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POSITIVE AND ASSOCIATIVE IMPLICATIVE FILTERS OF RESIDUATED LATTICE WAJSBERG ALGEBRAS

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Abstract: In this paper, we introduce the notions of positive and associative implicative filters of residuated lattice Wajsberg algebra with illustrations. Also, we prove every positive and an associative implicative filter are an implicative filter. Moreover, we investigate some properties and obtain equivalent conditions of positive and associative implicative filters.

Keywords: Wajsberg algebra; Lattice Wajsberg algebra; Residuated lattice Wajsberg algebra; Implicative filter; Positive implicative filter; Associative implicative filter.

Mathematical Subject classification 2010: 03G10, 03G25

1.Introduction

Intelligent information processing is one important research direction in artificial intelligence. Information processing dealing with certain information is based on the classical logic. However, non-classical logics including logics behind fuzzy reasoning handle information with various facts uncertainty such as fuzziness, randomness, etc. Therefore, non-classical logic has become as a formal and useful tool for computer science to deal with uncertain information. In the field of many valued logic, lattice valued logic plays an important role for the following two aspects: One is that is extends chain type truth valued field of some well-known present logic to some relatively general lattice. The other is that incompletely comparable property of truth value characterized by general lattice can more efficiently reflect uncertainty of human beings thinking, judging and decision. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a

frame of residuated lattices [13]. Residuated lattices were introduced by Ward and Dilworth [9].Xu and Qin [10] introduced the notions of filter and implication filter in lattice implication algebras. Young Bae Jun, Yang Xu and Keyun Qin [11] proposed the concept of positive implicative and associative filters of lattice implication algebras. Mordchaj Wajsberg [8] introduced the concept of Wajsberg algebras in 1935. Font, Rodriguez and Torrens [3] defined lattice structure of Wajsberg algebras and introduced the notion of an implicative filter of lattice Wajsberg algebras. Basheer Ahamed and Ibrahim [1,2,4] introduced the notions of fuzzy and an anti-fuzzy implicative filter, positive and an associative implicative filters of lattice Wajsberg algebras and obtained some properties. Recently, the authors [5,6,7] introduced the notion of implicative filter, fuzzy implicative filter, anti-fuzzy implicative filter of residuated lattice Wajsberg algebras and discussed some of their properties.
In the present paper, we introduce the notions of positive and an associative implicative filters of residuated lattice Wajsberg algebra, and obtain some equivalent conditions. Also, we find a characterization of positive and an associative implicative filter. Further, we prove every positive and an associative implicative filter is an implicative filter of residuated lattice Wajsberg algebra.

2. Preliminaries

In this section, we recall some basic definitions and their properties which are helpful to develop our main results.

Definition 2.1 *[3]*. Let $(A,\to,*,1)$ be a algebra with binary operation "→"and aquasi complement "*"is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x,y,z\in A$,

(i) $1\to x=x$
(ii) $(x\to y)\to\big((y\to z)\to(x\to z)\big)=1$
(iii) $(x\to y)\to y=(y\to x)\to x$
(iv) $(x^*\to y^*)\to(y\to x)=1$.

Proposition 2.2 *[3]*. A Wajsberg algebra $(A,\to,*,1)$ satisfies the following axioms for all$x,y,z\in A$,

(i) $x\to x=1$
(ii) if $(x\to y)=(y\to x)=1$ $then$ $x=y$.
(iii) $x\to 1=1$
(iv) $(x\to(y\to x))=1$
(v) If $(x\to y)=(y\to z)=1$ $then$ $x\to z=1$
(vi) $(x\to y)\to\big((z\to x)\to(z\to y)\big)=1$
(vii) $x\to(y\to z)=y\to(x\to z)$
(viii) $x\to 0=x\to 1^*=x^*$
(ix) $(x^*)^*=x$
(x) $(x^*\to y^*)=y\to x$.

Definition 2.3 *[3]*. A Wajsberg algebra A is called a lattice Wajsberg algebra if it satisfies the following conditions for all $x,y\in A$,

(i) The partial ordering " $\leq$ " on a lattice Wajsberg algebra A such that $x\leq y$ if and only if $x\to y=1$
(ii) $(x\vee y)=(x\to y)\to y$
(iii) $(x\wedge y)=(x^*\to y^*)\to y^*)^*$.

Thus,$(A,\vee,\wedge,\to,*,0,1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4 *[3]*. A Wajsberg algebra $(A,\to,*,1)$ satisfies the following axioms for all $x,y,z\in A$,

(i) If $x\leq y$ then $x\to z\geq y\to z$ and $z\to x\leq z\to y$
(ii) $x\leq y\to z$ if and only if $y\leq x\to z$

(iii) $(x \vee y)^* == (x^* \wedge y^*)$
(iv) $(x \wedge y)^* = (x^* \vee y^*)$
(v) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
(vi) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
(vii) $(x \to y) \vee (y \to x) = 1$
(viii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$
(ix) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$
(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$
(xi) $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

Definition 2.5 *[9]* A residuated lattice is a algebra $(A, \vee, \wedge, \odot, \to, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following conditions,
(i) $(A, \vee, \wedge, 0, 1)$is a bounded lattice
(ii) $(A, \odot, 1)$is a commutative monoid
(iii) $x \odot y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in A$.

Proposition 2.6 *[9]* A residuated lattice$(A, \vee, \wedge, \odot, \to, 0, 1)$ satisfies the following axioms for all $x, y, z \in A$,
(i) $x \to (y \to z) = (x \odot y) \to z$
(ii) $x \odot 0 = 0$
(iii) $(x \to y) \odot x \leq y$
(iv) $x \leq y \to (x \odot y)$
(v) $x \to y \leq (x \odot z) \to (y \odot z)$
(vi) $(x \to y) \odot (y \to z) \leq x \to z$
(vii) $x \odot x^* = 0$
(viii) $x \to y^* = (x \odot y)^*$.

Definition 2.7 *[3]* Let $(A, \vee, \wedge, \to, *, 0, 1)$be a lattice Wajsberg algebra. If a binary operation "$\odot$"on A satisfies$x \odot y = (x \to y^*)^*$ for all$x, y \in A$.Then $(A, \vee, \wedge, \odot, \to, * 0, 1)$ is called a residuatedlattice Wajsberg algebra.

Definition 2.8 *[1]* Let$A$ be a Wajsberg algebra, a subset F of A is called an implicative filter of A,if it satisfies the following axioms for all $x, y \in A$,
(i) $1 \in F$
(ii) $x \in F$and$x \to y \in F$imply $y \in F$.

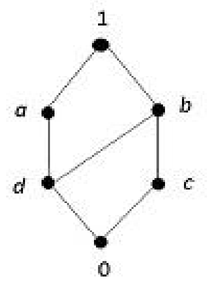
Proposition 2.9 *[1]* Let Fbe a non empty subset of A. Then F is a filter of A if and only if it satisfies $x \leq y \to z$ implies $z \in F$ for all $x, y, z \in A$.

3.Main results
In this section, we introduce the definitions of positive and an associative implicative filters of residuated lattice Wajsberg algebra and obtain some useful results with illustrations.

3.1 Properties of positive implicative filter
Definition 3.1.1. Let A be a residuated lattice Wajsberg algebra. A non-empty subset F of A is called a positive implicative filter of A if for all $x, y, z \in A$,
(i) $1 \in F$
(ii) $(y \odot x) \to z \in F$and$x \to y \in F$ imply $x \to z \in F$
(iii) $(x \to y) \to y \in F$and$x \in F$ imply $y \in F$.

Example 3.1.2 Let $A = \{0,\ a,\ b,\ c,\ d,\ 1\}$ be a set with Figure 3.1 as a partial ordering. Define binary operation"→" and quasi-complement "*"on A as in Tables 3.1and 3.2.Define ∨,∧ and ⊙ operations on $(A,\ \vee,\ \wedge,\ 0, 1)$ as follow
$(x \vee y) = (x \to y) \to y$
$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$; $x \odot y = (x \to y^*)^*$ forall $x, y \in A$.
Then, $(A,\ \vee,\ \wedge, \odot, \to, *\ 0, 1)$ is a residuated lattice Wajsberg algebra.

Figure 3.1



x	x^*
0	1
a	c
b	d
c	a
d	b
1	0

Table: 3.1
Complement

→	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	b	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table: 3.2
Implication

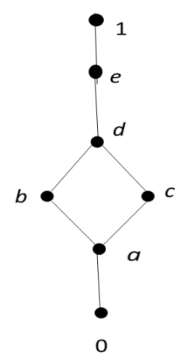
Itis easy to verify that $F_1 = \{\ a,\ b, c, d, 1\}$ is a positive implicative filter of residuated lattice Wajsberg algebra.
But, $F_2 = \{0, a, b, c, 1\}$is not a positive implicative filter of residuated lattice Wajsberg algebra, since $(b \odot a) \to 0 = d \notin\ F_2$.

Example 3.1.3. Let $A = \{0, a, b, c, d, e, 1\}$ be a set with Figure 3.2 as a partial ordering. Define binary operation"→" anda quasi-complement "*"on A as in Tables 3.3 and 3.4.
Define ∨,∧ and ⊙ operations on $(A,\ \vee,\ \wedge,\ 0, 1)$ as follow

$$(x \vee y) = (x \to y) \to y$$
$$(x \wedge y) = ((x^* \to y^*) \to y^*)^*; \ x \odot y = (x \to y^*)^* \text{forall } x, y \in A.$$

Then $(A,\ \vee,\ \wedge,\ \odot, \to, *\ 0, 1)$ is a residuated lattice Wajsberg algebra.

Figure 3.2



x	x^*
0	1
a	d
b	c
c	b
d	a
e	a
1	0

Table:3.3
Complement

→	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	d	1	d	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	d	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

Table: 3.4
Implication

It is easy to verify that $F_3 = \{\ c, d, e, 1\}$ is a positive implicative filter of residuated lattice Wajsberg algebra. But, $F_4 = \{a, b, c, 1\}$is not a positive implicative filter of residuated lattice Wajsberg algebra. Since $(1 \odot b) \to a = d \notin\ F_4$.

Proposition 3.1.4 Every positive implicative filter of a residuated lattice Wajsberg algebra A is an implicative filter.

Proof. Let F be a positive implicative filter of A and let $x \to y \in F$and $x \in F$, then $(1 \odot x) \to y \in F$and $1 \to x \in F$.
From (ii) of definition 3.1.1, we have $1 \to y \in F$. That is, $y \in F$, and$1 \in F$.
Hence,F is an implicative filter. ■

Proposition 3.1.5 Let F be a implicative filter of A. Then F is a positive implicative filter of A if and only if for all $x, y \in A (x \to y) \to x \in F$ implies $x \in F$.
Proof. Let F be a positive implicative filter of A and let $(x \to y) \to x \in F$for all $x, y \in A$. Then, we have $1 \to \big((x \to y) \to x\big) = (x \to y) \to x \in F$,and $1 \in F$ and from (iii) of definition 3.1.1. we get, $x \in F$ .Thus, $(x \to y) \to x \in F$ implies $x \in F$.
Conversely, suppose that F satisfies $(x \to y) \to x \in F$ implies $x \in F$.Let $(x \to y) \to y \in F$ and $x \in F$ for all $x, y \in A$. Then, $y \in F$ by (iii) of definition 3.1.1.Hence F is a positive implicative filter of A. ■

Proposition 3.1.6 Let$F$ be a subset of A.If F is a positive implicative filter of A,then it is an implicative filter of A.
Proof. Let$x \to (y \to z) \in F$and $x \to y \in F$ and $x \to y \in F$ for all $x, y, z \in A$.
Then $x \to (y \to z) = y \to (x \to z)$[From (vii) of proposition 2.2]
$\leq (x \to y) \to (x \to (x \to z))$
[From (vii) of proposition 2.2 and (ii) of definition 2.1]
Since, F is an implicative filter of A. From the proposition 2.9.we have,$x \to (x \to z) \in F$,
and$\big((x \to z) \to z\big) \to (x \to z) = (x \to \Big(\big((x \to z) \to z\big) \to z\Big)$
[From (iii) of definition 2.1]
$= x \to (\big(z \to (x \to z)\big) \to (x \to z))$
[From (vii) of proposition 2.2]
$= x \to (\big(x \to (z \to z)\big) \to (x \to z))$
[From (iii) of definition 2.1]
$= x \to ((x \to 1) \to (x \to z))$
[From (i) of proposition 2.2]
$= x \to (1 \to (x \to z))$ [From (i) of definition 2.1]
$= x \to (x \to z) \in F$.
Thus, $(x \to z) \in F$. ■ [From the proposition 3.1.3]

Proposition 3.1.7 Let F be a implicative filter of A. Then F is a positive implicative filter of A if and only if $(x \odot x) \to y \in F$ implies $x \to y \in F$ for all $x, y \in A$.
Proof. Let$F$ be a positive implicative filter of A.
If $(x \odot x) \to y \in F$, then $x \to y \in F$ and $x \to x \in F$ for all $x, y \in A$.
[From (ii) of definition 3.1.1]
Hence $(x \odot x) \to y \in F$implies$x \to y \in F$.
Conversely, let F be a implicative filter and $(x \odot x) \to y \in F$ implies $x \to y \in F$ for all $x, y \in A$.We have $1 \in F$.Thus, (i) of definition 3.1.1 hold.
If $(y \odot x) \to z \in F$ and $x \to y \in F$, then $\big((y \odot x) \to z\big) \odot (x \to y) \in F$.
Since,$\big((y \odot x) \to z\big) \odot (x \to y) = \big(y \to (x \to z)\big) \odot (x \to y)$ [From (i) of proposition 2.6]
$\leq x \to (x \to z)$ [From (vi) of proposition 2.2]
$= (x \odot x) \to z$ [From (i) of proposition 2.6]
We have, $(x \odot x) \to z \in F$implies $x \to z \in F$. Thus, (ii) of definition 3.1.1 hold.
Therefore,F is a positive implicative filter of A. ■ [From the definition 3.1.1]

3.2. Properties of associative implicativefilter
Definition 3.2.1.Let A be a residuated lattice Wajsberg algebra. A non-empty subset F of A is called a associative implicative filter of A if for all $x, y, z \in A$,
(i) $1 \in F$
(ii) $x \odot (y \odot z) \in Fandx \odot y \in F$ imply $z \in F$

(iii) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $z \in F$.

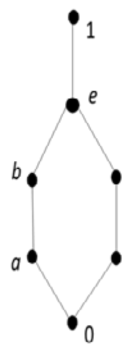
Note: In associative filter the (ii) of definition 3.2.1 always exists in a residuated lattice Wajsberg algebra. So, it is enough verifying (i) and (iii) of definition 3.2.1 in the illustrations and properties.

Example 3.2.2 Let $A = \{0, a, b, c, d, e, 1\}$ be a set with Figure 3.3 as a partial ordering. Define a binary operation"→" and a quasi-complement "*" on A as in Tables 3.5and 3.6. Define ∨, ∧ and ⊙ operations on $(A, \vee, \wedge, 0, 1)$ as follow

$$(x \vee y) = (x \to y) \to y$$
$$(x \wedge y) = ((x^* \to y^*) \to y^*)^*; \; x \odot y = (x \to y^*)^* \text{ for all } x, y \in A.$$

Then, $(A, \vee, \wedge, \odot, \to, *\, 0, 1)$ is a residuated lattice Wajsberg algebra.

Figure 3.3



x	x^*
0	1
a	d
b	c
c	b
d	a
e	0
1	0

Table: 3.5 Complement

→	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	a	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	e	e	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

Table: 3.6 Implication

It is easy to verify that $F_5 = \{a, b, 1\}$ is anassociative implicative filter of residuated lattice Wajsberg algebra.

But, $F_6 = \{b, c, 1\}$is not an associative implicative filter of residuated lattice Wajsberg algebra, since $(b \to (c \to 1)) = 1 \in F_6$but $b \to c = d \notin F_6$.

Proposition 3.2.3 Every associative implicative filter of a residuated lattice Wajsberg algebra A is an implicative filter.

Proof. Let F be an associative implicative filter of A and let $x \to y \in F$ and $x \in F$ for all $x, y \in A$.Then, $1 \to x \in F$ and$1 \to (x \to y) = x \to y \in F$and $x \in F$. From (iii) of definition3.2.1, we have$y \in F$.Hence,F is an implicative filter. ■

Proposition 3.2.4 Every associative implicative filter with respect to x contains x itself.

Proof. If $x = 0$ and$x = 1$ then it is trivial. If $x \neq 0$.Let F be an associative implicative filter of A with respect to x, we have $x \to (1 \to x) = x \to x = 1 \in F$ and $x \to 1 \in F$.From (iii) of definition 3.2.1, we have $x \in F$. ■

Proposition 3.2.5 Let F be an implicative filter of A. Then F is an associative implicative filter if and only if$x \to (y \to z) \in F$ implies $(x \to y) \to z \in F$ for all$x, y, z \in A$.(3.2.1)

Proof. If an implicative filter $F$of A satisfies $x \to (y \to z) \in F$ implies$(x \to y) \to z \in F$ for all $x, y, z \in A$.Then, we get F is an associative implicative filter.

Conversely, Let$F$ be an associative implicative filter of $A$and let $x \to (y \to z) \in F$for all$x, y, z \in A$.Then, we have

$$x \to \big((y \to z) \to (x \to y) \to z\big) = (y \to z) \to (x \to ((x \to y) \to z)$$

[From(vii) of proposition 2.2]

$$= (y \to z) \to ((x \to y) \to (x \to z))$$

[From (vii) of proposition 2.2]

$$= 1 \in F.$$

[From (vii) of proposition 2.2 and(ii) of definition 2.1]

Which implies from (iii) of definition 3.2.1, we have $(x \to y) \to z \in F$.■

Proposition 3.2.6 Let F be an implicative filter of A. Then F is an associative implicative filter if and only if it satisfies $x \to (x \to y) \in F$ implies $y \in F$ for all $x, y \in A$.(3.2.2)

Proof. Let F be an implicative filter of A. It is enough to show that (3.2.1) and (3.2.2) are equivalent. Putting $x = y$ in (3.2.1) and using (i) of definition 2.1 and (i), (ii) of proposition 2.2, we get (3.2.2.).If (3.2.2) holds and let $x \to (y \to z) \in F$ for all $x, y, z \in A$. Using (i) and (ii) of definition 2.1, (i) and (vii) of proposition 2.2 and (i) ,(ii) of proposition 2.4, we have $\big(x \to (y \to z)\big) \to (x \to (x \to \big((x \to y) \to z)\big))$

$= 1 \to \big((x \to (y \to z)\big) \to (x \to (x \to \big((x \to y) \to z)\big))$ [From (i) of definition 2.1] $= \Big((y \to z) \to \Big(x \to \big((x \to y) \to z\big)\Big)\Big) \to ((x \to (y \to z)) \to (x \to \Big(x \to \big((x \to y) \to z\big)\Big)$

$= \big(x \to (y \to z)\big) \to \Big(\Big((y \to z) \to x \to \big((x \to y) \to z\big)\Big)\Big) \to (x \to \Big(x \to \big((x \to y) \to z\big)\Big)$

$\geq \big(x \to (y \to z)\big) \to \big(x \to (y \to z)\big) = 1$[From (i) of proposition 2.2]

This implies that $\big(x \to (y \to z)\big) \to (x \to (x \to \big((x \to y) \to z\big))) = 1 \in F$.

Since, $x \to (y \to z) \in F$ and F is an implicative filter, it follows that

$x \to (x \to \big((x \to y) \to z\big)) \in F$.

By using (3.2.2), we get $(x \to y) \to z \in F$ for all $x, y \in A$. ■

4. Conclusion

We have proposed the notions of a positive and an associative implicative filter in residuated lattice Wajsberg algebra, discussed an equivalent condition that every implicative filter is a positive implicative filter. We have shown that every associative implicative filter is an implicative filter. Also, we have given some illustrations for positive and associative implicative filters. Finally, we have concluded that an equivalent condition for an implicative filter is an associative implicative filter. In future, we can extend these concepts in fuzzy environment.

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