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1-NEIGHBORLY EDGE IRREGULAR GRAPHS

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Abstract A simple graph $G(V,E)$ is 1- Neighbourly edge irregular graph(1NEI) if no two adjacent edges of G have same number of edges at edge distance one. In this paper, we prove a necessary and sufficient condition for a graph to be 1-NEI graph and some results on it. We study some properties of 1-NEI and several methods to construct 1-NEI graph from a given1-NEI graph. It is shown that every graph is an induced subgraph of some 1-Neighbourly irregular graph.

Keywords: Neighbourly irregular graphs, *m*-Neighbourly irregular graphs, Edge irregular graphs, Neighbourly edge irregular fuzzy graphs, pairable vertices, support of a graph.

AMS Subject Classification: Primary: 05C12, Secondary: 03E72, 05C72.

1. Introduction

Throughout this paper we consider finite, simple connected graphs. Let G be a graph with n vertices and m edges. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to v and is denoted by $d_G(v)$ or simply $d(v)$.The concept of Neighbourly irregular graphs was introduced and studied by S. Gnaana Bhragsam and S.K.Ayyaswamy [2]. N.R. SanthiMaheswari and C.Sekar introduced the concept of m-Neighbourly irregular graphs [6] and neighborly edge irregular fuzzy graphs [12]. The degree of an edge $e = (u,v)$ as the number of edges which have a common vertex with the edge e. (i.e)$deg(e) = deg(u) + deg(v) -2$[5]. Edge regular graphs are those graphs for which each edge has the same degree.The distance between two edges $e_1 = (u_1,v_1)$ and $e_2 = (u_2,v_2)$ is defined as $ed(e_1,e_2)= \min\{d(u_1,u_2),d(u_1,v_2), d(v_1,u_2),d(v_1,v_2)\}$. If $ed(e_1,e_2) = 0$, these edges are neighbour edges[4]. The purpose of this paper is to introduce a new class of graphs based on distance property in edge sense. The concept of 1-Neighborly edge irregular graphs is analogous to *m*-Neighborly irregular graphs but considering the distance between the edges instead of vertices.This is the background to introduce 1- Neighborly edge irregular graphs.

2. Preliminaries

We present some known definitions and results for ready reference to go through the work presented in the paper.

Definition 2.1. A graph G is said to be neighborly irregular if no two adjacent vertices of G have the same degree.

Definition 2.2. A connected graph is said to be m-Neighborly Irregular(m-NI)graph if no two adjacent vertices of G have the same number of vertices at a distance m away from it.

Definition 2.3. A graph G is said to be Neighborly edge irregular if no two adjacent edges of G have the same edge degree.

Definition 2.4. Let be G be a graph. For any two distinct vertices u and v in G, u is pairable with v if N[u] = N[v] in G. A vertex in G is called a pairable vertex if it is pairable with a vertex in G.

Definition 2.5. Let G be a graph. A full vertex of G is a vertex in G which is adjacent to all other vertices of G.

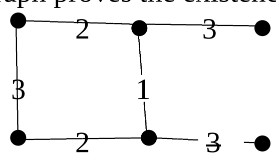
Definition 2.6. The support $s_G(v)$ or simply s(v) of a vertex v is the sum of degrees of its neighbors. That is, $s(v) = \sum_{u\in N(v)} d(u)$.

3. 1-Neighborly edge irregular graphs(1-NEI)

In this section, we introduce 1-Neighborly edge irregular graphs and study some properties of these graphs.

Definition 3.1. A simple graph G(V,E) is 1- Neighborly edge irregular graph(1-NEI) if no two adjacent edges of G have same number of edges at edge distance one.

Example 3.2. The following graph proves the existence of 1-NEI graphs.



2.1. *Figure* 1

Results:

- There are no 1-NEI graph of order n = 3,5and 7. For n = 4, P_4 is the 1-NEI graph
- Let G be a 1-NEI graph. Then there will not be more than one pendant edges at any vertex.
- Let G be a 1-NEI graph. Then there is no P_5 with internal vertices of degree 2 and external vertices of same degree as an induced subgraph.
- For any edge uvinE(G), $ed_1(uv)=s(u)+s(v)-\sum_{x\epsilon N(u)\cap N(v)} d(x)$ -2ed(uv)-m-2 where m is the number of edges in the induced subgraph<N(u)∪N(v)>.

The following theorem proves a necessary and sufficient condition for a graph to be 1-NEI graph.

Theorem 3.3 A graph G is a 1-NEI graph if and only if for any two adjacent edges uv and vw , then $(s(u) - s(w)) - (d( ) - d(w)) \neq \sum_{x \varepsilon N(u) \cap N(v)} d(x) - \sum_{y \varepsilon N(v) \cap N(w)} d(y) + (a - b)$, where a and b are the number edges in the induced subgraphs< N(u)∪N(v)\{u,v} > and < N(v)∪N(w)\{v,w} > respectively.

Proof: .Let G be a 1-NEI graph. Let uvand vwbe any two adjacent edges.Then $ed_1(uv) \neq ed_1(vw)$ which implies:

$$s(u)+s(v)-\sum x \varepsilon N_{(u)} \cap N_{(v)} d(x) -2ed(uv)-a-2 \neq s(v)+s(w)-\sum y \varepsilon N_{(v)} \cap N_{(w)} d(y) -2ed(vw)-b-2$$

where a and b are the number edges in the induced subgraphs < N(u)∪N(v)\{u,v} > and <N(v)∪N(w)\{v,w} > respectively, since ed(uv)=d(u)+d(v)-2 and ed(vw)=d(u)+d(w)-2,$(s(u) - s(w)) - (d(u) - d(w)) = \neq \sum_{x \varepsilon N(u) \cap N(v)} d(x) - \sum_{y \varepsilon N(v) \cap N(w)} d(y) + (a - b)$.

Conversely suppose that $(s(u) - s(w)) - (d(u) - d(w)) \neq \sum_{x \varepsilon N(u) \cap N(v)} d(x) - \sum_{y \varepsilon N(v) \cap N(w)} d(y) + (a - b)$, for some two adjacent edges uv and vw, that is:

$$s(u)+s(v)-\sum x \varepsilon N_{(u)} \cap N_{(v)} d(x) -2ed(uv)-a-2 \neq s(v)+s(w)-\sum y \varepsilon N_{(v)} \cap N_{(w)} d(y) -2ed(vw)-b-2.$$

This implies that $ed_1(uv) \neq ed_1(vw)$. Hence G is a 1-NEI graph. □

Theorem 3.4. . Let G be a 1-NEI graph without triangles . If there are two adjacent edges uv and vw in E(G) with d(u) = d(w), then $s(u) \neq s(w)$.

Proof: Let *G* be a 1-NEI graph with girth at least 5. Let *uv* and *vw* be two adjacent edges with *d*(*u*) = *d*(*w*). Then:

$$ed_1(uv) = s(u) + s(v) - \sum x \varepsilon N_{(u)} \cap N_{(v)} d(x) \text{ -2ed(uv)-a-2 and } ed_1(vw) = s(v) + s(w) - \sum y \varepsilon N_{(v)} \cap N_{(w)} d(y) \text{ -2ed(vw)-b-2.}$$

Since girth of *G* is at least 5, $|N(u) \cap N(v)| = |N(v) \cap N(w)| = 0$ and $a=b=0$. If $s(u)=s(w)$, then $ed_1(uv) = ed_1(vw)$, which is a contradiction. □

Corollary 3.5. If G is a 1-NEI graph without triangles, then there is no P_3 (say uvw) such that d(u) = d(w) and s(u) = s(w).

Theorem 3.6. Let G be a 1-NEI graph without triangles. For any two adjacent edges uv and vw , then $N(u) \neq N(w)$.

Proof: .Let *G* be a 1-NEI graph without triangles. suppose there are some adjacent edges *uv* and *vw* such that *N*(*u*) = *N*(*w*), then *d*(*u*) = *d*(*w*) and *s*(*u*)=*s*(*w*),since girth of *G* is at least 5, $|N(u) \cap N(v)| = |N(v) \cap N(w)| = 0$ and $a=b=0$. Then $ed_1(uv) = ed_1(vw)$, which is a contradiction. □

Theorem 3.7. A graph with a pairable vertex is not 1-NEI graph

Proof: Let *G* be a graph with a pairable vertex *u*, pairable with *v*, Then *N*[*u*] = *N*[*v*]. If *N*(*u*) = *N*(*v*) = $\{u_1,u_2,...,u_d\}$, then $ed_1(uu_i) = ed_1(vu_i)$ for $1 \leq i \leq d$, which is a contradiction. □

Theorem 3.8 . Any graph with more than one full vertex is not 1-NEI graph.

Proof.Suppose *G* has more than one full vertex say *u* and *v*. Then *N*[*u*] = *N*[*v*], If *N*(*u*) = *N*(*v*) = $\{u_1,u_2,...,u_d\}$, then $ed_1(uu_i) = ed_1(vu_i)$ for $1 \leq i \leq d$,*G*is not 1-NEI graph. □

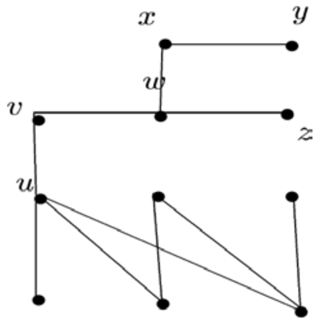
Theorem 3.9. Let G be a 1-NEI graph. Then there is no cycle in G with vertices $v_1,v_2,...,v_m$such that $d(v_i) = d(v_{i+2}) = 2$ and $d(v_{i-1}) = d(v_{i+3})$ for some $1 \leq i \leq m$.

Proof*:* .If there is a cycle in *G* with vertices $v_1,v_2,...,v_m$such that $d(v_i) = d(v_{i+2}) = 2$ and $d(v_{i-1}) = d(v_{i+3})$ for some $1 \leq i \leq m$, then $ed_1(v_iv_{i+1})= ed_1(v_{i+1}v_{i+2})$, which is a contradiction. □

Yousefalavi [1] proved that for every positive integer *n* = 36 ,5,7, there exists a highly irregular graph of order *n*.

Theorem 3.10. For every positive even integer n = 2d,d ≥ 3, there exists a 1-NEI graph of order n and it is denoted by 1-$NEI_{(n)}$.

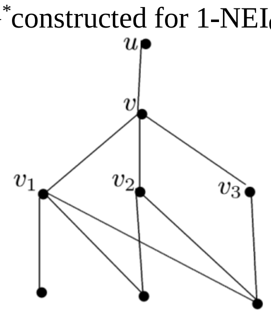
Corollary 3.11. For every positive odd integer n ≥ 11, there exists a 1-NEI graph of order n. For, we can construct G*from 1-$NEI_{(n-5)}$ by attaching a graph as in Figure 2 at u_1 or $v_{(n-5)}$. $ed_1(e)$ in G*=$ed_1(e)$+1 in G, $ed_1(u_1v)$ in G*=$ed_1(u_1v_1)$ + 2 in G, $ed_1(vw)$=d(u) + 1 in G, $ed_1(wz)$=ed(xy) = 2 and $ed_1(wx)$ = 1. Then G*is also a 1-NEI graph of order n.



2.2. Figure 2

Remark. We can construct a 1-NEI graph of order n+2 from 1−$NEI_{(n)}$ by attaching 2 new vertices u and v and joining the edges uv and vv_i, 1 ≤ i ≤ d.The resulting graph G^* is a 1 −NEI graph of order n + 2.

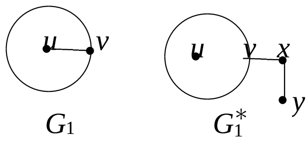
For illustration, the graph G^*constructed for 1-NEI_6 is given in Figure 3.



2.3. Figure 3

Results:

- Let G_1 be a 1-NEI graph. Let v be a vertex of degree 1 which is adjacent to the vertex u s.t d(u) ≥ 3 as in Figure 4,which satisfies $ed_1(uv) + 1 \neq ed(uv)$ and $ed_1(uu_i) \neq ed_1(u_iv_j) + 1$ for $u_i$inN(u) and $v_j$inN(u_i). We can construct $G_1^*$by introducing P_3 at v .Then $ed_1(u_iv_j)\ \ in\ \ G_1^* =\ ed_1(u_iv_j)$ in $G_1 \neq ed_1(uu_i)+1$ in G_1 = $ed_1(uu_i)$ in G_1^*, $ed_1(vx)$ in G_1^*= ed(uv) in $G_1\ \ \neq ed_1(uv)+1\ \ in\ \ G_1 = ed_1(uv)\ \ in\ \ G_1^*$ and $ed_1(xy)$ = 1.Hence G_1^* is alsoa 1-NEI graph.

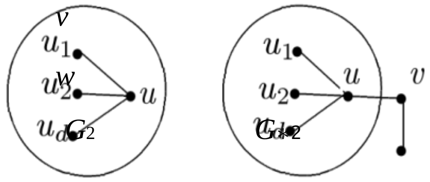


2.4. Figure 4

- Let G_2 be a triangle-free 1-NEI graph. Let u be a vertex of the graph G_2 having degree d which is adjacent to the vertices $u_1,u_2,...,u_d$ s.t $ed_1(uu_i) + 1 \neq s(u) - d(u)$ and $2d(u) \neq s(u)$ in G_2 as in Figure 5. We can construct G_2^* by introducing P_3 at u. Then for $1 \leq i \leq d$ and w_j in $N(v_i)$,

$$ed_1(u_iw_j) \text{ in } G_2^* = ed_1(u_iw_j) + 1 \text{ in } G_2$$
$$\neq ed_1(uu_i) + 1 \text{ in } G_2$$
$$= ed_1(uu_i) \text{ in } G_2^*,$$
$$ed_1(uv) \text{ in } G_2^* = s(u) - d(u) \text{ in } G_2$$
$$\neq ed_1(uu_i) + 1 \text{ in } G_2$$
$$= ed_1(uu_i) \text{ in } G_2^*,$$
$$ed(uv) \text{ in } G_2^* = s(u) - d(u) \text{ in } G_2$$
$$\neq 2d(u) - d(u) \text{ in } G_2$$
$$= d(u) \text{ in } G_2$$
$$= ed_1(vw) \text{ in } G_2^*.$$

Therefore, G_2^* is also a 1-NEI graph.

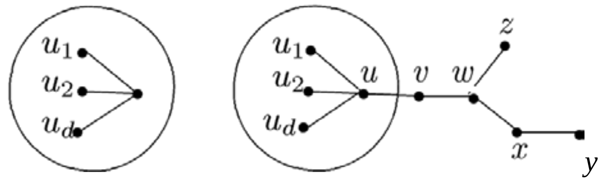


2.5. Figure 5

- Let G_3 be a triangle-free 1-NEI graph. Let u be a vertex of the graph G_3 having degree d which is adjacent to the vertices $u_1,u_2,...,u_d$ s.t $ed_1(uu_i)+1 \neq s(u)-d(u)$ and $2d(u)-1 \neq s(u)$ in G_3 as in Figure 6. We can construct G_3^* by attaching a graph at u as in Figure 6. Then for $1 \leq i \leq d$, w_j in $N(v_i)$, $ed_1(u_iw_j)$ in $G_3^* = ed_1(u_iw_j) + 1$ in G_3

$$\neq ed_1(uu_i) + 1 \text{ in } G_3$$
$$= ed_1(uu_i) \text{ in } G_3^*,$$
$$ed_1(uv) \text{ in } G_3^* = s(u) - d(u) \text{ in } G_3$$
$$\neq ed_1(uu_i) + 1 \text{ in } G_3$$
$$= ed_1(uu_i) \text{ in } G_3^*,$$
$$ed_1(uv) \text{ in } G_3^* = s(u) - d(u) \text{ in } G_3$$
$$\neq 2d(u) - 1 - d(u) \text{ in } G_3$$
$$= d(u) - 1 \text{ in } G_3$$
$$= ed_1(vw) \text{ in } G_3^*,$$
$$ed_1(wz) = ed_1(xy) = 2 \text{ and } ed_1(wx) = 1.$$

Therefore, G_3^* is also a 1-NEI graph.



G_3 G_3^*

2.6. Figure 6

Theorem 3.12. Every complete bipartite graph $K_{r,r}$ is an induced subgraph of a 1-NEI graph of order 4r.

Proof. Let $u_1, u_2, \ldots u_r$ and $v_1, v_2, \ldots v_r$ are two partites of $K_{r,r}$. Introduce the vertices $u'_1, u'_2, \ldots u'_r$ and $v'_1, v'_2, \ldots v'_r$. Join the vertices $u_i u'_j$, $1 \leq i \leq r$, $i \leq j \leq r$ and $v_i v'_j$, $1 \leq i \leq r$, $i \leq j \leq r$. The resulting graph contains $K_{r,r}$ as an induced subgraph.

Figure 7 illustrates the theorem 3.20 for $K_{3,3}$

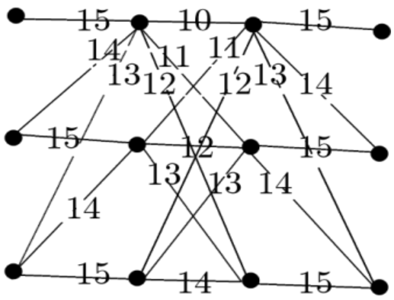


Figure 7

□

Theorem 3.13. Every complete graph of order n ≥ 3 is an induced subgraph of a 1-NEI graph of order 2n(n+1) if n is even n(2n+1) if n is odd.

Proof: Let G be a complete graph of order $n \geq 3$. Let $u_1, u_2, \ldots, u_n$ be the vertices of G. For each $1 \leq i \leq n$, we add new vertices v_{ij} and w_{ij}, $1 \leq j \leq [\frac{n}{2}] + i$. The vertices $u_i (1 \leq i \leq n)$ and the vertices v_{ij} and w_{ij}, $1 \leq i \leq n, 1 \leq j \leq [\frac{n}{2}] + 1$ constitute the vertex set of the required graph H .Along with the edges of G, we add several edges to complete the constitution of H . For $1 \leq i \leq n, 1 \leq j \leq [\frac{n}{2}] + i$, we join u_i and v_{ij} and for $j \leq k \leq [\frac{n}{2}] + i$, v_{ij} and w_{ik}. The resulting graph H contains G as an induced subgraph.

Moreover:

$ed_1(u_i u_{ij}) = (n-1)C_2 + ([\frac{n}{2}]+i+1)C_2 - ([\frac{n}{2}]+i-(j-1)) + n[\frac{n}{2}] + ni + nC_2 - ([\frac{n}{2}]+i)$, $ed_1(v_{ij} w_{ik}) = ([\frac{n}{2}]+i)C_2 + (j-k) + ([\frac{n}{2}]+i-1) + n-1$ for all $1 \leq i \leq n, 1 \leq j \leq [\frac{n}{2}]+1, 1 \leq k \leq [\frac{n}{2}]+1$, $ed_1(u_p u_q) = (p+1)C_2 + ([\frac{n}{2}]+q+1)C_2 + (n-2)C_2 + n[\frac{n}{2}] + (n+1)C_2 - (([\frac{n}{2}]+i) - ([\frac{n}{2}]+p))$. Order of H,

$$O(H) = n + 2([\frac{n}{2}]+1+[\frac{n}{2}]+2+\ldots[\frac{n}{2}]+n) = n + 2(n[\frac{n}{2}]) + \frac{n(n+1)}{2} = n + 2n[\frac{n}{2}] + n(n+1) = \begin{cases} 2n(n+1) \textit{ if } n \textit{ is even} \\ n(2n+1) \textit{ if } n \textit{ is odd} \end{cases}$$

Figure 8 illustrates the theorem 3.13 for K_3 .

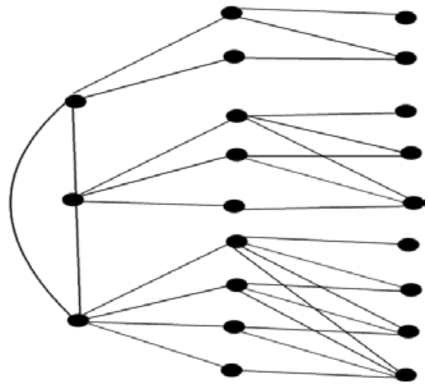


Figure 8

Theorem 3.14. Every connected graph of order ≥ 5 is an induced subgraph of 1-NEI graph.

Proof*:* Let G be a graph of order $n \geq 5$. Let G^0 be another copy of of$G$, where $V(G) = \{v_1^1, v_2^1, ..., v_n^1\}$ and $V(G^0) = \{v_1^2, v_2^2, ..., v_n^2\}$ and v_i^1 corresponds to $v_i^2 (1 \leq i \leq n)$. Join the edges $v_j^1 v_i^2, v_j^2 v_i^1 : v_j^1 v_i^1 \notin E(G), 1 \leq j \leq n, j+1 \leq i \leq n$ and $v_k^1 v_k^2, 1 \leq k \leq n$. Consider n is any of the form $5m + l$, $0 \leq l \leq 4$. For each v_i^1 and $v_i^2, 1 \leq i \leq n$, introduce the new vertices u^1_{ij}, w_{ij}^1 and u^2_{ij}, w_{ij}^2 , the values of j are 1≤j≤3+12(m-1)+(i-1) if l=0, 1≤j≤5+12(m-1)+(i-1) if l=1, 1≤j≤8+12(m-1)+(i-1) if l=2, 1≤j≤10+12(m-1)+(i-1) if l=3, and 1≤j≤12+12(m-1)+(i-1) if l=4, for $1 \leq i \leq n$, join the edges $u_{ij} w_{ik}$, for above mentioned$j$ and$k \geq m$ and join the vertices v_i^1 with u^1_{ij} and v_i^2 with u^2_{ij} for all i and$j$.

The resulting graph contains G as an induced subgraph and it is 1-NEI graph of order $2n + 2(n(k + 12(m - 1) + nC_2) = n(n + 1) + 2n(k + 12(m - 1))$ where k = 3,5,8,10,12 for l = 0,1,2,3,4 respectively.

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