epsilon k(1-k))\}\\
&*\cdots*\{E(\lambda_{p-1},\lambda_{p-1},\lambda_p,\epsilon k^{p-q-1}(1-k))*E(\mu_{p-1},\mu_{p-1},\mu_p,\epsilon k^{p-q-1}(1-k))\} \qquad (2.9.5)
\end{aligned}$$

$$\begin{aligned}
&F(\lambda_q,\lambda_q,\lambda_p,\epsilon)\diamond F(\mu_q,\mu_q,\mu_p,\epsilon)\\
&\leq\{F(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))\diamond F(\lambda_{q+1},\lambda_{q+1},\lambda_{q+2},\epsilon k(1-k))\diamond\cdots\diamond F(\lambda_{p-1},\lambda_{p-1},\lambda_p,\epsilon k^{p-q-1}(1-k))\\
&\diamond F(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))\diamond F(\mu_{q+1},\mu_{q+1},\mu_{q+2},\epsilon k(1-k))\diamond\cdots\diamond F(\mu_{p-1},\mu_{p-1},\mu_p,\epsilon k^{p-q-1}(1-k))\}\\
&=\{F(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))\diamond F(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))\}\\
&\diamond\{F(\lambda_{q+1},\lambda_{q+1},\lambda_{q+2},\epsilon k(1-k))\diamond F(\mu_{q+1},\mu_{q+1},\mu_{q+2},\epsilon k(1-k))\}\\
&\diamond\cdots\diamond\{F(\lambda_{p-1},\lambda_{p-1},\lambda_p,\epsilon k^{p-q-1}(1-k))\diamond F(\mu_{p-1},\mu_{p-1},\mu_p,\epsilon k^{p-q-1}(1-k))\} \qquad (2.9.6)
\end{aligned}$$

We put $t=(1-k)\epsilon k^i$ in (2.9.3) and (2.9.4), we get, for all $q \geq 0$,$i \geq 1$,

$$\begin{aligned}
&E(\lambda_{q+i},\lambda_{q+i},\lambda_{q+i+1},(1-k)\epsilon k^i)*E(\mu_{q+i},\mu_{q+i},\mu_{q+i+1},(1-k)\epsilon k^i)\\
&\geq E(\lambda_q,\lambda_q,\lambda_{q+1},(1-k)\epsilon)*E(\mu_q,\mu_q,\mu_{q+1},(1-k)\epsilon),\\
&F(\lambda_{q+i},\lambda_{q+i},\lambda_{q+i+1},(1-k)\epsilon k^i)\diamond F(\mu_{q+i},\mu_{q+i},\mu_{q+i+1},(1-k)\epsilon k^i)\\
&\leq F(\lambda_q,\lambda_q,\lambda_{q+1},(1-k)\epsilon)\diamond F(\mu_q,\mu_q,\mu_{q+1},(1-k)\epsilon)
\end{aligned}$$

From the above, and using (2.9.5) and (2.9.6), with $p > q$, we get

$$\begin{aligned}
&E(\lambda_q,\lambda_q,\lambda_p,\epsilon)*E(\mu_q,\mu_q,\mu_p,\epsilon))\\
&\geq \{E(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))*E(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))\}\\
&*\{E(\lambda_{q+1},\lambda_{q+1},\lambda_{q+2},\epsilon(1-k))*E(\mu_{q+1},\mu_{q+1},\mu_{q+2},\epsilon(1-k))\}\\
&*\cdots*\{E(\lambda_{p-1},\lambda_{p-1},\lambda_p,\epsilon(1-k))*E(\mu_{p-1},\mu_{p-1},\mu_p,\epsilon(1-k))\}\\
&F(\lambda_q,\lambda_q,\lambda_p,\epsilon)\diamond F(\mu_q,\mu_q,\mu_p,\epsilon))\\
&\leq \{F(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))\diamond F(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))\\
&\diamond\{F(\lambda_{q+1},\lambda_{q+1},\lambda_{q+2},\epsilon(1-k))*F(\mu_{q+1},\mu_{q+1},\mu_{q+2},\epsilon(1-k))\}\\
&\diamond\cdots\diamond\{F(\lambda_{p-1},\lambda_{p-1},\lambda_p,\epsilon(1-k))\diamond F(\mu_{p-1},\mu_{p-1},\mu_p,\epsilon(1-k))\}
\end{aligned}$$

$$E(\lambda_q,\lambda_q,\lambda_p,\epsilon)*E(\mu_q,\mu_q,\mu_p,\epsilon)\geq *^{p-q}\{E(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))*E(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))\} \quad (2.9.7)$$

$$F(\lambda_q,\lambda_q,\lambda_p,\epsilon)\diamond H(\mu_q,\mu_q,\mu_p,\epsilon)\leq \diamond^{p-q}\{F(\lambda_q,\lambda_q,\lambda_{q+1},\epsilon(1-k))\diamond F(\mu_q,\mu_q,\mu_{q+1},\epsilon(1-k))\} \quad (2.9.8)$$

By equicontinuity of t-norm at 1, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$*^{m-n}(s) > 1-\lambda \qquad (2.9.9)$$

and by the equicontinuity of t-conorm at 0, we have

$$\diamond^{m-n}(u) < \lambda \qquad (2.9.10)$$

whenever $1 \geq s,u > \eta(\lambda)$, where $0 < \lambda < 1$.

Since $E(\lambda_0,\lambda_0,\lambda_1,u) \to 1, F(\lambda_0,\lambda_0,\lambda_1,u) \to 0$ as $u \to \infty$, there exists $\phi(\epsilon,\lambda)$ such that

$$E\left(\lambda_0,\lambda_0,\lambda_1,\frac{(1-k)\epsilon}{k^n}\right) * E\left(\mu_0,\mu_0,\mu_1,\frac{(1-k)\epsilon}{k^n}\right) > \eta(\lambda) \quad (2.9.11)$$

$$F\left(\lambda_0,\lambda_0,\lambda_1,\frac{(1-k)\epsilon}{k^n}\right) \lozenge F\left(\mu_0,\mu_0,\mu_1,\frac{(1-k)\epsilon}{k^n}\right) < \eta(\lambda) \quad (2.9.12)$$

Whenever $n \geq N(\epsilon,\lambda)$.

From (2.9.3), (2.9.4), (2.9.11) and (2.9.12) with $q = 0$, $i = n \geq \phi(\epsilon,\lambda)$ and $t = (1-k)\epsilon$, we have

$$\begin{aligned}
&E(\lambda_n,\lambda_n,\lambda_{n+1},(1-k)\epsilon)*E(\mu_n,\mu_n,\mu_{n+1},(1-k)\epsilon)>\eta(\lambda)\\
&F(\lambda_n,\lambda_n,\lambda_{n+1},(1-k)\epsilon)\diamond F(\mu_n,\mu_n,\mu_{n+1},(1-k)\epsilon)<\eta(\lambda).
\end{aligned}$$

Then, from (2.9.9), (2.9.10) with

$$\begin{aligned}
&s=E(\lambda_n,\lambda_n,\lambda_{n+1},(1-k)\epsilon)*E(\mu_n,\mu_n,\mu_{n+1},(1-k)\epsilon)\\
&u=F(\lambda_n,\lambda_n,\lambda_{n+1},(1-k)\epsilon)\diamond F(\mu_n,\mu_n,\mu_{n+1},(1-k)\epsilon),
\end{aligned}$$

and with $m > n \geq \phi(\epsilon,\lambda)$, we have

$$*^{(m-n)}\{E(\lambda_n,\lambda_n,\lambda_{n+1},(1-k)\epsilon) * E(\mu_n,\mu_n,\mu_{n+1},(1-k)\epsilon) > 1-\lambda$$

$$\lozenge^{(m-n)}\{F(\lambda_n,\lambda_n,\lambda_{n+1},(1-k)\epsilon) \lozenge F(\mu_n,\mu_n,\mu_{n+1},(1-k)\epsilon) < \lambda$$

By (2.9.7) and (2.9.8), for all $m > n \geq N(\epsilon,\lambda)$, we obtain

$$E(\lambda_n,\lambda_n,\lambda_m,\epsilon)*E(\mu_n,\mu_n,\mu_m,\epsilon))>1-\lambda \text{ and}$$
$$F(\lambda_n,\lambda_n,\lambda_m,\epsilon)\diamond F(\mu_n,\mu_n,\mu_m,\epsilon))<\lambda,$$

which imply that

$$E(\lambda_n,\lambda_n,\lambda_m,\epsilon) > 1-\lambda \text{ and } F(\lambda_n,\lambda_n,\lambda_m,\epsilon) < \lambda,$$

$E(\mu_n, \mu_n, \mu_m, \epsilon) > 1 - \lambda$ and $F(\mu_n, \mu_n, \mu_m, \epsilon) < \lambda$ for all $m, n \geq N(\epsilon, \lambda)$.

As $\epsilon > 0$ and λ are arbitrary within their range, λ_n and μ_n are Cauchy sequences. □

3. Main Results

Theorem 3.1.

Let $(X, E, F, *, \lozenge)$ be a generalized intuitionistic fuzzy metric space having Hadžićtype t-norm and t-conorm. Suppose that $E(x, y, z, s) \to 1$ and $F(x, y, z, s) \to 0$ as $s \to \infty$, for all $x,y,z \in X$ and partial order $\preccurlyeq$ on X. Let $\Delta:X \times X \to X$ and $\Gamma:X \to X$ be two functions of which Δ has mixed Γ-monotone property and satisfy the conditions:

$$E(\Delta(\lambda, \mu), \Delta(\lambda, \mu), \Delta(\nu, \xi), ks) * E(\Delta(\mu, \lambda), \Delta(\mu, \lambda), \Delta(\xi, \nu), ks) \geq E(\Gamma(\lambda), \Gamma(\lambda), \Gamma(u), s) * E(\Gamma(\mu), \Gamma(\mu), \Gamma(v), s), \quad (3.1.1)$$

$$F(\Delta(\lambda, \mu), \Delta(\lambda, \mu), \Delta(\nu, \xi), ks) \diamond F(\Delta(\mu, \lambda), \Delta(\mu, \lambda), \Delta(\xi, \nu), ks) \leq F(\Gamma(\lambda), \Gamma(\lambda), \Gamma(u), s) \diamond F(\Gamma(\mu), \Gamma(\mu), \Gamma(v), s), \quad (3.1.2)$$

for all $\lambda,\mu,\nu,\xi \in X, s > 0$ with $\Gamma(\lambda) \preceq \Gamma(u)$ and $\Gamma(\mu) \succeq \Gamma(v)$, where $0 < k < 1$ and $\Delta(X \times X) \subseteq \Gamma(\lambda)$, (Γ,Δ) is compatible. Suppose either (a) Δ is continuous or (b) If $\{z_p\} \to z$ is such that $z_p \preceq z_{p+1}$, for every $p \geq 0$, then $z_p \preceq z$ for every $p \geq 0$; If $\{z_p\} \to z$ is such that $z_p \succeq z_{p+1}$, for every $p \geq 0$, then $z_p \succeq z$ for every $p \geq 0$.

If there exist $\lambda_0,\mu_0 \in X$ for which $\Gamma(\lambda_0) \preceq \Delta(\lambda_0, \mu_0), \Gamma(\mu_0) \succeq \Delta(\mu_0, \lambda_0)$, then there exist $\lambda,\mu \in X$ for which $\Gamma(\lambda) = \Delta(\lambda,\mu)$ and $\Gamma(\mu) = \Delta(\mu,\lambda)$.

Proof. Let $\lambda_0,\mu_0 \in X$ for which $\Gamma(\lambda_0) \preccurlyeq \Delta(\lambda_0, \mu_0), \Gamma(\mu_0) \succcurlyeq \Delta(\mu_0, \lambda_0)$). Consider the sequences $\{\lambda_p\}$ and $\{\mu_p\}$ in X given by

$$\Gamma(\lambda_{p+1}) = \Delta(\lambda_p,\mu_p) \text{ and } \Gamma(\mu_{p+1}) = \Delta(\mu_p,\lambda_p) \text{ for all } p \geq 0, \quad (3.1.3)$$

Then it follows that for all $p \geq 0$,

$$\Gamma(\lambda_p) \preceq \Gamma(\lambda_{p+1}) \text{ and} \quad (3.1.4)$$

$$\Gamma(\mu_p) \succeq \Gamma(\mu_{p+1}), \quad (3.1.5)$$

Let $s > 0, p \geq 0$ and $k \geq 1$. By (3.1.3), (3.1.4) and (3.1.5), and, from (3.1.1) and (3.1.2), we have

$E(\Gamma(\lambda_p),\Gamma(\lambda_p),\Gamma(\lambda_{p+1}),ks) * E(\Gamma(\mu_p),\Gamma(\mu_p),\Gamma(\mu_{p+1}),ks)$

$= E(\Delta(\lambda_{p-1}),\mu_{p-1},\Delta(\lambda_{p-1},\mu_{p-1}),\Delta(\lambda_p,\mu_p),ks) * E(\Delta(\mu_{p-1},\lambda_{p-1}),\Delta(\mu_{p-1},\lambda_{p-1}),\Delta(\mu_p,\lambda_p),ks)$

$\geq E(\Gamma(\lambda_{p-1}),\Gamma(\lambda_{p-1}),\Gamma(\lambda_p),ks) * E(\Gamma(\mu_{p-1}),\Gamma(\mu_{p-1}),\Gamma(\mu_p),ks)$ and

$$F(\Gamma(\lambda_p), \Gamma(\lambda_p), \Gamma(\lambda_{p+1}), ks) \diamond F(\Gamma(\mu_p), \Gamma(\mu_p), \Gamma(\mu_{p+1}), ks)$$
$$= F(\Delta(\lambda_{p-1}, \mu_{p-1}, F(\lambda_{p-1}, \mu_{p-1}), \Delta(\lambda_p, \mu_p), ks) \diamond F(\Delta(\mu_{p-1}, \lambda_{p-1}), \Delta(\mu_{p-1}, \lambda_{p-1}), \Delta(\mu_p, \lambda_p), ks)$$
$$\leq F(\Gamma(\lambda_{p-1}), \Gamma(\lambda_{p-1}), \Gamma(\lambda_p), ks) \diamond F(\Gamma(\mu_{p-1}), \Gamma(\mu_{p-1}), \Gamma(\mu_p), ks),$$

that is,

$$E(\Gamma(\lambda_p), \Gamma(\lambda_p), \Gamma(\lambda_{p+1}), s) * E(\Gamma(\mu_p), \Gamma(\mu_p), \Gamma(\mu_{p+1}), s) \geq E(\Gamma(\lambda_{p-1}), \Gamma(\lambda_{p-1}), \Gamma(\lambda_p), \tfrac{s}{k}) * E\left(\Gamma(\mu_{p-1}), \Gamma(\mu_{p-1}), \Gamma(\mu_p), \tfrac{s}{k}\right) \quad (3.1.6)$$

$$F(\Gamma(\lambda_p), \Gamma(\lambda_p), \Gamma(\lambda_{p+1}), s) \lozenge F(\Gamma(\mu_p), \Gamma(\mu_p), \Gamma(\mu_{p+1}), s) \leq F(\Gamma(\lambda_{p-1}), \Gamma(\lambda_{p-1}), \Gamma(\lambda_p), \tfrac{s}{k}) \lozenge F\left(\Gamma(\mu_{p-1}), \Gamma(\mu_{p-1}), \Gamma(\mu_p), \tfrac{s}{k}\right) \quad (3.1.7)$$

(3.1.6), (3.1.7) together with Lemma (2.9) exhibits that $\{\Gamma(\lambda_p)\}$ and $\{\Gamma(\mu_p)\}$ are Cauchy sequences and hence converge as X is complete. Therefore there exist $\lambda,\mu \in X$ such that

$$\lim_{p\to\infty} \Gamma(\lambda_p) = \lambda \text{ and } \lim_{p\to\infty} \Gamma(\mu_p) = \mu \quad (3.1.8)$$

Therefore,

$$\lim_{p\to\infty} \Gamma(\lambda_{p+1}) = \lim_{p\to\infty} \Delta(\lambda_p,\mu_p) = \lambda \text{ and } \lim_{p\to\infty} \Gamma(\mu_{p+1}) = \lim_{p\to\infty} \Delta(\mu_p,\lambda_p) = \mu$$

Since the pair (Γ,Δ) is compatible, the continuity of Γ and Definition (2.7) imply that

$$\Gamma(\lambda) = \lim_{p\to\infty} \Gamma(\Gamma(\lambda_{p+1})) = \lim_{p\to\infty} \Gamma(\Delta(\lambda_p,\mu_p)) = \lim_{p\to\infty} \Delta(\Gamma(\lambda_p),\Gamma(\mu_p)) \quad (3.1.9)$$

$$\Gamma(\mu) = \lim_{p\to\infty} \Gamma(\Gamma(\mu_{p+1})) = \lim_{p\to\infty} \Gamma(\Delta(\mu_p,\lambda_p)) = \lim_{p\to\infty} \Delta(\Gamma(\mu_p),\Gamma(\lambda_p)) \quad (3.1.10)$$

Suppose that (a) holds. Then from (3.1.9), (3.1.10) and by using (3.1.8), we have that

$$\begin{aligned}
\Gamma(\lambda) &= \lim_{p\to\infty} \Gamma(\Delta(\lambda_p,\mu_p)) \\
&= \lim_{p\to\infty} \Delta(\Gamma(\lambda_p),\Gamma(\mu_p)) \\
&= \Delta(\lim_{p\to\infty}\Gamma(\lambda_p), \lim_{p\to\infty} \Gamma(\mu_p)) = \Delta(\lambda,\mu) \text{ and} \\
\Gamma(\mu) &= \lim_{p\to\infty} \Gamma(\Delta(\mu_p,\lambda_p)) \\
&= \lim_{p\to\infty} \Delta(\Gamma(\mu_p),\Gamma(\lambda_p)) \\
&= \Delta(\lim_{p\to\infty} \Gamma(\mu_p), \lim_{p\to\infty} \Gamma(\lambda_p)) \\
&= \Delta(\mu,\lambda).
\end{aligned}$$

Therefore $\Gamma(\lambda) = \Delta(\lambda,\mu)$ and $\Gamma(\mu) = \Delta(\mu,\lambda)$.
Suppose that (b) holds. By (3.1.4), (3.1.5) and (3.1.8) we have that, for all $p \geq 0$,

$$\Gamma(\lambda_p) \preceq \lambda \text{ and } \Gamma(\mu_p) \succeq \mu.$$

Since Γ is monotonic increasing,

$$\Gamma(\Gamma(\lambda_p)) \preceq \Gamma(\lambda) \text{ and } \Gamma(\Gamma(\mu_p)) \succeq \Gamma(\mu) \quad (3.1.11)$$

Now, for all $s > 0, p \geq 0$, we have

$$\begin{aligned}
&E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Delta(\lambda_p,\mu_p)),s) \\
&\geq E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Gamma(\lambda_{p+1})),ks) * E(\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Delta(\lambda_p,\mu_p)),(s-ks)) \\
&F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Delta(\lambda_p,\mu_p)),s) \\
&\leq F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Gamma(\lambda_{p+1})),ks) \diamond F(\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Delta(\lambda_p,\mu_p)),(s-ks)).
\end{aligned}$$

Letting $p \to \infty$, we get, for all $s > 0$,

$$\begin{aligned}
&\lim_{p\to\infty} E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Delta(\lambda_p,\mu_p)),s) \\
&\geq \lim_{p\to\infty} E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Gamma(\lambda_{p+1})),ks) * E(\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Delta(\lambda_p,\mu_p)),(s-ks)), \\
&\lim_{p\to\infty} F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(F(\lambda_p,\mu_p)),s) \\
&\leq \lim_{p\to\infty} F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Gamma(\lambda_{p+1})),ks) \diamond F(\Gamma(\Gamma(\lambda_{p+1}1)),\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Delta(\lambda_p,\mu_p)),(s-ks))
\end{aligned}$$

That is,

$E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\lambda),s)$

$= \lim_{p\to\infty} E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Delta(\lambda_p,\mu_p)),ks) * E(\Gamma(\Gamma(\lambda_{p+1})),\Gamma(hg(\lambda_{p+1})),\Gamma(\lambda),(s-ks))$

$= E(\Delta(\lambda,\mu),\Delta(\lambda,\mu), \lim_{p\to\infty p\to\infty} \Gamma(\Delta(\lambda_p,\mu_p)),ks) * E(\lim_{p\to\infty} \Gamma(\Gamma(\lambda_{p+1})), \lim \Gamma(\Gamma(\lambda_{p+1})),\Gamma(\lambda),(s-ks))$

$= E(\Delta(\lambda,\mu),\Delta(\lambda,\mu), \lim_{p\to\infty} \Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),ks) * E(\Gamma(\lambda),\Gamma(\lambda),\Gamma(\lambda),s-ks)$ by (3.1.9)

$= \lim_{p\to\infty} E(\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),\Delta(\lambda,\mu),\Delta(\lambda,\mu),ks) * 1$

$= \lim_{p\to\infty} E(\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),\Delta(\lambda,\mu),\Delta(\lambda,\mu),ks)$

Therefore $E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\lambda),s) \geq \lim_{p\to\infty} E(\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),\Delta(\lambda,\mu),\Delta(\lambda,\mu),ks)$.(3.1.12)

$$F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\lambda),s)$$
$$= \lim_{p\to\infty} \{F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\Delta(\lambda_p,\mu_p)),ks) \diamond F(\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\Gamma(\lambda_{p+1})),\Gamma(\lambda),(s-ks))\}$$
$$= F(\Delta(\lambda,\mu),\Delta(\lambda,\mu), \lim_{p\to\infty} \Gamma(\Delta(\lambda_p,\mu_p)),ks) \diamond F(\lim_{p\to\infty} \Gamma(\Gamma(\lambda_{p+1})), \lim_{p\to\infty} \Gamma(\Gamma(\lambda_{p+1})),\Gamma(\lambda),(s-ks))\}$$
$$= F(\Delta(\lambda,\mu),\Delta(\lambda,\mu), \lim_{p\to\infty})\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),ks) \diamond F(\Gamma(\lambda),\Gamma(\lambda),\Gamma(\lambda),(s-ks))\text{by (3.1.9)}$$
$$= \lim_{p\to\infty} F(\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),\Delta(\lambda,\mu),\Delta(\lambda,\mu),ks) \diamond 0$$
$$= \lim_{p\to\infty} F(\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),\Delta(\lambda,\mu),\Delta(\lambda,\mu),ks)$$

Therefore $F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\lambda),s) \leq \lim_{p\to\infty} F(\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),\Delta(\lambda,\mu),\Delta(\lambda,\mu),ks)$. (3.1.13)

Similarly, we obtain for all $s > 0$

$E(\Delta(\mu,\lambda),\Delta(\mu,\lambda),\Gamma(\mu),s) \geq \lim_{p\to\infty} E(\Delta(\Gamma(\mu_p),\Gamma(\lambda_p)),\Delta(\mu,\lambda),\Delta(\mu,\lambda),ks)$ and (3.1.14)

$F(\Delta(\mu,\lambda),\Delta(\mu,\lambda),\Gamma(\mu),s) \leq \lim_{p\to\infty} F(\Delta((\mu_p),\Gamma(\lambda_p)),\Delta(\mu,\lambda),\Delta(\mu,\lambda),ks)$. (3.1.15)

From (3.1.14) and (3.1.15), using (3.1.10) and (3.1.11), for all $s > 0$, we have

$E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Gamma(\lambda),s) * E(\Delta(\mu,\lambda),\Delta(\mu,\lambda),\Gamma(\mu),s)$

$\geq \lim_{p\to\infty} E(\Delta(\Gamma(\lambda_p),\Gamma(\mu_p)),\Delta(\lambda,\mu),\Delta(\lambda,\mu),ks) * \lim_{p\to\infty} E(\Delta(\Gamma(\mu_p),\Gamma(\lambda_p)),\Delta(\mu,\lambda),\Delta(\mu,\lambda),ks)$

$\geq \lim_{p\to\infty} E(\Gamma(\Gamma(\lambda_p)), \Gamma(\lambda),), \Gamma(\lambda), s) * \lim_{p\to\infty} E(\Gamma(\Gamma(\mu_p)), \Gamma(\mu),), \Gamma(\mu), s)$ (since $*$ is continuous)

$= E(\lim_{p\to\infty} (\Gamma(\Gamma(\lambda_p)),\Gamma(\lambda),\Gamma(\lambda),s) * E(\lim_{p\to\infty} E(\Gamma(\Gamma(\mu_p)),\Gamma(\mu),\Gamma(\mu),s)$

$= E(\Gamma(\lambda),\Gamma(\lambda),\Gamma(\lambda),s) * E(\Gamma(\mu),\Gamma(\mu),\Gamma(\mu),s))$ by (3.1.9)

$= 1 * 1 = 1.$

$F(\Delta(\lambda, \mu), \Delta(\lambda, \mu), \Gamma(\lambda), s) * F(\Delta(\mu, \lambda), \Delta(\mu, \lambda),\Gamma(\mu), s)$

$\leq \lim_{p\to\infty} F(\Delta(\Gamma(\lambda_p), \Gamma(\mu_p)), \Delta(\lambda, \mu), \Delta(\lambda, \mu), ks) \lozenge \lim_{p\to\infty} F(\Delta(\Gamma(\mu_p), \Gamma(\lambda_p)), \Delta(\mu, \lambda), \Delta(\mu, \lambda), ks)$

$\leq \lim_{p\to\infty} F(\Gamma(\Gamma(\lambda_p)), \Gamma(\lambda),), \Gamma(\lambda), s) \lozenge \lim_{p\to\infty} F(\Gamma(\Gamma(\mu_p)), \Gamma(\mu),), \Gamma(\mu), s)$ (since $\lozenge$ is continuous)

$= F(\lim_{p\to\infty} (\Gamma(\Gamma(\lambda_p)), \Gamma(\lambda), \Gamma(\lambda), s) * F(\lim_{p\to\infty} E(\Gamma(\Gamma(\mu_p)), \Gamma(\mu), \Gamma(\mu), s)$

$= F(\Gamma(\lambda) ,\Gamma(\lambda), \Gamma(\lambda) ,s) \lozenge F(\Gamma(\mu), \Gamma(\mu), \Gamma(\mu), s))$ by (3.1.9)

$= 0 \lozenge 0 = 0.$

That is,

$$E(\Delta(\lambda,\mu),\Gamma(\lambda),\Gamma(\lambda),s) * E(\Delta(\mu,\lambda),\Gamma(\mu),\Gamma(\mu),s) \geq 1$$
$$F(\Delta(\lambda,\mu),\Gamma(\lambda),\Gamma(\lambda),s) \diamond F(\Delta(\mu,\lambda),\Gamma(\mu),\Gamma(\mu),s) \leq 0;,$$

which implies that $\Gamma(\lambda) = \Delta(\lambda,\mu)$ and $\Gamma(\mu) = \Delta(\mu,\lambda)$.Hence the proof. □

Corollary 3.2.

Let$(X, E, F,*,\lozenge)$be a generalized intuitionistic fuzzy metric space having Hadžićtype t-norm and t-conorm such that $E(x, y, z, s) \to 1$ and $F(x, y, z, s) \to 0$ as $s \to \infty$, for all $x,y,z\in X$.Let $\preccurlyeq$ be a partial order on X. Let $\Delta:X \times X \to X$ and $\Gamma:X \to X$ be two functions of which Δhas mixed Γ-monotone property and with the following properties:

$$\begin{aligned}&E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Delta(\nu,\xi),ks) * E(\Delta(\mu,\lambda),\Delta(\mu,\lambda),\Delta(\xi,\nu),ks)\\&\geq E(\Gamma(\lambda),\Gamma(\lambda),\Gamma(u),s) * E(\Gamma(\mu),\Gamma(\mu),\Gamma(v),s),\\&F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Delta(\nu,\xi),ks) \diamond F(\Delta(\mu,\lambda),\Delta(\mu,\lambda),\Delta(\xi,\nu),ks)\\&\leq F(\Gamma(\lambda),\Gamma(\lambda),\Gamma(u),s) \diamond F(\Gamma(\mu),\Gamma(\mu),\Gamma(v),s),\end{aligned}$$

for all $\lambda,\mu,\xi,\nu\in X$,$s >0$ with $\Gamma(\lambda) \preceq \Gamma(u)$ and $\Gamma(\mu) \succeq \Gamma(\nu)$, where $0 < k <1$ and$\Delta(X \times X) \subseteq \Gamma(\lambda)$, Γ is continuous and monotonic increasing, (Δ,Γ) is a commuting pair. Suppose either (a) $\Delta$is continuous or (b) If$\{z_p\} \to z$ is such that$z_p \preceq z_{p+1}$, for everyp ≥ 0, then $z_p \preceq z$ for every $p \geq 0$; If $\{z_p\} \to z$ is such that$z_p \succeq z_{p+1}$, for every $p \geq 0$, then $z_p \succeq z$ for every $p \geq 0$.

If there are $\lambda_0,\mu_0 \in X$ for which $\Gamma(\lambda_0) \preceq \Delta(\lambda_0,\mu_0), \Gamma(\mu_0) \succeq \Delta(\mu_0,\lambda_0)$, then there exist $\lambda,\mu \in X$ for which $\Gamma(\lambda) = \Delta(\lambda,\mu)$ and $\Gamma(\mu) = \Delta(\mu,\lambda)$.

Proof. As commuting pairs are also compatible pairs, the result follows from Theorem (3.1). It is established with an example that the Corollary (3.2) is actually contained within Theorem(3.1). □

Corollary 3.3.

Let $(X, E, F,*,\lozenge)$be a generalized intuitionistic fuzzy metric space having Hadžićtype t-norm and t-conorm. Suppose that $E(x, y, z, s) \to 1$ and $F(x, y, z, s) \to 0$ as $s \to \infty$, for all $x,y,z\in X$. Let$\preccurlyeq$ be a partial order on X. Let $\Delta:X \times X \to X$ and $\Gamma:X \to X$ be two functions of which Δhas mixed Γ-monotone property and with the following properties:

$$\begin{aligned}&E(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Delta(\nu,\xi),ks) * E(\Delta(\mu,\lambda),\Delta(\mu,\lambda),\Delta(\xi,\nu),ks)\\&\geq E(\Gamma(\lambda),\Gamma(\lambda),\Gamma(u),s) * E(\Gamma(\mu),\Gamma(\mu),\Gamma(v),s),\\&F(\Delta(\lambda,\mu),\Delta(\lambda,\mu),\Delta(\nu,\xi),ks) \diamond F(\Delta(\mu,\lambda),\Delta(\mu,\lambda),\Delta(\xi,\nu),ks)\\&\leq F(\Gamma(\lambda),\Gamma(\lambda),\Gamma(u),s) \diamond F(\Gamma(\mu),\Gamma(\mu),(v),s),\end{aligned}$$

for all $\lambda,\mu,\xi,\nu\in X$,$s >0$ with $\lambda \preceq \mu$ and $\nu \succeq \xi$, where $0 < k <1$ and $\Delta(X \times X) \subseteq \Gamma(\lambda)$.
Suppose either (a) $\Delta$is continuous or (b) If $\{z_p\} \to z$ is such that$z_p \preceq z_{p+1}$, for every $p \geq 0$, then$z_p \preceq z$ for every $p \geq 0$; If $\{z_p\} \to z$ is such that$z_p \succeq z_{p+1}$, for every $p \geq 0$, then $z_p \succeq z$ for every $p \geq 0$.

If there are $\lambda_0,\mu_0 \in X$ for which $\Gamma(\lambda_0) \preceq \Delta(\lambda_0,\mu_0), \Gamma(\mu_0) \succeq \Delta(\mu_0,\lambda_0)$, then there exist $\lambda,\mu \in X$ such that $\Gamma(\lambda) = \Delta(\lambda,\mu)$ and $\Gamma(\mu) = \Delta(\mu,\lambda)$.

Proof. Taking $\Gamma = I$ in Theorem (3.1), the result follows.

Example 3.4.

Let $(X,\preceq)$ be a partially ordered set with $X = [0,1]$ and the usual relation ordering $\leq$ on real numbers. Let for all $s >0$,$p,q,z\in X$,

$$\Delta(p,q,z,s)=\exp\{\frac{-(|p-q|+|q-z|+|z-p|)}{s}\}$$

$$H(p,q,z,s)=\frac{\exp\{\frac{-(|p-q|+|q-z|+|z-p|)}{s}\}-1}{\exp\{\frac{-(|p-q|+|q-z|+|z-p|)}{s}\}}.$$

Let $u * v = \min\{u,v\}$ and $u \diamond v = min\{u+v,1\}$. Then $(X, E, F, *, \lozenge)$ is a complete generalized intuitionistic fuzzy metric space such that $E(p,q,z,s) \to 1$ and $F(x, y, z, s) \to 0$ as $s \to \infty$, for all $p,q \in X$. Let the $\Gamma: X \to X$ and $\Delta: X \times X \to X$ be defined by

$$\Gamma(p)=\frac{5}{6}p^2 \text{ for all } p \in X,\ \Delta(p,q)=\frac{p^2-q^2}{4}.$$

Then $\Delta(X \times X) \subseteq \Gamma(\lambda)$ and Δ satisfies the mixed Γ-monotone property.

Let $\{t_n\}$ and $\{r_n\}$ be sequences in X such that

$\lim_{p\to\infty} \Delta(t_n,r_n) = a, \lim_{p\to\infty} \Gamma(t_n) = a, \lim_{p\to\infty} \Delta(r_n,t_n) = b$ and $\lim_{p\to\infty} \Gamma(r_n) = b$.

Now, for all $n \geq 0, \Gamma(t_n)=\frac{5}{6}r_n^2, \Gamma(r_n)=\frac{5}{6}r_n^2, \Delta(t_n,r_n)=\frac{t_n^2-r_n^2}{4}$ and $\Delta(r_n,t_n)=\frac{r_n^2-t_n^2}{4}$.

Then a and b must be zero. Lemma(2.5) gives then, for all $s > 0$,

$$\lim_{p\to\infty} E(\Gamma(\Delta(t_n,r_n)),\Gamma(F(t_n,r_n)),\Delta(g(t_n),\Gamma(r_n),s) = 1,$$

$$\lim_{p\to\infty} F(\Gamma(\Delta(t_n,r_n)),\Gamma(F(t_n,r_n)),\Delta(g(t_n),\Gamma(r_n),s) = 0, \text{ and}$$

$$\lim_{p\to\infty} E(\Gamma(\Delta(r_n,t_n)),\Gamma(F(r_n,t_n)),\Delta(g(r_n),\Gamma(t_n),s) = 1,$$

$$\lim_{p\to\infty} F(\Gamma(\Delta(r_n,t_n)),\Gamma(F(r_n,t_n)),\Delta(g(r_n),\Gamma(t_n),s) = 0.$$

Therefore (Δ,Γ) is a compatible pair.

Now we show that the inequalities (3.1.1) and (3.1.2) hold.

$$2|\Delta(p,q) - \Delta(u,v)| \geq |\Gamma(p) - \Gamma(u)| + |\Gamma(q) - \Gamma(v)|,\ p \geq u,\ q \leq v, \text{ and} \quad (3.4.1)$$

$$2|\Delta(q,p) - \Delta(v,u)| \geq |\Gamma(q) - \Gamma(v)| + |\Gamma(p) - \Gamma(u)|,\ p \geq u,\ q \leq v. \quad (3.4.2)$$

From (3.4.1), for all $s > 0$ and $0 < k < 1$, we have Similarly from (3.4.2), we get

$$\exp\frac{-(|\Delta(q,p)-\Delta(v,u)|)}{ks} \geq min\{E(\Gamma(p),\Gamma(p),\Gamma(u),s),\ E(\Gamma(q),\Gamma(q),\Gamma(v),s)\} \quad (3.4.3$$

$$\exp\frac{-(|\Delta(q,p)-\Delta(v,u)|)}{ks} \leq max\{F(\Gamma(p),\Gamma(p),\Gamma(u),s),\ F(\Gamma(q),\Gamma(q),\Gamma(v),s)\} \quad (3.4.4$$

From (3.4.3) and (3.4.4), we obtain that

$$\min\{E(\Delta(p,q),\Delta(p,q),\Delta(\nu,\xi),ks),\ E(\Delta(q,p),\Delta(q,p),\Delta(\xi,\nu),ks)\}$$
$$\geq \min\{E(\Gamma(p),\Gamma(p),\Gamma(u),s),\ E(\Gamma(q),\Gamma(q),\Gamma(v),s)\} \text{ and}$$

$$\max\{F(\Delta(p,q),\Delta(p,q),\Delta(\nu,\xi),ks),\ F(\Delta(q,p),\Delta(q,p),\Delta(\xi,\nu),ks)\}$$
$$\leq \max\{F(\Gamma(p),\Gamma(p),\Gamma(u),s),\ F(\Gamma(q),\Gamma(q),\Gamma(v),s)\}.$$

Therefore

$$E(\Delta(p,q),\Delta(p,q),\Delta(\nu,\xi),ks) * E(\Delta(q,p),\Delta(q,p),\Delta(\xi,\nu),ks)$$
$$\geq E(\Gamma(p),\Gamma(p),\Gamma(u),s) * E(\Gamma(q),\Gamma(q),\Gamma(v),s),$$
$$F(\Delta(p,q),\Delta(p,q),\Delta(\nu,\xi),ks) \diamond F(\Delta(q,p),\Delta(q,p),\Delta(\xi,\nu),ks)$$
$$\leq F(\Gamma(p),\Gamma(p),\Gamma(u),s) \diamond E(\Gamma(q),\Gamma(q),\Gamma(v),s).$$

Thus (3.1.1) follows.Similarly, the remaining cases follow. Thus Theorem 3.1 can be applied.It can also be found that (0,0) is the coupled coincidence point of (Δ,Γ).

References

[1] Alaca C, Turkoglu D and Yildiz C 2006 Fixed oints in intuitionistic fuzzy metric space, chaos, soliton and fractals **29 (5)**, 1073–78

[2] Atanassov K, 1986 Intuitionistic fuzzy sets, fuzzy sets and system **20** 87–96

[3] Bhaskar T G and Lakshmikantham V 2006 Fixed point theorems in partially ordered metric spaces and applns. Nonlinear Anal. **65**1379–93

[4] Choudhury B S and Das P 2016 Coupled coincidence point results in partially ordered generalized fuzzy metric spaces with applications to integral eqns. Math. Sci. **10(1-2)** 23-32

[5] Choudhury B S, Kumar S, Asha and Das K P 2012 Some fixed point theorems in G-metric spaces Math. Sci. Lett. **1** 25–31

[6] Choudhury B S, Das K P and Das P 2013 Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces Fuzzy Sets Syst. **222** 84–97

[7] Choudhury B S and Das P 2014 Coupled coincidence point results for compatible mappings in partially ordered probabilistic metric spaces Asian-European Journal of Mathematics **7(02)** 1450009

[8] George A and Veeramani P 1994 On some results in fuzzy metric spaces Fuzzy Sets and Syst. **64(3)** 395–399

[9] Gregori V and Sapena A 2002 On fixed point theorems in fuzzy metric spaces it Fuzzy Sets and Syst. **125** 245–252

[10] HadžićO and Pap E 2001 Fixed Point Theory in Probabilistic Metric Spaces Kluwer Academic Publishers, Dordrecht

[11] Lakshmikantham V and Ciric L 2009 Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces it Nonlinear Anal. **70** 4341–49

[12] Mohiuddine S A and Alotaibi A 2013 Coupled coincidence point theorems for compatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces Fixed Point Theory and Applications 265(2013)

[13] Mustafa Z and Sims B 2006 A new approach to generalized metric spaces J. Nonlinear Convex. Anal. **7** 289–297

[14] Park J H 2004 Intuitionistic fuzzy metric space, Chaos, Solitions and Fractals **22** 1039-46

[15] Saadati R, Vaezpour S M, Vetro P and Rhoades B E 2010 Fixed point theorems in generalized partially ordered G-metric spaces Math. Comput. Model **52** 797–801.

[16] Sushil Sharma, BhavanaDeshpande and Suresh Chouhan 2011 DS-Weak commutativity condition and common fixed point in intuitionistic menger spaces The Pure and Applied Mathematics **18(3)** 201-17.

[17] Sun G and Yang K 2010 Generalized fuzzy metric spaces with properties Res. J. Appl. Sci. Eng. Technol. **2** 673–78.

[18] Turkoglu D, CihangirAlaca, Cho Y J and Yildiz C 2006 Common fixed point theorems in intuitionistic fuzzy metric spaces Journal of Applied Mathematics and Computing **22(1-2)** 411-24.

[19] Wu Jun 2014 Some fixed point theorems for mixed monotone operators in partially ordered probabilistic metric spaces Fixed Point Theory and Applications 49(2014) [20] Zadeh L 1965 Fuzzy sets Inf. and Control **8(3)** 338–53.