Advances in Mathematics: Scientific Journal 8 (2019), no.3, 292–299
(Special issue on ICRAPAM)
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

POSITIVE IMPLICATIVE AND ASSOCIATIVE *WI*-IDEALS OF LATTICE WAJSBERG ALGEBRAS

A. Ibrahim[1] and C. Shajitha Begum[2]

[1]*Assistant Professor, PG and Research Department of Mathematics, H. H. The Rajah's College, Pudukkotai, Affiliated to Bharathidasan University, Tiruchirappalli, Tamil Nadu, India. Email: ibrahimaadhil@yahoo.com; dribrahimaadhil@gmail.com*

[2] *Research Scholar, PG and Research Department of Mathematics, H. H. The Rajah's College, Pudukkotai, Affiliated to Bharathidasan University, Tiruchirappalli, Tamilnadu, India. Email: nousheen2407@gmail.com.*

Abstract: In this paper, we introduce the notions of positive implicative *WI*-ideal and an associative *WI*-ideal of lattice Wajsberg algebras and investigate some of their properties. Also, we prove that every positive implicative *WI*-ideal is an implicative *WI*-ideal, and hence a *WI*-ideal, and that every associative *WI*-ideal is a *WI*-ideal.

Keywords: Wajsberg algebra; Lattice Wajsberg algebra; *WI*-ideal; Implicative *WI*-ideal; Positive implicative *WI*-ideal; Associative *WI*-ideal.

Mathematical Subject classification 2010: 03G10, 03G20, 03G25, 06B10.

1. Introduction

Wajsberg algebras are formulated in terms of the operations "implication" and "quasi complement" Wajsberg algebra concept was first proposed by Mordchaj Wajsberg [13] in 1935, and analyzed by Font, Rodriguez, and Torrens [1] in 1984, but they were also considered earlier by Komari [11,12] under the name of *CN* algebras, they were the model of $\aleph_0$-valued Łukasiewicz logic too. Font, Rodriguez and Torrens [1] introduced a lattice structure of Wajsberg algebra. The authors [2] introduced the notion of *WI*-ideal of lattice Wajsberg algebra and discussed some related properties. Further, the authors [3,4,5,6,7,8,9,10] introduced the notions of fuzzy *WI*-ideal, normal fuzzy *WI*-ideal, intuitionistic fuzzy *WI*-ideal, annihilator, implicative *WI*-ideal, fuzzy implicative *WI*-ideal, anti fuzzy *WI*-ideal, intuitionistic anti fuzzy *WI*-ideal of lattice Wajsberg algebra and also investigated their properties with suitable illustrations.

In this paper, we introduce the notions of positive implicative *WI*-ideal and associative *WI*-ideal of lattice Wajsberg algebras. Further, we investigate some of their properties. Also, we prove that every positive implicative *WI*-ideal is an implicative *WI*-ideal, and hence a *WI*-ideal, and that every associative *WI*-ideal is a *WI*-ideal. Moreover, we provide the condition equivalent for both positive implicative *WI*-ideal and associative *WI*-ideal.

2. Preliminaries

In this section, we recall some basic definitions and their properties that are useful to develop our main results.

Definition 2.1 *[1]*. Let (A, $\rightarrow$, *****, 1) be an algebra with binary operation '$\rightarrow$'and a quasi complement '*' is said to be Wajsberg algebra if it satisfies the following axioms for all $x, y, z \in A$

(i) $1 \rightarrow x = x$;

(ii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;

(iii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$.

Proposition 2.2 *[1]*.A Wajsberg algebra (A, $\rightarrow$, *****, 1) satisfies the following axioms for all $x, y, z \in A$,

(i) $x \rightarrow x = 1$;

(ii) If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$;

(iii) If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$;

(iv) $(x \rightarrow (y \rightarrow x)) = 1$;

(v) $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$;

(vi) $x \rightarrow 1 = 1$;

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$;

(ix) $(x^*)^* = x$;

(x) $(x^* \rightarrow y^*) = y \rightarrow x$.

Definition 2.3 *[1]*. A Wajsberg algebra (A, $\rightarrow$, *****, 1) is called a lattice Wajsberg algebra if it satisfies the following conditions for all $x, y \in A$,

(i) The partial ordering '$\leq$' on a lattice Wajsberg algebra *A*, such that $x \leq y$ if and only if $x \rightarrow y = 1$;

(ii) $(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$;

(iii) $(x \vee y) = (x \rightarrow y) \rightarrow y$.

Note. From Definition 2.3 an algebra (A, $\vee$, $\wedge$, *****, 0, 1) is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4 *[1]*. A lattice Wajsberg algebra (A, $\vee$, $\wedge$, *****, 0, 1) satisfies the following axioms for all $x, y, z \in A$,

(i) If $x \leq y$ then $x \rightarrow z \geq y \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$;

(ii) $x \leq y \rightarrow z$ if and only if $y \leq x \rightarrow z$;

(iii) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$;

(iv) $(x \vee y)^* = (x^* \wedge y^*)$;

(v) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(vi) $x \to (y \vee z) = (x \to y) \vee (x \to z)$;
(vii) $(x \to y) \vee (y \to x) = 1$;
(viii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$;
(ix) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$;
(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$;
(xi) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.5 *[1]*. Let *A* be a lattice. An ideal *I* of *A* is a non empty subset of *A* is said to be a lattice ideal if it satisfies the following axioms,
(i) $x \in I$, $y \in A$ and $y \leq x$ imply $y \in I$ for all $x, y \in I$;
(ii) $x, y \in I$ implies $x \vee y \in I$ for all $x, y \in I$.

Definition 2.6 *[2]*. The lattice Wajsberg algebra *A* is said to be lattice *H*-Wajsberg algebra, if $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in A$. In a lattice *H*-Wajsberg algebra *A* the following conditions hold,
(i) $x \to (y \to z) = (x \to y) \to (x \to z)$;
(ii) $x \to (x \to y) = (x \to y)$.

Definition 2.7 *[2]*. Let *A* be a lattice Wajsberg algebra. Let *I* be a nonempty subset of *A*. Then, *I* is said to be *WI*-ideal of lattice Wajsberg algebra *A* satisfies,
(i) $0 \in I$;
(ii) $(x \to y)^* \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in A$.

Definition 2.8 *[6]***.** Let *I* be a non-empty subset of lattice Wajsberg algebra *A*. Then, *I* is called an implicative *WI*-ideal of *A*, if it satisfies the following conditions,
(i) $0 \in I$;
(ii) $((x \to y)^* \to z^*) \in I$ and $(y \to z)^* \in I$ imply $(x \to z)^* \in I$ for all $x, y, z \in A$.

3. Main Results

3.1. Positive Implicative *WI*-ideals of Lattice Wajsberg algebras

Definition 3.1.1. A non-empty subset *I* of a lattice Wajsberg algebra *A* is called a positive implicative *WI*-ideal of *A* if it satisfies the following,
(i) $0 \in I$;
(ii) $((y \to (z \to y)^*)^* \to x)^* \in I$ and $x \in I$ imply $y \in I$ for all $x, y, z \in A$.

Example 3.1.2. Let $A = \{0, p, q, r, s, t, 1\}$ be a partial ordering set as given in figure 3.1. Define a binary operation '$\to$' and a quasi complement '$*$' on *A* as in tables 3.1 and 3.2.

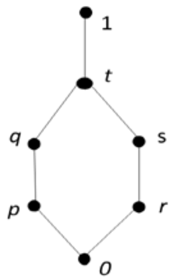

Figure 3.1
Lattice Diagram

x	x^*
0	1
p	s
q	s
r	q
s	q
t	0
1	0

Table 3.1
Complement

$\to$	0	p	q	r	s	t	1
0	1	1	1	1	1	1	1
p	s	1	1	s	s	1	1
q	s	t	1	s	s	1	1
r	q	q	q	1	1	1	1
s	q	q	q	t	1	1	1
t	0	q	q	s	s	1	1
1	0	p	q	r	s	t	1

Table 3.2
Implication

Define ' $\wedge$ 'and ' $\vee$ 'operations on A as follows,

$(x \wedge y) = (((x^* \to y^*) \to y^*)^*)$,

$(x \vee y) = (x \to y) \to y$ for all $x, y \in A$.

Then, (A, $\vee$, $\wedge$, *****, 0, 1) is a lattice Wajsberg algebra. It is easy to check that, $I_1 = \{0, r, s\}$ is a positive implicative *WI*-ideal of lattice Wajsberg algebra *A*. But, $I_2 = \{0, p, s\}$ is not a positive implicative *WI*-ideal of *A*. Since, $((p \to (s \to p)^*)^* \to 0)^* = q \notin I_2$.

Proposition 3.1.3. Let *I* be a non-empty subset of *A*. If *I* is a positive implicative *WI*-ideal of *A*, then *I* is a *WI*-ideal of *A*.

Proof. Let *I* be a positive implicative *WI*-ideal of *A* then from the definition 3.1.1 we have $0 \in I$ and replace $x = y$ and $z = x$ for all $x, y, z \in A$ in (ii) of definition 3.1.1, $(((x \to (x \to x)^*)^* \to y)^*) \in I$ and $y \in I$ imply $x \in I$ for all $x, y, z \in A$

$(((x \to 0)^* \to y)^*) \in I$ and $y \in I$ imply $x \in I$ for all $x, y, z \in A$

$((x \to y)^*) \in I$ and $y \in I$ imply $x \in I$ for all $x, y, z \in A$

Thus, *I* is a *WI*-ideal of *A*.■

Proposition 3.1.4. Let *I* be a *WI*-ideal of *A*. Then *I* is a positive implicative *WI*-ideal of *A* if and only if $(x \to (y \to x)^*)^* \in I$ implies $x \in I$ for all $x, y \in A$.

Proof. Let *I* be a positive implicative *WI*-ideal of *A* and let $x = 0, y = x, z = y$ in $(((y \to (z \to y)^*)^* \to x)^*) \in I$ and $x \in I$ imply $y \in I$ then, we have $(((x \to (y \to x)^*)^* \to 0)^*) \in I$ and $0 \in I$ imply $x \in I$, which implies that, $(((x \to (y \to x)^*)^*) \in I$ implies $x \in I$.

Conversely, since *I* is a *WI*-ideal of *A*, $(((y \to (z \to y)^*)^*) \in I$ .Thus, we have $y \in I$.■

Proposition 3.1.5. Let *I* be a non-empty subset of lattice Wajsberg algebra *A*. If *I* is a positive implicative *WI*-ideal of *A*, then it is an implicative *WI*-ideal of *A*.

Proof. Let *I* be a positive implicative *WI*-ideal of *A*.

We need to prove**:** *I* is an implicative *WI*-ideal of *A*. Let $((x \to y)^* \to z^*) \in I$ and $(y \to z)^* \in I$. It is enough to show that $(x \to z)^* \in I$.

Here, $((x \to y)^* \to z)^* = (z^* \to (x \to y))^*$

$= (x \to (z^* \to y))^*$ [From (vii) of proposition 2.2]

$= (x \to (y^* \to z))^*$ [From (x) of proposition 2.2]

$= (y^* \to (x \to z))^*$ [From (vii) of proposition 2.2]

$= ((x \to z)^* \to y)^*$ [From(x) of proposition 2.2]

Therefore, $((x \to y)^* \to z)^* = ((x \to z)^* \to y)^*$

We prove that, $(x \to z)^* \to y \le ((y \to z) \to ((x \to z)^* \to z)$

then $((y \to z) \to ((x \to z)^* \to z))^* \le ((x \to z)^* \to y)^*$

$(((x \to z)^* \to z)^* \to (y \to z)^*)^* \le ((x \to z)^* \to y)^*$

Since ($((x \to z)^* \to y)^*), (y \to z)^* \in I$, we have $((x \to z)^* \to z)^* \in I$.

Also,

$$((x \to z)^* \to z)^* = (((x \to z)^* \to 0)^* \to z)^*$$

$$= (((x \to z)^* \to (0 \to z)^*)^* \to z)^*$$

$$= (((x \to z)^* \to ((x \to x)^* \to z)^*)^* \to z)^* \quad \text{[From (i) of proposition 2.2]}$$

$$= (((x \to z)^* \to ((x \to z)^* \to x)^*)^* \to z)^*$$

From (iii) of definition 2.1, we have

$$((x \to (x \to (x \to z)^*)^*)^* \to z)^* = ((x \to z)^* \to (x \to (x \to z)^*)^*)^*$$

Thus, we have $(x \to z)^* \in I$.■

Proposition 3.1.6. Let I be a non-empty subset of lattice H-Wajsberg algebra A. If I is an implicative *WI*-ideal of A, then I is a positive implicative *WI*-ideal of A.

Proof. Let I be an implicative *WI*-ideal of lattice H-Wajsberg algebra A,

Then, we have $(y \to (z \to y)^*)^* \in I$

Thus, we get $(y \to (z \to y)^*)^* = ((z \to y) \to y^*)^* = ((y^* \to z^*) \to y^*)^* = (y^*)^*$

Since, A is a H-Wajsberg algebra, we get $y = (y \to (z \to y)^*)^* \in I$. ■

Proposition 3.1.7. Let M and N be two *WI*-ideals of lattice Wajsberg algebra A with $M \subseteq N$. If M is a positive implicative *WI*-ideal of A then so is N.

Proof. Let $(x \to (y \to x)^*)^* \in N$. Take $r = (x \to (y \to x)^*)^*$, $X = (x \to r)^*$ and $Y = x$.

Then, $(Y \to X)^* = (x \to (x \to r)^*)^*$

$$= (x \to (x \to (x \to (y \to x)^*)^*)^*)^*$$

$$= ((x \to (y \to x)^*)^*)^* = r^*$$

Therefore, $(Y \to X)^* = r^*$

So, $(X \to (Y \to X)^*)^* = ((x \to r)^* \to r^*)^*$

$$= (r \to (x \to r))^*$$

$$= (x \to (r \to r))^*$$

$$(X \to (Y \to X)^*)^* = 0 \in M$$

and so $x \in M$ by M is a positive implicative *WI*-ideal of A.

Since $M \subseteq N$, $(x \to r)^* = X \in N$, $r \in N$ implies that $x \in N$. Thus, N is a positive implicative *WI*-ideal of A.■

3.2. Associative *WI*-ideals of Lattice Wajsberg algebras

Definition 3.2.1. A subset *I* of *A* is said to be an associative *WI*-ideal of *A* with respect to *x*, where *x* is a fixed element of *A*, if it satisfies following condition,

(i) $0 \in I$

(ii) $(y \to x)^* \in I$ and $((z \to y)^* \to x)^* \in I$ imply $z \in I$ for all $x, y, z \in A$ and $x \neq 1$.

An associative *WI*-ideal with respect to 1 is whole algebra *A.* An associative *WI*-ideal with respect to 0 coincides with *WI*-ideal.

Example 3.2.2. Let $A = \{0, l, m, n, 1\}$ be a partial ordering set as in figure 3.2. Define "$\to$" and "$*$" on *A* as in table 3.3 and table 3.4.

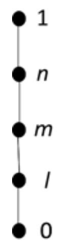

Figure 3.2
Lattice Diagram

x	x^*
0	1
l	n
m	m
n	l
1	0

Table 3.3
Complement

$\to$	0	l	m	n	1
0	1	1	1	1	1
l	n	1	1	1	1
m	m	n	1	1	1
n	l	m	1	1	1
1	0	l	m	n	1

Table 3.4
Implication

Here, $(A, \wedge, \vee, *, 0, 1)$ is a lattice Wajsberg algebra. It is easy to verify that, $I_3 = \{0, m, n\}$ is an associative *WI*-ideal of lattice Wajsberg algebra *A*.

Proposition 3.2.3. Every associative *WI*-ideal with respect to *x* contains *x* itself.
Proof. Let *I* be an associative *WI*-ideal of *A*.
If $x = 0$ then $(y \to 0)^* \in I$ and $((z \to y)^* \to 0)^* \in I$ imply $z \in I$.
So $y \in I$ and $(z \to y)^* \in I$ imply $z \in I$.
Hence, we have *I* is a *WI*-ideal of *A* that contain 0. If $x = 1$ then *I* = *A*. If $x \neq 0,1$, take $y = 0$ and $z = x$ then $((x \to 0)^* \to x)^* = (x \to x)^* = 0 \in I$ and $(0 \to x)^* = 0 \in I$ imply $x \in I$.■

Proposition 3.2.4. Every associative *WI*-ideal is a *WI*-ideal of lattice Wajsberg algebra *A*.
Proof. If $y \in I$ and $(x \to y)^* \in I$ then $(y \to 0)^* \in I$ and $(x \to y)^* \to 0)^* \in I$. Since *I* is an associative *WI*-ideal of *A* then $x \in I$. ■

Proposition 3.2.5. Let *I* be a *WI*-ideal of *A*. *I* is an associative *WI*-ideal if and only if $(((z \to y)^* \to x)^*) \in I$ implies $((z \to (y \to x)^*)^*) \in I$.
Proof. If $(((z \to y)^* \to x)^*) \in I$ and $((y \to x)^*) \in I$ then $((z \to (y \to x)^*)^*) \in I$ and $((y \to x)^*) \in I$
Since *I* is a *WI*-ideal of *A*, then $z \in I$ .Conversely, Let $(((z \to y)^* \to x)^*) \in I$ then

$$((((z \to (y \to x)^*)^* \to (z \to y)^*)^* \to x)^*) = ((((z \to (y \to x)^*)^* \to x) \to (z \to y)^*)^*)$$
$$= ((((z \to x)^* \to (y \to x)^*)^* \to (z \to y)^*)^*)$$

$$= 1^* = 0 \in I$$

Hence, $((((z \to (y \to x)^*)^* \to (z \to y)^*)^* \to x)^*) \in I$ (3.2.1)

equation (3.2.1) comes from $(z \to y) \le (y \to x) \to (z \to x)$

Which implies $(((z \to x)^* \to (y \to x)^*)^*) \le ((z \to y)^*)$

From our assumption that, $(((z \to y)^* \to x)^*) \in I$ and I is an associative *WI*-ideal.

Thus, we have $((z \to (y \to x)^*)^*) \in I$.■

Proposition 3.2.6. Let I be a *WI*-ideal of *A*. I is an associative *WI*-ideal if and only if $((y \to x)^* \to x)^* \in I$ implies $y \in I$.

Proof. If $((y \to x)^* \to x)^* \in I$ then $((y \to (x \to x)^*)^* \in I$ .So, $((y \to 0)^* = y \in I$.

Conversely, $((((z \to (y \to x)^*)^* \to x)^* \to x)^* \to ((z \to y)^* \to x)^*)^*$

$= (((((z \to (y \to x)^*)^* \to x)^* \to x)^* \to ((z \to y)^* \to x)^*)^* \to 0)^*$

$= (((((z \to (y \to x)^*)^* \to x)^* \to x)^* \to ((z \to y)^* \to x)^*)^* \to (((z \to x)^* \to (y \to x)^*)^* \to (z \to y)^*)^*)^*$ (3.2.2)

$= (((((z \to (y \to x)^*)^* \to x)^* \to x)^* \to ((z \to y)^* \to x)^*)^* \to (((z \to (y \to x)^*)^* \to x)^* \to (z \to y)^*)^*)^*$

$= (((((z \to (y \to x)^*)^* \to x)^* \to x)^* \to (((z \to (y \to x)^*)^* \to x)^* \to (z \to y)^*)^*)^* \to ((z \to y)^* \to x)^*)^*$

$\le (((z \to y)^* \to x)^* \to ((z \to y)^* \to x)^*)^* = 0$

Hence, $((((z \to (y \to x)^*)^* \to x)^* \to x)^* \to ((z \to y)^* \to x)^*)^* \in I$, $((((z \to (y \to x)^*)^* \to x)^* \to x)^* \in I$

From the given condition, we have $((y \to x)^* \to x)^* \in I$.

From proposition 3.2.4, we have I is an associative *WI*-ideal.

Equation (3.2.2) comes from $(z \to y) \le (y \to x) \to (z \to x)$ so $((z \to x)^* \to (y \to x)^*)^* \le (z \to y)^*$ that, $(((z \to x)^* \to (y \to x)^*)^* \to (z \to y)^*) = 0$ and the inequality in (3.2.2) from $(x \to y) \le (z \to x) \to (z \to y)$ then $((z \to y)^* \to (z \to x)^*)^* \le (x \to y)^*$.■

4. Conclusion

In this paper, we have introduced the notions of positive implicative *WI*-ideal and associative *WI*-ideal of lattice Wajsberg algebras. We have investigated some of their properties. Also, we have analyzed the relationship of positive implicative *WI*-ideal with implicative *WI*-ideal and *WI*-ideal, and hence an associative *WI*-ideal with *WI*-ideal. Moreover, we provide the condition equivalent for both positive implicative *WI*-ideal and associative *WI*-ideal.

References

[1] Font, J. M., Rodriguez, A. J., and Torrens, A., *Wajsberg algebras*, STOCHASTICA, Volume 8, Number 1 (1984), 5-31.

[2] Ibrahim, A., and Shajitha Begum, C., *On WI-Ideals of lattice Wajsberg algebras*, Global Journal of Pure and Applied Mathematics, Volume 13, Number 10 (2017), 7237-7254.

[3] Ibrahim, A., and Shajitha Begum, C., *Fuzzy and Normal Fuzzy WI-Ideals of Lattice Wajsberg Algebras*, International Journal of Mathematical Archive, Volume 8, Number 11 (2017), 122-130.

[4] Ibrahim, A., and Shajitha Begum, C., *Intuitionistic Fuzzy WI-Ideals of Lattice Wajsberg Algebras*, International Journal of Fuzzy Mathematical Archive, Volume 15, Number 1 (2018), 7-17.

[5] Ibrahim, A., and Shajitha Begum, C., *Properties of Annihilators in Lattice Wajsberg Algebras*, Journal of Advances in Mathematics and Computer Science, Volume 26, Number 1(2018), 1-11.

[6] Ibrahim, A., and Shajitha Begum, C., *Ideals and implicative WI-ideals of Lattice Wajsberg Algebras*, IPASJ International Journal of Computer Science, Volume 6, Issue 4 (2018), 30-38.

[7] Ibrahim, A., and Shajitha Begum, C., *Fuzzy Implicative WI-ideals of Lattice Wajsberg Algebras*, Journal of Computer and Mathematical Science, Volume 9, Number 8 (2018), 1026-1035.

[8] Ibrahim, A., and Shajitha Begum, C., *Characterizations of Implicative and Fuzzy Implicative WI-ideals of Lattice Wajsberg Algebras*, Journal of Applied Science and Computations, Volume 5, Issue 9 (2018), 672-683.

[9] Ibrahim, A., and Shajitha Begum, C., *Anti fuzzy WI-ideals of Lattice Wajsberg Algebras*, International Journal of Research in Advent Technology, Volume 6, Issue 11 (2018), 2957-2960.

[10] Ibrahim, A., and Shajitha Begum, C., *Intuitionistic Anti fuzzy WI-ideals of Lattice Wajsberg Algebras*, International Journal of Research and Analytical Reviews, Volume 5, Issue 4 (2018), 364-370.

[11] Komari, Y., *The Separation Theorem of the* $\aleph_0$*-valued Łukasiewicz Propositional logic*, Reports of Faculty of Science, Shizuoka University 12 (1978), 1-5.

[12] Komari, Y., *Super-Łukasiewicz Propositional Logics*, Nagoya Math. J. 84 (1981), 119-133.

[13] Wajsberg, M., *Beitragezum Metaaussagenkalkul* l, Monat. Math. phys. 42 (1935), 221-242.