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HIGHER ORDER DERIVATIVE RUNGE KUTTA METHOD FOR SOLVING DELAY DIFFERENTIAL EQUATIONS

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Abstract. This article proposes Runge-Kutta method with higher order derivative approximations for solving delay differential equations. This method includes new terms with higher order derivatives of $f$in the Runge–Kutta$k_i$terms (i >1) to get better accuracy without increasing number of evaluations of f, but with the addition of approximations of f'.The delay term is approximated by using Lagrange interpolation. Stability polynomial of the proposed method is determined and obtained its corresponding stability region. Numerical examples of linear and nonlinear delay differential equations are given to illustrate the effectiveness of the newly proposed method. The numerical results are compared with classical Runge-Kutta method.

*Keywords:*ddes, higher order derivatives, rk, stability polynomial, stability region.

2010 Mathematics Subject Classification: 34K, 65L, 68N

1. Introduction

Delay Differential Equations (DDEs) are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. DDEs are classified into constant delay, time dependent delay and state dependent delay. DDEs emerge in chemical kinetics [7], control systems [9], population dynamics [13], traffic models [4] and in several areas. DDEs are discussed by Bellman Cooke [2], Norkin [14], Hale [11], Driver [5]. Various numerical methods have been proposed by the researchers for the solution of DDEs. Some well-known methods for DDEs are Runge-Kutta Method [10], Adomain Decomposition Method [8], Chebyshev Method [6], Direct Lyapunov Method [1], Predictor – Corrector Block Method [12], Variational Iteration Method [15].

The aim of this work is to propose a third order Runge-Kutta(RK) method with two stages to solve DDEs. The delay term is approximated using Lagrange interpolation. This paper has been organized as follows: In Section 2, the higher order derivative method for solving delay differential equations has been discussed. In Section 3, the stability analysis of this method has been derived. In Section 4, numerical examples of linear and non-linear DDEs have been provided to demonstrate the efficiency of the proposed method.

2. Higher Order Derivative Runge-KuttaMethod of Order 3

David Goken and Johnson [3] discussedRunge-Kutta method with higher order derivative approximations which requires less evaluations of f to get higher order accuracy for solving $y'(t) = f\big(t, y(t)\big)$. This method permits to introduce new Runge-Kutta parameters which increase the order of accuracy of the solution.

A third order Runge-Kutta method with two functional evaluations is given by

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2, \tag{1}$$

where

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\big(t_n + c_2 h, y_n + a_{21} k_1 + h a_{22}(f_y(t_n, y_n) k_1 + h f_t(t_n, y_n))\big)$$

The Taylor series expansion of the above gives the following system of equations:

$$b_1 + b_2 = 1,$$

$$b_2 c_2 = \frac{1}{2},$$

$$b_2 a_{21} = \frac{1}{2},$$

$$b_2 a_{22} = \frac{1}{6}.$$

Solving this system of equations we get,

$$b_1 = \frac{1}{4}, b_2 = \frac{3}{4}, c_2 = a_{21} = \frac{2}{3}, a_{22} = \frac{2}{9}.$$

Using these values in equation (1), we get the higher order derivative RK method of order 3

$$y_{n+1} = y_n + \frac{1}{4}k_1 + \frac{3}{4}k_2$$

where

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{2}{3}h, y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 f'\right) \qquad (2)$$

Here $hf' = f_y(t_n, y_n)k_1 + hf_t(t_n, y_n)$.

To test the convergency of the proposed method, consider the following test problem

$$y'(t) = \lambda y(t), \quad y(0) = 1 \qquad (3)$$

$$k_1 = (h\lambda)y_n$$

$$k_2 = (h\lambda + a_{21}(h\lambda)^2 + a_{22}(h\lambda)^3)y_n$$

$$y_{n+1} = (1 + h\lambda(b_1 + b_2) + a_{21}b_2(h\lambda)^2 + a_{22}b_2(h\lambda)^3)y_n \qquad (4)$$

Applying the initial condition $y(0) = 1$, the solution is

$$y_n = \lim_{h\to 0}(1 + h\lambda(b_1 + b_2) + a_{21}b_2(h\lambda)^2 + a_{22}b_2(h\lambda)^3)^n \qquad (5)$$

To examine the convergence of this at t_n, keep t_n fixed as $h \to 0$. Now,

$$y_n = (1 + h\lambda(b_1 + b_2) + a_{21}b_2(h\lambda)^2 + a_{22}b_2(h\lambda)^3)^{\frac{t_n}{h}}$$

Taking log on both sides,

$$ln\, y_n = \frac{t_n}{h} ln(1 + h\lambda(b_1 + b_2) + a_{21}b_2(h\lambda)^2 + a_{22}b_2(h\lambda)^3)$$

ApplyingL'Hospital rule, we get

$$\lim_{h\to 0}\frac{1}{h} ln(1 + h\lambda(b_1 + b_2) + a_{21}b_2(h\lambda)^2 + a_{22}b_2(h\lambda)^3)$$

$$= \lim_{h\to 0}\frac{\lambda(b_1+b_2)+2a_{21}b_2h\lambda^2+3a_{22}b_2h^2\lambda^3}{1+h\lambda(b_1+b_2)+a_{21}b_2(h\lambda)^2+a_{22}b_2(h\lambda)^3}$$

$$= \lambda(b_1 + b_2)$$

Hence,

$$\lim_{h\to 0} ln\, y_n = \lambda(b_1 + b_2)t_n$$

and thus,

$$\lim_{h\to 0} y_n = e^{\lambda(b_1+b_2)t_n} \quad (6)$$

Since $b_1 + b_2 = 1$, the method is consistent and convergent to $O(h^3)$, if from (5) the order condition $a_{22}b_2 = \frac{1}{6}$ is satisfied.

In this paper we adapt the above discussed method to solve the DDEs.

Consider the first order DDEs of the form

$$y'(t) = f\big(t,\ y(t),\ y(t-\tau)\big), \quad t > t_0, \quad (7)$$

$$y(t) = \emptyset(t), \quad t \le t_0$$

where the delay τ is a positive constant and $\emptyset(t)$ is the initial function.

Then the higher order derivative RK method of order 3 is given by

$$y_{n+1} = y_n + \frac{1}{4}k_1 + \frac{3}{4}k_2$$

where

$$k_1 = hf\big(t_n, y_n, y(t_n - \tau)\big)$$

$$k_2 = hf\left(t_n + \frac{2}{3}h, y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 f', y(t_n + \frac{2}{3}h - \tau)\right) \quad (8)$$

Here Lagrange interpolation is used to approximate the delay term.

3. Stability Analysis of Proposed Method

The stability of the proposed method has been considered with respect to the following linear test equation

$$y'(t) = \lambda y(t) + \mu y(t-\tau), \quad t \geq 0 \tag{9}$$

$$y(t) = g(t), \quad -\tau \leq t \leq 0$$

whereλ and μ are complex numbers, τ is a positive constant delay and g is a specified initial function. Let the numerical solution has been obtained up to the point t_n, with uniform step size $h = \frac{t}{m}$, where m is a positive integer.

By approximating the delay term using Lagrange interpolation, we get

$$y(t_n + c_i h - \tau) = y(t_{n-m} + c_i h) = \sum_{l=r1}^{s1} L_l(c_i) y_{n-m+l} \tag{10}$$

wherey_{n-m+l} is the calculated value of $y(t_{n-m+l})$ and$L_l(c_i) = \prod_{j1=r1}^{s1} \frac{c_i - j_1}{l - j_1}, j_1 \neq l$ and $r_1, s_1 > 0$. (11)

When 2-stage higher order derivative RK method of order 3 is adapted to DDE (1) with delay $\tau = 1$, the following equations are obtained:

$$k_1 = hf(t_n, y_n, \sum_{l=r1}^{s1} L_l(c_1) y_{n-m+l})$$

$$k_2 = hf\left(t_n + \frac{2}{3}h, y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 f', \sum_{l=r1}^{s1} L_l(c_2) y_{n-m+l}\right) \tag{12}$$

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2$$

Define $\mathrm{u} = (1, \ldots, 1)^{\mathrm{T}}$,$\mathrm{k} = (k_1, k_2, \ldots, k_q)^{\mathrm{T}}$, $\mathrm{b} = (b_1, b_2, \ldots, b_q)^{\mathrm{T}}$ and $L_l(c) = (L_l(c_1), \ldots, L_l(c_q))^T$.

For $n \geq m$, considering f as in (9), equation (12) can be written as

$$\mathrm{k} = \lambda(y_n \mathrm{u} + hAk) + \mu(\sum \mathrm{L_l(c)} y_{n-m+l}) \tag{13}$$

$$y_{n+1} = y_n + h\mathrm{b}^{\mathrm{T}}\mathrm{k} \tag{14}$$

From equation (14),

$$\mathrm{k} = \lambda y_n \mathrm{u}[I - \lambda hA]^{-1} + \mu[I - \lambda hA]^{-1} \sum \mathrm{L_l(c)} y_{n-m+l}$$

$$h\mathrm{k} = \alpha y_n \mathrm{u}\eta + \beta\eta \sum \mathrm{L_l(c)} y_{n-m+l} \tag{15}$$

where$\alpha = \lambda h, \beta = \mu h, \eta = [I - \lambda h A]^{-1}$ and I is the identity matrix.

Substituting (15) in (14),

$$y_{n+1} = y_n + \alpha \mathrm{b}^{\mathrm{T}} \eta y_n \mathrm{u} + \beta \mathrm{b}^{\mathrm{T}} \eta \sum \mathrm{L_l(c)} y_{n-m+l} \qquad (16)$$

Taking $\mathrm{Y_n} = (y_n, h\mathrm{k})^T$, equations (15) and (16) can be written as the recurrence

$Y_{n+1} = XY_n + ZY_{n-m+l}$where

$$\mathbf{X} = \begin{bmatrix} 1+\alpha \mathbf{b}^{\mathrm{T}} \eta \mathbf{u} & 0,\dots,0 \\ & 0 \\ & . \\ \alpha \eta \mathbf{u} & . \\ & . \\ & 0 \end{bmatrix} \text{ and } \mathbf{Z} = \begin{bmatrix} \beta \mathbf{b}^{\mathrm{T}} \eta \sum \mathbf{L_l(c)} & 0 \\ & 0 \\ & . \\ \beta \eta \sum \mathbf{L_l(c)} & . \\ & . \\ & 0 \end{bmatrix}$$

By putting $n - m + l = 0$, the stability polynomial will be in the standard form. The recurrence is stable if the zero ζ_i of the stability polynomial

$$S(\alpha, \beta, \zeta) = \det \left[\zeta^{n+1} I - \zeta^n X - \sum_{l=r_1}^{s_1} \zeta^{1+l} Z_l \right]$$

satisfies the root condition $|\zeta_i| \le 1$.

To obtain the stability region of the method, we used three points interpolation to evaluate $y(t_n + c_i h - 1)$. Then the stability polynomial for the method is,

$$S(\alpha, \beta : \zeta) = \zeta^{n+1} - (1 + \alpha \mathrm{b}^T \eta \mathrm{u}) \zeta^n - \beta \mathrm{b}^T \eta (L_{-1}(c) + L_0(c)\zeta + L_1(c)\zeta^2)$$

The stability polynomial of proposed method is given by

$$S_p(\alpha, \beta, \xi) = \xi^4 - \left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6}\right)\xi^3 - \left(\frac{5}{12}\beta\right)\xi^2 - \left(\frac{2}{3}\beta + \frac{1}{2}\alpha\beta\right)\xi + \left(\frac{1}{3}\beta\right)$$

The stability region of proposed method using Lagrange interpolation is shown in Figure 1.

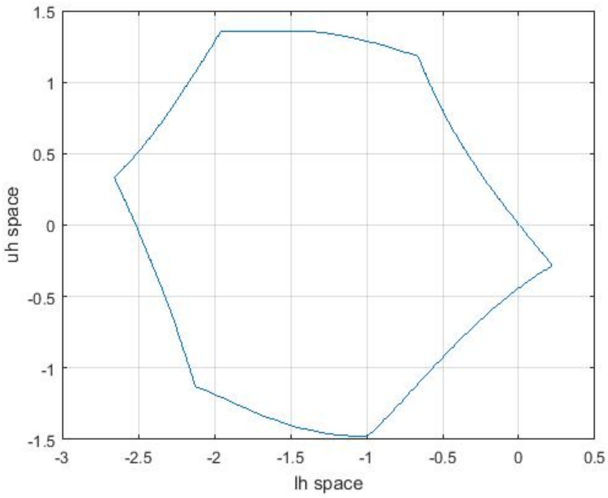

Figure 1: Stability Region of Proposed Method

4. Numerical Examples

Example 4.1

Consider the linear DDE of the form

$$y'(t) = 5y(t) + y(t-1),$$

$$y(t) = 5, \text{ for } t \leq 0$$

with exact solution $y(t) = 6e^{5t} - 1$.

Example 4.2

Consider the non-linear DDE of the form

$$y'(t) = y^2(t) + y(t-1) - t^4 + t + 1,$$

$$y(t) = t, \text{for } t \leq 0$$

with exact solution $y(t) = t^2$.

The numerical results by proposed method of order 3 with 2 stages have been compared with the results by the classical third order Explicit RungeKutta (ERK3) method with three stages. The absolute error results of Examples 4.1 and 4.2 are given in Table 4.1.

Table 4.1: Absolute Error Results

t	Absolute Error in Example 4.1		Absolute Error in Example 4.2	
	Proposed Method	ERK3	Proposed Method	ERK3
0.1	2.56e-08	2.56e-08	1.16e-08	1.95e-09
0.2	8.46e-08	8.46e-08	9.12e-08	7.24e-09
0.3	2.09e-07	2.09e-07	3.07e-07	1.58e-08
0.4	4.59e-07	4.59e-07	7.34e-07	2.78e-08
0.5	9.47e-07	9.47e-07	1.46e-07	4.31e-08
0.6	1.87e-06	1.87e-06	2.59e-06	6.17e-08
0.7	3.60e-06	3.60e-06	4.30e-06	8.36e-08
0.8	6.79e-06	6.79e-06	6.79e-06	1.09e-07
0.9	1.26e-05	1.26e-05	1.04e-06	1.37e-07
1	2.55e-05	2.55e-05	7.54e-06	8.50e-06

5. Conclusion

In this paper, higher order derivative Runge-Kutta method of order three with two stages has been proposed to obtain the numerical solution of DDEs. Stability polynomial of the method and its stability region is obtained. The effectiveness of the proposed method has been illustrated by considering numerical examples of linear and nonlinear DDEs. The numerical results by proposed method have been compared with the classical Runge-Kuttamethod.From the results, it is evident that the third order accuracy of results is obtained in two stages. Thus it is concluded that the proposed method is suitable for solving linear and nonlinear DDEs.

References

[1] Alexander V. Prasolov, A direct Lyapunov method for delay differential equations, International Journal of Pure and Applied Mathematics, 105 (2015) 823-834.

[2] Bellman R andCooke K L, Differential Difference Equations, Academic Press, New York, 1963.

[3] David Goken and Olin Johnson, Runge-Kutta with higher order derivative approximations, Applied Numerical Mathematics, 34 (2000) 207-218.

[4] Davis C L, Modification of the optimal velocity traffic model to include delay due to driver reaction time, Physica, 319 (2002) 557-567.
[5] Driver R D, Ordinary and Delay Differential Equations, Springer, New York, 1977.
[6] El Safty Aand Hussien M A, Chebyshev solution for stiff delay differential equations, International Journal of Computer Mathematics, 68 (1998) 323-335.
[7] Epstein Iand Luo Y, Differential delay equations in chemical kinetics: non-linear models; the cross shaped phase diagram and the Oregonator, Journal of Chemical Physics, 95 (1991) 24-254.
[8] Evans D Jand Raslan K R, The Adomian decomposition method for solving delay differential equation, International Journal of Computer Mathematics, (2004) 1-6.
[9] Fridman E, Fridman L andShustin E, Steady models in realy control systems with a time delay and periodic disturbance, Journal of Dynamical Systems and Measurement Control, 122 (2000) 732-737.
[10] Fudziah Ismail, RaedAli Al Khasawneh, Aung San Lwin and Mohamed Suleiman, Numerical treatment of delay differential equations by Runge-Kutta method using Hermite interpolation, Mathematika, 18 (2002) 79-90.
[11] Hale J K, Theory of Functional Differential Equations, Springer, New York, 1977.
[12] Ishak F, Suleiman M B andOmar Z, Two Point Predictor-Corrector Block methods for solving delay differential equations, Mathematika, 24 (2008) 131-140.
[13] Kuang Y, Delay Differential Equations with Applications in Population Biology, Academic Press, New York, 1993.
[14] Norkin S B andElsgolts L E, Introduction to the Theory and Application of Differential Equations with Deviating Arguments, Academic Press, New York, 1973.
[15] Syed Tauseef, Mohyud Din andAhmetYildirim, Variational Iteration Method for delay differential equations using He’s Polynomial, Z. Naturforsch, 65 (2010) 1045-1048.