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FIXED POINT THEOREM IN GENERALIZED FUZZY METRIC SPACES FOR IDEMPOTENT MAPPINGS

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Abstract: In this paper, we generalized a common fixed point theorem in generalized fuzzy metric spaces by using the relationship between reciprocal continuity for idempotent maps into generalized fuzzy metric space.

Key words: Fixed Point, Generalized Fuzzy Metric Space, Weak Compatibility,Idempotent.

AMS Mathematics Subject Classification: 47H10, 54A40.

1. Introduction

The concept of fuzzy set was given by Zadeh [14], which laid the foundation of fuzzy mathematics. Later on, the concept of fuzzy metric space was introduced by Kramosil and Michalek[6] which is modified by George and Veeramani[2]. Also Grabiec [3] proved some fixed point results for fuzzy metric space which was developed extensively by many authors and used in various fields. Sessa [10] introduced the tradition of improving commutative condition in fixed point theorems by introducing the notion of weak commuting property. In 2006 the concept of weakly compatible maps is given Jungck and Rhodes[5] which is more generalized than compatible maps after that R - weak commutatibity of mappings of fuzzy metric space is defined by Vasuki[12] and he also proved the fuzzy version of Pant's[8] theorem. In 2000, Singh and Chauhan introduced the concept of compatibility in fuzzy metric space. Here we will prove a fixed point theorem in fuzzy metric space by defining weak commuting in fuzzy metric space and reciprocal continuity for idempotent maps in generalized fuzzy metric spaces.

2. Preliminaries

Definition 2.1.

A binary operation *: [0, 1] × [0, 1] → [0, 1] is a continuous t - norm, if it satisfies the following conditions:

(i) * is associative and commutative,
(ii) * is continuous,
(iii) a * 1 = a for all a ∈ [0, 1],
(iv) a * b ≤ c * d whenever a ≤ c and b ≤ d, for each a, b, c, d ∈[0, 1].

Definition 2.2

A 3-tuple (X, $\mathcal{M}$, ∗) is called $\mathcal{M}$- fuzzy metric space if X is an arbitrary non empty set, ∗ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions: for eachx, y, z, a ∈ X and t, s > 0.

($\mathcal{M}$1) $\mathcal{M}(x, y, z, t) > 0$,
($\mathcal{M}$2)$\mathcal{M}(x, y, z, t) = 1$ if and only if x = y = z,
($\mathcal{M}$3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, when p is the permutation function,
($\mathcal{M}$4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
($\mathcal{M}$5) $\mathcal{M}(x, y, z, .) : [0, \infty) \to [0,1]$ is left continuous,
($\mathcal{M}$6) $\lim_{t\to\infty}\mathcal{M}(x, y, z, t) = 1$ for all x, y, z ∈ X.

Definition 2.3

Two self mappings f and g of a generalized fuzzy metric space (X,$\mathcal{M}$,∗) is called weakly commuting if $\mathcal{M}(f^2g^2x, g^2f^2x, g^2f^2x, t) \geq \mathcal{M}(f^2x, g^2x, g^2x, t)$.

Remark 2.4

Weak commutative reduced to weak commuting pair (f, g) that is

$\mathcal{M}(f^2g^2x, g^2f^2x, g^2f^2x, t) \geq \mathcal{M}(f^2gx, g^2fx, g^2fx, t) \geq \mathcal{M}(fg^2x, gf^2x, gf^2x, t)$
$\geq \mathcal{M}(fgx, gfx, gfx, t) \geq \mathcal{M}(f^2x, g^2x, g^2x, t)$

If f and g are idempotent map that is $f^2 = f$, $g^2 = g$.

Definition 2.5

Two self mappings f and g of a generalized fuzzy metric space (X, $\mathcal{M}$, ∗) into itself which are idempotent maps that is f^2=f and g^2= g are called reciprocally continuous on X if $\lim_{n\to\infty} f^2g^2x_n = f^2x$ and $\lim_{n\to\infty} g^2f^2x = g^2x$whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f^2x_n = \lim_{n\to\infty} g^2x_n = x$ for some x in X.

That is $\mathcal{M}(f^2g^2x_n, g^2f^2x_n, g^2f^2x_n, t) \geq \mathcal{M}(f^2gx_n, g^2fx_n, g^2fx_n, t)$
$\geq \mathcal{M}(fg^2x_n, gf^2x_n, gf^2x_n, t)$
$\geq \mathcal{M}(fgx_n, gfx_n, gfx_n, t)$
$\geq \mathcal{M}(f^2x_n, g^2x_n, g^2x_n, t)$
$\geq \mathcal{M}(f^2x, g^2x, g^2x, t)$,

whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty}\mathcal{M}(f^2x_n, g^2x_n, g^2x_n, t) \geq \mathcal{M}(f^2x, g^2x, g^2x, t)$ for all t > 0 thus if two self mappings are weak commuting then they are reciprocally continuous as well.

Lemma 2.6

Let $\{y_n\}$ be a sequence in generalized fuzzy metric space $(X, \mathcal{M}, *)$ with the condition $\lim_{n\to\infty} \mathcal{M}(x, y, z, t) = 1$ and if there exist a number $k \in (0, 1)$ such that

$\mathcal{M}(y_{2n+2}, y_{n+1}, y_{n+1}, kt) \geq \mathcal{M}(y_{2n+1}, y_n, y_n, t)$ for all $t > 0$, then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.7

Let f and g be two mappings on a complete generalized fuzzy metric space $(X, \mathcal{M}, *)$ into itself such that for some $k \in (0, 1)$, $\mathcal{M}(fx, gx, gx, kt) \geq \min\{\mathcal{M}(x, y, z, t), \mathcal{M}(fx, x, x, t)\}$ for all x, y, z $\in$ X and for all $t > 0$. Then f and g have a unique common fixed point in X.

3. Main Result

Theorem 3.1

Let $(X, \mathcal{M}, *)$ be a complete generalized fuzzy metric space and P, Q and R be continuous mappings of X in X. Let A, B and C be self mappings of X satisfying [A, P],[B, Q] and [C, R] are weak commuting and

(i) $A(X) \subseteq P(X)$, $B(X) \subseteq Q(X)$ and $C(X) \subseteq R(X)$.

(ii) $\mathcal{M}(A^2x, B^2y, C^2z, t) \geq r[\min\{\mathcal{M}(P^2x,Q^2y, R^2z,t),\mathcal{M}(P^2x,A^2x,Q^2z,t),\mathcal{M}(Q^2y,P^2y,C^2z,t),$ $\mathcal{M}(R^2z,C^2z,Q^2x,t),\mathcal{M}(R^2x, Q^2y, R^2z, t),\mathcal{M}(P^2x,Q^2x,C^2y,t)\}]$.

For all $x,y,z \in X$, where $r:[0,1] \to [0,1]$ is continuous function such that $r(t) > t$ for each $0 \leq t \leq 1$ and $r(1) = 1$. The sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X are such that $x_n \to x, y_n \to y, z_n \to z$ $\Rightarrow \mathcal{M}(x_n, y_n, z_n, t) \to \mathcal{M}(x, y, z, t)$ where $t > 0$ then A, B, C, P, Q and R have a unique common fixed point in X.

Proof: We define sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that $y_{2n}=A^2x_{2n}=P^2x_{2n+1}$, $y_{2n+1}=B^2x_{2n+1}=Q^2x_{2n+2}$ and $y_{2n+2}= C^2x_{2n+2}=R^2x_{2n+3}$, for n=1,2... now we shall prove that $\{x_n\}$ is a Cauchy sequence .

Let $\mathcal{M}_{2n} = \mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n}, t)= \mathcal{M}(A^2x_{2n+2}, B^2x_{2n+1}, C^2x_{2n}, t)$

$\geq r(\min\{\mathcal{M}(P^2x_{2n+2}, Q^2x_{2n+1}, R^2x_{2n}, t), \mathcal{M}(P^2x_{2n+2}, A^2x_{2n+2}, Q^2x_{2n}, t),$

$\mathcal{M}(Q^2x_{2n+1}, P^2x_{2n+1},C^2x_{2n}, t), \mathcal{M}(R^2x_{2n}, C^2x_{2n}, Q^2x_{2n+2}, t),$

$\mathcal{M}(R^2x_{2n+2}, Q^2x_{2n+1}, R^2x_{2n}, t), \mathcal{M}(P^2x_{2n+2}, Q^2x_{2n+2}, C^2x_{2n+1}, t)\})$

$\geq r(\min\{\mathcal{M}(y_{2n+1}, y_{2n}, y_{2n-1}, t), \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n}, t), \mathcal{M}(y_{2n}, y_{2n}, y_{2n},t),$

$\mathcal{M}(y_{2n-1}, y_{2n}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n-1}, t),\mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+1}, t)\})$

$\geq r(\min\{\mathcal{M}_{2n-1}, \mathcal{M}_{2n}, 1,\mathcal{M}_{2n-1}, \mathcal{M}_{2n-1}, t\})$ (3.1.1)

If $\mathcal{M}_{2n-1} \geq \mathcal{M}_{2n}$ then $\mathcal{M}_{2n} \geq r[\mathcal{M}_{2n-1}] > \mathcal{M}_{2n-1}$

which is a contradiction therefore $\mathcal{M}_{2n-1} \geq \mathcal{M}_{2n}$

From (3.1.1) we have $\mathcal{M}_{2n} \geq r[\mathcal{M}_{2n-1}] > \mathcal{M}_{2n-1}$ (3.1.2)

Thus $\{\mathcal{M}_{2n}, n \geq 0\}$ is increasing sequence of positive real numbers in [0,1] and therefore approaches to $l_1 \geq 1$ it is clear that $l_1 = 1$ because if $l_1 < 1$ then on taking limit as $n \to \infty$ in (3.1.2) we get $l_1 \geq r[l_1] > l_1$ a contradiction hence $l_1 = 1$. Now for any integer m,

$\mathcal{M}(y_n, y_{n+m}, y_{n+m}, t) \geq \mathcal{M}(y_n, y_{n+m}, y_{n+m+1}, t/m) * \ldots * \mathcal{M}(y_{n+m-1}, y_{n+m}, y_{n+m+1}, t/m)$

Therefore, $\lim_{n\to\infty} \mathcal{M}(y_n,y_{n+m}, y_{n+m+1}, t) \geq 1 * 1 * \ldots * 1$ because $\lim_{n\to\infty} \mathcal{M}(y_n, y_{n+1}, y_{n+2}, t) = 1$

for $t > 0$. Thus $\{y_n\}$ is a Cauchy sequence and by the completeness of X.

$\{y_n\}$ converges to $u \in X$ so its subsequence $\{A^2x_{2n+2}\},\{B^2x_{2n+1}\}, \{C^2x_{2n}\}, \{p^2x_{2n+2}\}, \{Q^2x_{2n+1}\}, \{R^2x_{2n}\}$ also converges to same point u. Since [A, P] is weak commuting so,

$\mathcal{M}(A^2P^2 x_{2n+2}, A^2P^2 x_{2n+2}, A^2P^2 x_{2n+2}, t) \geq \mathcal{M}(A^2 x_{2n+2}, P^2 x_{2n+1}, P^2 x_{2n+1}, t)$. On taking limit $n \to \infty$

$A^2P^2 x_{2n+1} = P^2 A^2 x_{2n+1} = P^2u$.

Now, we will show that $P^2u = u$.

First suppose that $P^2u \neq u$ then there exists $t > 0$ such that $\mathcal{M}(P^2u,u,u,t) < 1$.

Now,
$\mathcal{M}(A^2x_{2n+2}, B^2x_{2n+1}, C^2x_{2n}, t) \geq r(\min\{\mathcal{M}(P^3x_{2n+2}, Q^2x_{2n+1}, R^2x_{2n}, t), \mathcal{M}(P^3x_{2n+2}, A^2P^2x_{2n+2}, Q^2x_{2n}, t),$
$\mathcal{M}(Q^2x_{2n+1}, P^3x_{2n+1}, C^2x_{2n}, t), \mathcal{M}(R^2x_{2n}, C^2x_{2n}, Q^2x_{2n+2}, t),$
$\mathcal{M}(R^2x_{2n+2}, Q^2x_{2n+1}, R^2x_{2n}, t), \mathcal{M}(P^3x_{2n+2}, Q^2x_{2n+2}, C^2x_{2n+1}, t)\})$
$\Rightarrow \mathcal{M}(P^2u, u, u, t) \geq r(\min\{\mathcal{M}(P^2u, u, u, t), \mathcal{M}(P^2u, P^2u, u, t), \mathcal{M}(u, P^2u, u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t),$
$\mathcal{M}(P^2u, u, u, t)\})$
$\Rightarrow \mathcal{M}(P^2u, u, u, t) \geq r[\mathcal{M}(P^2u, u, u, t)] > \mathcal{M}(P^2u, u, u, t).$
Which is a contradiction. Therefore, $P^2u = u$.
Thus u is a fixed point of P. Similarly we can prove that u is also a fixed point of A.*i.e.* $A^2u = u$.
Now to prove that u is a fixed point of Q, suppose u is not a fixed point of Q then for any $t > 0$,
$\mathcal{M}(u, Q^2u, u, t) < 1$. Now,
$\mathcal{M}(A^2u, B^2Q^2x_{2n+1}, C^2x_{2n}, t) \geq r(\min\{\mathcal{M}(P^2u, Q^3x_{2n+1}, R^2x_{2n}, t), \mathcal{M}(P^2u, A^2u, Q^3x_{2n}, t),$
$\mathcal{M}(Q^3x_{2n+1}, P^2x_{2n+1}, R^2x_{2n}, t), \mathcal{M}(R^2x_{2n}, C^2x_{2n}, Q^3u, t),$
$\mathcal{M}(R^2u, Q^3x_{2n+1}, R^2x_{2n}, t), \mathcal{M}(P^2u, Q^3u, C^2x_{2n+1}, t)\})$
$\Rightarrow \mathcal{M}(u, Q^2u, u, t) \geq r(\min\{\mathcal{M}(u, Q^2u, u, t), \mathcal{M}(u, u, Q^2u, t), \mathcal{M}(Q^2u, u, u, t), \mathcal{M}(u, Q^2u, u, t),$
$\mathcal{M}(u, Q^2u, u, t), \mathcal{M}(u, Q^2u, u, t)\})$
$\Rightarrow \mathcal{M}(u, Q^2u, u, t) \geq r[\mathcal{M}(u, Q^2u, u, t)] > \mathcal{M}(u, Q^2u, u, t).$
Which is a contradiction. Therefore $Q^2u = u$.So,u is a fixed point of Q. i.e., $B^2u = u$.
Now we prove that u is a fixed point of R then for any $t > 0$, $\mathcal{M}(u, u, R^2u, t) < 1$. Now,
$\mathcal{M}(A^2u, B^2u, C^2x_{2n}, t) \geq r(\min\{\mathcal{M}(P^2u, Q^2u, R^3x_{2n}, t), \mathcal{M}(P^2u, A^2u, Q^2u, t), \mathcal{M}(Q^2u, P^2u, R^3x_{2n}, t),$
$\mathcal{M}(R^3x_{2n}, C^2R^2x_{2n}, Q^2u, t), \mathcal{M}(R^3u, Q^2u, R^3x_{2n}, t), \mathcal{M}(P^2u, Q^2u, C^2R^2u, t)\})$
$\Rightarrow \mathcal{M}(u, u, R^2u, t) \geq r(\min\{\mathcal{M}(u, u, R^2u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, R^2u, t), \mathcal{M}(R^2u, u, R^2u, t),$
$\mathcal{M}(R^2u, u, R^2u, t), \mathcal{M}(u, u, R^2u, t)\})$
$\Rightarrow \mathcal{M}(u, u, R^2u, t) \geq r[\mathcal{M}(u, u, R^2u, t)] > \mathcal{M}(u, u, R^2u, t).$
Which is a contradiction. Therefore $R^2u = u$.So,u is a fixed point of R. i.e., $C^2u = u$.
Similarly, we can show that s is a common fixed point of A, B, C, P, Q and R.

Uniqueness: Suppose another fixed point $u \neq w$.Then
$\mathcal{M}(A^2u, B^2w, C^2w, t) \geq r(\min\{\mathcal{M}(P^2u, Q^2w, R^2w, t), \mathcal{M}(P^2u, A^2u, Q^2u, t), \mathcal{M}(Q^2w, P^2w, C^2w, t),$
$\mathcal{M}(R^2w, C^2w, Q^2u, t), \mathcal{M}(R^2u, Q^2w, R^2w, t), \mathcal{M}(P^2u, Q^2u, C^2w, t)\})$
$\geq r(\min\{\mathcal{M}(u, w, w, t), \mathcal{M}(u, u, u, t), \mathcal{M}(w, w, w, t), \mathcal{M}(w, w, u, t),$
$\mathcal{M}(u, w, w, t), \mathcal{M}(u, u, w, t)\})$
$\Rightarrow \mathcal{M}(u, w, w, t) \geq r[\mathcal{M}(u, w, w, t)].$
Which is contradiction, therefore,$u = w$.
Hence A, B, C, P, Q and R have unique common fixed point.

References

[1] Chauhan, M.S. et.al., Common Fixed Point of Semi Compatible Maps in Fuzzy Metric Spaces, Kathmandu UniversityJournal of Science, Engineering and Technology, Vol. 6(1),(2010), 70-78.
[2] George, A. and Veeramani, P., On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64,(1994), 395-399.
[3] Grabiec. M., Fixed point in Fuzzy metric spaces, Fuzzy sets & systems, 27, (1988), 245 – 252.
[4] Jungck, G., Compatible mappings and common fixed points, Int. J. Math.& Math. Sci. 9,(1986), 771-779.
[5] Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, Indian J. Pure and Appl.Math.,29(3), (1998), 227-238.
[6] Kramosil, O. and Michalek, J., Fuzzy metric and statistical metric spaces, Kybernetika11, (1975), 326-334.
[7] Mishra, M.K. et.al., An application of Fixed Point Theorems in Fuzzy Metric Spaces,

International Journal ofAdvanced Engineering Sciences and Technologies Vol. 1(2), (2010), 123-129.

[8] Pant, R. P., Common fixed points of noncommuting mappings, J. Math. Anal. Appl., 188(2), (1994), 436-440.

[9] Sedghi. S and Shobe. N, Fixed point theorem is $\mathcal{M}$-fuzzy metric spaces with Property (E), Advances In Fuzzy Mathematics, 1(1),(2006), 55 – 65.

[10] Sessa, S., On a weak commutativity condition in fixed points consideration Publ. Inst. Math (Beogard), 46(2), (1986), 149-135.

[11] Singhi J. et.al, Fixed point theorem in fuzzy metric spaces, Int. math. forum**,** 5(30), (2010), 1473-1480.

[12] Vasuki, R., Common fixed points for R-weakly commuting maps in fuzzy Metric spaces, Indian J. Pure Appl.Math. 30,(1999), 419-423.

[13] Veerapandi. T, Jeyaraman. M, Paul Raj Josph. J, Some fixed point and coincident point theorem in generalized M-fuzzy metric space, Int. Journal of Math. Analysis, Vol. 3, 2009, no. 13 16, 627-635.

[14] Zadeh, L.A., Fuzzy sets, Inform and Control, 8, (1965), 338-353.