Advances in Mathematics: Scientific Journal 8 (2019), no.3, 300–306
(Special issue on ICRAPAM)
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

ANTI FUZZY *WI*-IDEALS OF RESIDUATED LATTICE WAJSBERG ALGEBRAS

A. Ibrahim[1] and R. Shanmugapriya[2]

[1]P.G.and Research Department of Mathematics, H.H.The Rajah's College, Pudukkotai,Affiliated to Bharathidasan University, Tiruchirappalli,Tamilnadu, India. Email:ibrahimaadhil@yahoo.com;dribrahimaadhil@gmail.com

[2]Department of Mathematics, Coimbatore Institute of Engineering and Technology, Narasipuram, Coimbatore,Research Scholar,P.G and Research Department of Mathematics, H. H. The Rajah's College, Pudukkotai, Affiliated to Bharathidasan University, Tamilnadu, India.Email:priyasanmu@gmail.com

Abstract--In this paper, we introduce the notions of an anti fuzzy *WI*- ideal (anti FWI-ideal) and an anti fuzzy lattice ideal of residuated lattice Wajsberg algebra. Further, we discuss the relationship between an anti FWI-ideal and anti fuzzy lattice ideal. Also, we investigate some of its related properties.

Keywords--Wajsberg algebra; Lattice Wajsberg algebra; Residuated lattice Wajsberg algebra; *WI*–ideal; *FWI* –ideal; Anti *FWI* –ideal; Fuzzy lattice ideal; Anti fuzzy lattice ideal.

***Mathematical Subject classification 2010*:**03G10, 03E72, 06B10.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh [11] in 1965. Fuzziness occurs when the boundary of a piece of information is not clear-cut. Classical set theory allows the membership of the elements in the set in binary terms. Fuzzy set theory permits membership function valued in the interval [0, 1]. Mordchaj Wajsberg [10] introduced the concept of Wajsberg algebra in 1935 and studied by Font, Rodriguez and Torrens [3]. Also, they [3] defined lattice structure of Wajsberg algebra. Further, they introduced the notion of an implicative filter of lattice Wajsberg algebra and discussed their properties. Basheer Ahamed and Ibrahim [1, 2] introduced the definitions of fuzzy implicative and an anti fuzzy implicative filters of lattice Wajsberg algebra and investigated some properties. Ibrahim and Shajitha Begum [4, 5]introduced the notions of *WI*-ideal, fuzzy *WI*-ideal, normal fuzzy *WI*-ideal of lattice Wajsberg algebra and studied some related properties. The authors [7,

8] introduced the notions of *WI*-ideal, *FWI*-ideal of residuated lattice Wajsberg algebra and discussed some related properties.
In this paper, we introduce the definitions of an anti *FWI*-ideal and an anti fuzzy lattice ideal of residuated lattice Wajsberg algebra. Further, we discuss the relationship between an anti *FWI*-ideal and an anti fuzzy lattice ideal in residuated lattice Wajsberg algebra as well as in residuated lattice*H*-Wajsberg algebra. Moreover, we investigate some of its related properties.

2. Preliminaries

In this section, we recall some basic definitions and properties which are helpful to develop our main results.

Definition2.1*[6]***.** Let $(A, \rightarrow, *, 1)$ bean algebra with a binary operation "$\rightarrow$" and a quasi complement "$*$" is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x, y, z \in A$,
(i) $1 \rightarrow x = x$
(ii) $(x \rightarrow y) \rightarrow y = ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$
(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1.$

Definition 2.2 *[6]*. A Wajsberg algebra $(A, \rightarrow, *, 1)$satisfies the following axioms for all $x, y, z \in A$,
(i) $x \rightarrow x = 1$
(ii) If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$
(iii) $x \rightarrow 1 = 1$
(iv) $(x \rightarrow (y \rightarrow x)) = 1$
(v) If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$
(vi) $(x \rightarrow y) \rightarrow \big((z \rightarrow x) \rightarrow (z \rightarrow y)\big) = 1$
(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$
(ix) $(x^*)^* = x$
(x) $(x^* \rightarrow y^*) = y \rightarrow x.$

Definition 2.3 *[6]*. A Wajsberg algebra A is called a lattice Wajsberg algebra, if it satisfies the following conditions for all $x, y \in A$,
(i) The partial ordering "$\leq$" on a lattice Wajsberg algebra, such that $x \leq y$ if and only if $x \rightarrow y = 1$
(ii) $x \vee y = (x \rightarrow y) \rightarrow y$
(iii) $x \wedge y = ((x^* \rightarrow y^*) \rightarrow y^*)^*.$
Thus$(A, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4 *[6]*. A lattice Wajsberg algebra $(A, \rightarrow, *, 1)$ satisfies the following axioms for all $x, y, z \in A$,
(i) If $x \leq y$ then $x \rightarrow z \geq y \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$
(ii) $x \leq y \rightarrow z$ if and only if $y \leq x \rightarrow z$
(iii) $(x \vee y)^* = (x^* \wedge y^*)$
(iv) $(x \wedge y)^* = (x^* \vee y^*)$
(v) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
(vi) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
(vii) $(x \rightarrow y) \vee (y \rightarrow x) = 1$
(viii) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$
(ix) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$
(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xi) $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.5 *[6]*. A residuated lattice$(A, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ satisfiesthe following conditions for all $x, y, z \in A$,

(i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(ii) $(A, \otimes, 1)$ is commutative monoid,

(iii) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$.

Definition 2.6 *[6]*. Let $(A, \vee, \wedge, *, \rightarrow, 1)$ be a lattice Wajsberg algebra. If a binary operation " $\otimes$ " on A satisfies $x \otimes y = (x \rightarrow y^*)^*$ for all $x, y \in A$. Then $(A, \vee, \wedge, \otimes, \rightarrow, *, 0, 1)$ is called a residuated lattice Wajsberg algebra.

Definition 2.7 *[6]*. Let A be a lattice Wajsberg algebra. Let I be a nonempty subset of A, then I is called a *WI*-ideal of lattice Wajsberg algebra A satisfies for all $x, y \in A$,

(i) $0 \in I$

(ii) $(x \rightarrow y)^* \in I$and $y \in I$ imply $x \in I$.

Definition 2.8 *[6]*. Let L be a lattice. An ideal I of L is a nonempty subset of L is called a lattice ideal, if it satisfies the following axioms for all$x, y \in A$,

(i) $x \in I, y \in L$ and $y \leq x$ imply $y \in I$

(ii) $x, y \in I$ implies $x \vee y \in I$.

Definition 2.9 *[7]*. Let A be a residuated lattice Wajsberg algebra. Let I be a nonempty subset of A, then I is called a WI-ideal of residuated lattice Wajsberg algebra A satisfies the following axiomsfor all $x, y \in A$,

(i) $0 \in I$

(ii) $x \otimes y \in I$ and $y \in I$ imply $x \in I$

(iii) $(x \rightarrow y)^* \in I$ and $y \in I$ imply $x \in I$.

Definition 2.10 *[7]*. Let A be a set. A function$\mu: A \rightarrow [0, 1]$ is called a fuzzy subset on A, for each $x \in A$, the value of $\mu(x)$ describes a degree of membership of x in μ.

Definition 2.11 *[6]***.** Let A be a lattice Wajsberg algebra. A fuzzy subset μ of A is called a fuzzy WI-ideal of A, if for all $x, y \in A$,

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu((x \rightarrow y)^*), \mu(y)\}$.

Definition 2.12 *[6]*. A fuzzy subsetμ of a lattice Wajsberg algebra A is called a fuzzy lattice ideal if for all $x, y \in A$,

(i) If $y \leq x$ then $\mu(y) \geq \mu(x)$

(ii) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.13 *[6]*. A fuzzy subset μ of lattice Wajsberg algebra A is called an anti fuzzy lattice ideal if for all $x, y \in A$,

(i) If $y \leq x$ then $\mu(y) \leq \mu(x)$

(ii) $\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\}$.

Definition 2.14 *[7]* .Let A be a residuated lattice Wajsberg algebra. A fuzzy subset μ of A is called a FWI-ideal of residuated lattice Wajsberg algebra A if for all $x, y \in A$,

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu(x \otimes y), \mu(y)\}$

(iii) $\mu(x) \geq \min\{\mu((x \rightarrow y)^*), \mu(y)\}$.

Definition 2.15 *[6]*. The lattice Wajsberg algebra $A$is called a lattice *H*- Wajsberg algebra, if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in A$.
In a lattice *H*- Wajsberg algebra A, the following hold,

(i) $x \to (x \to y) = (x \to y)$

(ii) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Definition 2.16 *[6]*. Let μ be a fuzzy set in a set A. Then for $t \in [0, 1]$, the set $\mu_t = \{x \in A/\mu(x) \geq t\}$ is called level subset of μ.

Definition 2.17 *[6]*. Let μ be a fuzzy set in a set A. Then for $t \in [0, 1]$, the set $\mu^t = \{x \in A/\mu(x) \leq t\}$ is called the lower t-level cut of μ.

Definition 2.18 *[6]*. Let $\mathcal{F}(A)$ be the set of all anti fuzzy subset of lattice Wajsberg algebra A and μ, we define the set as $A_\mu = \{x \in A/\mu(x) = \mu(0)\}$.

3. Properties of anti *FWI*-ideals

In this section, we define an anti *FWI*-ideal in residuated lattice Wajsberg algebra and obtain some useful results with illustrations.

Definition 3.1 A fuzzy subset μ of residuated lattice Wajsberg algebra A is called an anti *FWI*-ideal of A if for any $x, y \in A$,

(i) $\mu(0) \leq \mu(x)$

(ii) $\mu(x) \leq \max\{\mu(x \otimes y), \mu(y)\}$

(iii) $\mu(x) \leq \max\{\mu((x \to y)^*), \mu(y)\}$.

Example 3.2. Consider a set A=$\{0, a, b, c, d, 1\}$. Define a partial ordering "$\leq$" on A, such that $0 \leq a \leq b \leq c \leq d \leq 1$ with a binary operations"$\otimes$" and "$\to$" and a quasi complement "$*$"on A as in following tables 3.1 and 3.2.

x	x^*
0	1
a	c
b	b
c	d
d	0
1	0

Table 3.1: Complement

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Table 3.2: Implication

Define $\vee$ and $\wedge$ operations on A as follows:

$$(x \vee y) = (x \to y) \to y,$$

$$(x \wedge y) = (x^* \to y^*) \to y^*)^*;\ x \otimes y = (x \to y^*)^* \text{ for all } x, y \in A.$$

Then, A is a residuated lattice Wajsberg algebra.

Consider the fuzzy subset μ on A as, $\mu(x) = \begin{cases} 0.2 & \text{if } x = 0 \\ 0.4 & \text{otherwise} \end{cases}$ for all $x \in A$

Then, we have μ is an anti FWI-ideal of residuated lattice Wajsberg algebra A.

Proposition 3.3 Every anti FWI-ideal μ of a residuated lattice Wajsberg algebra A is order preserving.

Proof. Let μ be an anti FWI-ideal of A and $x \leq y$

Then $x \otimes y = (x \to y^*)^* = 1^* = 0$ and $(x \to y)^* = 1^* = 0$ for all $x, y \in A$

Now, $\mu(x) \leq \max\{\mu(x \otimes y), \mu(y)\}$ [From (ii) of definition 3.1]

$= \max\{\mu(0), \mu(y)\} = \mu(y)$ [From (i) of definition 3.1]

We have $\mu(x) \leq \mu(y)$ for all $x, y \in A$

And $\mu(x) \leq \{\mu(x \to y)^*,\ \mu(y)\}$ [From (iii) of definition 3.1]

$= \max\{\mu(0), \mu(y)\}$

$= \mu(y)$ [From (i) of definition 3.1]

Thus, $\mu(x) \leq \mu(y)$ for all $x, y \in A$

Hence, μ is order preserving. ■

Proposition 3.4 Every anti FWI-ideal of residuated lattice Wajsberg algebra A is an anti fuzzy lattice ideal.

Proof. Let μ be an anti FWI-ideal of A.

Then $(x \vee y) \otimes x = \left(\left((\mathrm{x} \to \mathrm{y}) \to \mathrm{y}\right) \to x^*\right)$ [From (ii) of definition 2.3]

$= (1 \to y)^*$

$= x$ for all $x, y \in A$

And $((x \vee y) \to y)^* = ((x \to \mathrm{y}) \wedge (y \to \mathrm{y}))^*$ [From (v) of definition 2.4]

$= (x \to y)^*$

$\leq x$ for all $x, y \in A$

Now, we have $\mu(x \vee y) \leq \max\{\mu((x \vee y) \otimes x), \mu(y)\}$

$= \max\{\ \mu(x), \mu(y)\}$ [From (ii) of definition 2.13]

And $\mu(x \vee y) \leq \max\{\mu(x \vee y) \to y)^*, \mu(y)\}$

$\leq \max\{\ \mu(x), \mu(y)\}$ [From (ii) of definition 2.13]

Thus, $\mu(x \vee y) \leq \max\{\ \mu(x), \mu(y)\}$for all $x, y \in A$. ■

The following example shows that the converse of Proposition 3.4 is not true.

Example 3.5 Consider a set $A = \{0, e, f, g, 1\}$. Define a partial ordering "$\leq$" on A, such that $0 \leq e \leq f \leq g \leq 1$ with a binary operations"$\otimes$"and "$\to$"and a quasi complement "$*$"on A as in following tables 3.3 and 3.4.

x	x^*
0	1
e	g
f	f
g	e
1	0

Table 3.3: Complement

$\to$	0	e	f	g	1
0	1	1	1	1	1
e	g	1	1	1	1
f	f	g	1	1	1
g	e	f	g	1	1
1	0	e	f	g	1

Table 3.4: Implication

Define $\vee$ and $\wedge$ operations on A as follows:

$(x\ \vee\ y) = (x \to y) \to y,$

$(x\ \wedge\ y) = (x^* \to y^*) \to y^*)^*;\ x \otimes y = (x \to y^*)^*$ for all $x, y \in A$.

Then, A is a residuated lattice Wajsberg algebra.

Consider the fuzzy subset μ on A as, $\mu(x) = \begin{cases} 0.1 & \text{if } x \in \{0, f, 1\} \quad \text{for all } x \in A \\ 0.5 & \text{if } x \in \{e, f\} \quad \text{for all } x \in A \end{cases}$

Then, μ is an anti fuzzy lattice ideal of A, but not an anti FWI-ideal for $\mu(e) \nleq \max\{\mu(e \otimes f), \mu(f)\}$.

Definition 3.6 The residuated lattice Wajsberg algebra A is called a residuated lattice H –Wajsberg algebra if it satisfies $x \vee y \vee \left((x \wedge y) \to z\right) = 1$for all $x, y, z \in A$.

In a residuated lattice H-Wajsberg algebra A, the following hold,

(i) $x \otimes y \in A$,

(ii) $x \otimes (x \otimes y) = (x \otimes y); x \to (x \to y) = (x \to y)$.

(iii) $x \otimes (y \otimes z) = (x \otimes y) \otimes (x \otimes z); x \to (y \to z) = (x \to y) \to (x \to z)$

for all $x, y, z \in A$.

Proposition 3.7 Every anti fuzzy lattice ideal is an anti FWI-ideal in residuated lattice H-Wajsberg algebra A.

Proof. Let μ be an anti fuzzy lattice ideal of residuated lattice H-Wajsberg algebra A.
If $y \leq x$, then $\mu(y) \leq \mu(x)$ and
$\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\}$for all $x, y \in A$ [From (ii) of definition 2.13]
Since $0 \leq x$, it follows that $\mu(0) \leq \mu(x)$for all $x, y \in A$ [From (i) of definition 2.13]
Now, we have $\mu(x) \leq \mu(x \vee y)$
$= \mu(y \vee (x^* \vee y)^*)$
$= \mu(y \vee (x \rightarrow y^*)^*)$
$= \mu(y \vee (x \otimes \mathrm{y}))$
$\leq \max\{\mu(y), \mu(x \otimes \mathrm{y})\}$for all $x, y \in A$ [From (ii) of definition 2.13]
And $\mu(x) \leq \mu(x \vee y)$
$= \mu(y \vee (x^* \vee y^*)^*)$
$= \mu(y \vee (x \rightarrow y)^*)$
$\leq \max\{\mu(y), \mu(x \rightarrow y)^*\}$for all $x, y \in A$ [From (ii) of definition 2.13]
Hence, μ is an anti FWI-ideal. ■

Proposition 3.8. A fuzzy subset μ of a residuated lattice Wajsberg algebra A is an anti FWI-ideal if and only if μ_t is a WI-ideal when $\mu_t \neq \phi, t \in [0, 1]$.
Proof. Let μ be an anti FWI-ideal of A and let $t \in [0, 1]$ such that $\mu_t \neq \phi$.
Therefore, we have $0 \in \mu_t$.
Let $x, y \in A$. $x \otimes y \in \mu_t$, $(x \rightarrow y)^* \in \mu_t$ and $y \in \mu_t$.
Then, $\mu(x \otimes \mathrm{y}) \leq \mathrm{t}, \mu(x \rightarrow y)^* \leq t$ and $\mu(y) \leq t$. [From definition 2.17]
It follows that, $\mu(x) \leq \max\{\mu(x \otimes \mathrm{y}), \mu(\mathrm{y})\} \leq \mathrm{t}$. [From (ii) of definition 3.1]
We have $x \in \mu_t$.
Hence, μ_t is a WI-ideal of A.
Conversely, assume that $\mu_t(\, t \in [0, 1])$ is a WI-ideal of A.
When $\mu_t \neq \phi$ for any $x \in A, x \in \mu_{\mu(x)}$
It follows that $\mu_{\mu(x)}$ is a WI-ideal of A and hence $0 \in \mu_{\mu(x)}$.
That is $\mu(0) \leq \mu(x)$ for all $x, y \in A$. [From (i) of definition 3.1]
Let $t = \max\{\mu(x \otimes y), \mu(y)\}, t = \max\{\mu(x \rightarrow y)^*, \mu(y)\}$, it follows that μ_t is a WI-ideal
and $x \otimes y \in \mu_t, (x \rightarrow y)^* \in \mu_t, y \in \mu_t$.
This implies that $x \in \mu_t$ and $\mu(x) \leq t = \max\{\mu(x \otimes y), \mu(y)\}$, [From (ii) of definition 3.1]
$\mu(x) \leq t = \max\{\mu(x \rightarrow y)^*, \mu(y)\}$ for all $x, y \in A$. ■ [From (iii) of definition 3.1]

Proposition 3.9. Let A be a residuated lattice Wajsberg algebra. If μ and σ are non-empty anti fuzzy WI-ideals of A such that for any $x, y \in A, \mu(x) \geq \mu(y)$ if and only if $\sigma(x) \geq \sigma(y)$ then $\mu \circ \sigma$ is also a FWI-ideal of A and $A_{\mu \circ \sigma} = A_\mu \cap A_\sigma$ where $(\mu \circ \sigma)(x) = \mu(x)\sigma(x)$ for any $x \in A$.
Proof. Let μ and σ are non-empty anti fuzzy WI-ideals of A.
We need to prove: $\mu \circ \sigma$ is a WI-ideal of A and $A_{\mu \circ \sigma} \supseteq A_\mu \cap A_\sigma$
Let $x \in A_{\mu \circ \sigma}$. Then $(\mu \circ \sigma)(x) = \mu \circ \sigma(0)$ [From definition 2.18]
$$\mu(x)\sigma(x) = \mu(0)\sigma(0)$$
Hence, $\mu(x) \neq 0$ and $\sigma(x) \neq 0$ for all $x \in A$
If $\mu(x) > \mu(0)$, then $\mu(x)\sigma(x) > \mu(0)\sigma(0)$ for all $x \in A$
This is a contradiction.
Similarly, it is also a contradiction when $\sigma(x) > \sigma(0)$.
Hence, $\mu(x) = \mu(0)$ and $\sigma(x) = \sigma(0)$. [From definition 2.18]
That is $x \in A_\mu \cap A_\sigma$ for all $x \in A$. ■

4. Conclusion

In this paper, we have introduced the notions of an anti FWI- ideal and an anti fuzzy lattice ideal of residuated lattice Wajsberg algebra. Further, we have shown that every anti FWI- ideal of residuated lattice Wajsberg algebra is order preserving. Also, we have obtained some of its properties.

References

[1] Basheer Ahamed, M., and Ibrahim, A., *Fuzzy implicative filters of lattice WajsbergAlgebras*, Advances in Fuzzy Mathematics, volume 6, Number 2 (2011), 235-243.

[2] Basheer Ahamed, M., and Ibrahim, A., *Anti fuzzy implicative filters in lattice W-algebras*, International Journal of Computational Sciences and Mathematics, Volume 4, Number 1 (2012), 49-56.

[3] Font, J.M., Rodriguez, A. J., and Torrens, A., *Wajsberg algebras*, STOCHASTICA Volume 8, Number 1, (1984), 5-31.

[4] Ibrahim, A., and Shajitha Begum, C., *On WI- -ideals of lattice Wajsberg algebras*, Global Journal of Pure and Applied Mathematics, Volume 13, Number 10 (2017), 7237-7254.

[5] Ibrahim, A., and Shajitha Begum, C., *Fuzzy and Normal Fuzzy WI-ideals of lattice Wajsberg algebras*, International Journal of Mathematical Archeive, Volume 8, Number 11 (2017), 122-130.

[6] Ibrahim, A., and Shajitha Begum, C., *AntiFuzzy Ideals of Lattice Wajsberg Algebras*, International Journal of Research in Advent Technology, Volume 6, Number 11 (2018), 2321-9637.

[7] Ibrahim, A., and Shanmugapriya, R.,*WI- -ideals of Residuated Lattice Wajsberg Algebras*, Journal of Applied Science and Computations, Volume 5, Number 11 (2018) 1076-5131.

[8] Ibrahim, A., and Shanmugapriya, R., *FWI-ideals of Residuated Lattice WajsbergAlgebras*, Journal of Computer and Mathematical Sciences, Volume 10, Number 3 (2018) 491-501.

[9] Rose, A., and Rosser, J.B., *Fragments of many valued statement calculi*, Transaction of American Mathematical Society 87, (1958), 1-53.

[10] Wajsberg, M., *Beitrage zum Metaaussagenkalkul1*, Monat. Math. Phy. 42 (1935) 221-242.

[11] Zadeh, L. A., *Fuzzy Sets*, Information and Control 8 (1965), 338-353.