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# PSEUDO - BOOLEAN AND FUZZY PSEUDO – BOOLEAN IMPLICATIVE FILTERS OF LATTICE PSEUDO - WAJSBERG ALGEBRAS

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**ABSTRACT:** In this paper, we introduce the notion of a pseudo-Boolean implicative filter and a fuzzy pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra and to investigate some properties with illustrations. The relations between pseudo-Boolean implicative filter and fuzzy pseudo-Boolean implicative filter are discussed. Moreover, we show the characterizations of pseudo-Boolean implicative filter and fuzzy pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra.

**KEYWORDS:** Wajsberg algebra; Pseudo-Wajsberg algebra; Lattice pseudo-Wajsberg algebra; Pseudo-Boolean implicative filter; Fuzzy pseudo-Boolean implicative filter.

**Mathematical Subject classification 2010:** 03G25, 03E72, 06F35

## 1. Introduction

MordchajWajsbreg [6] introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrens in [3]. Basheer Ahamed and Ibrahim [1] introduced the definition of fuzzy implicative filters of lattice Wajsberg Algebras and obtained some properties with illustrations. Pseudo-

Wajsberg algebras were introduced by Ceterchi Rodica [2] with the explicit purpose of providing a concept categorically equivalent to that of pseudo-MV algebras. Pseudo-Wajsberg algebras are generalizations of Wajsberg algebras. The notion of fuzzy set was introduced by L.A. Zadeh [9] in 1965. Zadeh [9] extended the notion of binary membership to accommodate various "degrees of membership" on the real continuous interval [0, 1], where the endpoints of 0 and 1 conform to no membership and full membership, respectively. Fuzzy set theory has been developed in many directions by many researchers and has evoked great interest among mathematicians working in different fields of Mathematics. Recently, the authors [4, 5] introduced the notions of implicative filter, fuzzy implicative filter of lattice pseudo-Wajsberg algebra and discussed some of their properties.
The aim of this paper is to introduce the definition of pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra and obtain some related properties. Further, we define fuzzy pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra. Next, we show that every fuzzy pseudo-Boolean implicative filter is an implicative filter of lattice pseudo-Wajsberg algebra.

## 2. Preliminaries

In this section, we recall some basic definitions and properties which are helpful to develop the main results.

***Proposition 2.1*** *[1]*. A Wajsberg algebra $(A, \rightarrow, ^{-}, 1)$ satisfies the following axioms for all $x, y \in A$,

(i) $x \rightarrow x = 1$

(ii) If $x \rightarrow y = y \rightarrow x = 1$, then $x = y$

(iii) $x \rightarrow 1 = 1$

(iv) $x \rightarrow (y \rightarrow x) = 1$

(v) If $x \rightarrow y = y \rightarrow z = 1$, then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow \big((z \rightarrow x) \rightarrow (z \rightarrow y)\big) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

***Definition 2.2*** *[2]*. An algebra $(A, \rightarrow, \rightsquigarrow, ^{-}, ^{\sim}, 1)$ with a binary operations " $\rightarrow$ ", " $\rightsquigarrow$ " and quasi complements " $^{-}$ ", " $^{\sim}$ " is called a pseudo-Wajsberg algebra if it satisfies the following axioms for all $x, y, z \in A$,

(i) (a) $1 \rightarrow x = x$

(b) $1 \rightsquigarrow x = x$

(ii) $(x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x = (y \rightarrow x) \rightsquigarrow x = (x \rightarrow y) \rightsquigarrow y$

(iii) (a) $(x \rightarrow y) \rightarrow \big((y \rightarrow z) \rightsquigarrow (x \rightarrow z)\big) = 1$

(b) $(x \rightsquigarrow y) \rightsquigarrow \big((y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)\big) = 1$

(iv) $1^{-} = 1^{\sim} = 0$

(v) (a) $(x^{-} \rightsquigarrow y^{-}) \rightarrow (y \rightarrow x) = 1$

(b) $(x^{\sim} \rightarrow y^{\sim}) \rightsquigarrow (y \rightsquigarrow x) = 1$

(vi) $(x \rightarrow y^{-})^{\sim} = (y \rightsquigarrow x^{\sim})^{-}$.

***Definition 2.3*** *[2]*. An algebra $(A, \rightarrow, \rightsquigarrow, ^{-}, ^{\sim}, 1)$ is called a lattice pseudo-Wajsberg algebra if it satisfies the following axioms for all $x, y \in A$,

(i) A partial ordering "$\leq$" on a lattice pseudo-Wajsberg algebra $A$, such that $x \leq y$ if and only if $x \rightarrow y = 1 \Leftrightarrow x \rightsquigarrow y = 1$.

(ii) $x \vee y = (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x = (x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$

(iii) $x \wedge y = (x \rightsquigarrow (x \rightarrow y)^{\sim})^{-} = \big((x \rightarrow y) \rightarrow x^{-}\big)^{\sim} = (y \rightsquigarrow (y \rightarrow x)^{\sim})^{-} = \big((y \rightarrow x) \rightarrow y^{-}\big)^{\sim}$
$= (y \rightarrow (y \rightsquigarrow x)^{-})^{\sim} = \big((y \rightsquigarrow x) \rightsquigarrow y^{\sim}\big)^{-} = (x \rightarrow (x \rightsquigarrow y)^{-})^{\sim} = \big((x \rightsquigarrow y) \rightsquigarrow x^{\sim}\big)^{-}$.

***Proposition 2.4*** *[2]*. A lattice pseudo-Wajsberg algebra $(A, \rightarrow, \rightsquigarrow, ^{-}, ^{\sim}, 1)$ satisfies the following axioms for all $x, y \in A$,

(i) $x \rightarrow x = 1, x \rightsquigarrow x = 1$

(ii) $x \to (y \rightsquigarrow x) = 1,\ x \rightsquigarrow (y \to x) = 1$
(iii) $x \le y \Rightarrow z \to x \le z \to y\ ;\ z \rightsquigarrow x \le\ z \rightsquigarrow y$
(iv) $x \le y \Rightarrow y \to z \le x \to z\ ;\ y \rightsquigarrow z \le x \rightsquigarrow z$
(v) $x \le y \to x\ ;\ \ x \le y \rightsquigarrow x$
(vi) $x \to y \le (y \to z) \rightsquigarrow (x \to z)\ ;\ x \rightsquigarrow y \le (y \rightsquigarrow z) \to (x \rightsquigarrow z)$
(vii) $x \to y \le (z \to x) \to (z \to y)\ ;\ x \rightsquigarrow y \le (z \rightsquigarrow x) \rightsquigarrow\ (z \rightsquigarrow y)$
(viii) $x \to (y \rightsquigarrow z) =\ y \rightsquigarrow (x \to z)$
(ix) $x \to (y \vee z)=(x \to y) \vee (x \to z)$
(x) $x \rightsquigarrow (y \vee z)=(x \rightsquigarrow y) \vee (x \rightsquigarrow z)$
(xi) $x \le y \to z \Leftrightarrow y \le x \rightsquigarrow z$
(xii) $(x \vee y) \to z = (x \to z) \wedge (x \to z)$
(xiii) $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (x \rightsquigarrow z)$
(xiv) $x \vee y^- = x^- \wedge y^-$
(xv) $(x \vee y) \to x =\ y \to x$
(xvi) $(x \vee y) \rightsquigarrow x =\ y \rightsquigarrow x$
(xvii) $(x^- \rightsquigarrow 0) \to x = 1\ ;\ \ (x^\sim \to 0) \rightsquigarrow x = 1$
(xviii) $x \to 0 = x^-; x \rightsquigarrow 0 = x^\sim$
(xix) $0 \to x = 1\ ; 0 \rightsquigarrow x = 1$
(xx) $x^- \to x = x;\ x^\sim \rightsquigarrow x = x.$

***Definition 2.5*** *[4]*. Let $A$ be lattice pseudo-Wajsberg algebra. A non-empty subset $F$ of $A$ is called an implicative filter of $A$ if it satisfies the following axioms for all $x, y \in A$,
(i) $1 \in F$
(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$
(iii) $x \in F$ and $x \rightsquigarrow y \in F$ imply $y \in F$.

***Definition 2.6*** *[1]*. Let $A$ be a set. A function $\mu\colon A\ \to\ [0, 1]$ is called a fuzzy subset on $A$, for each $x \in A$, the value of $\mu(x)$ describes a degree of membership of $x$ in $\mu$.

***Proposition 2.7*** *[1]*. Every implicative filter $F$ of $A$ has the following property $x \le y$ and $x \in F$ imply $y \in F$.

***Proposition 2.8*** *[1]*. Let $\mu$ be a fuzzy implicative filter of lattice pseudo-Wajsberg algebra, then for all $x, y \in A, x \le y$ implies $\mu(x) \le \mu(y)$.

***Definition 2.9*** *[5]* **.**Let $A$ be lattice pseudo-Wajsberg algebra. A non empty fuzzy subset $\mu$ of$A$ is called an implicative filter of $A$ if it satisfies the following axioms for all $x, y \in A$,
(i) $\mu(1) \ge \mu(x)$
(ii) $\mu(y) \ge \min\{\mu(x \to y), \mu(x)\}$
(iii) $\mu(y) \ge \min\{\mu(x \rightsquigarrow y), \mu(x)\}$.

***Proposition 2.10*** *[4]*. Let $(A, \to\ ,\ \rightsquigarrow\ ,\ ^-\ ,\ ^\sim\ ,\ 1)$ be a Type-1implicative filter of lattice pseudo-Wajsberg algebra. Then if it satisfies the following axioms for all $x, y \in A$,

(i) $\big((y \rightsquigarrow x) \to x\big) = \Big(\big((y \rightsquigarrow x) \to x\big) \to x\Big) \to x$
(ii) $\big((y \to x) \rightsquigarrow x\big) = \Big(\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow x\Big) \rightsquigarrow x.$

## 3. Main results

### 3.1. Pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra

In this section, we define pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra, and obtain some results with illustrations.

***Definition 3.1.1.*** Let $A$ be lattice pseudo-Wajsberg algebra. An implicative filter $F$ of $A$ is called pseudo-Boolean if it satisfies $x \vee x^{-} \in F$ and $x \vee x^{\sim} \in F$ for all $x \in A$.

***Example 3.1.2.*** Consider a set $A = \{0, a, b, 1\}$. Define a partial ordering "≤" on $A$, such that $0 \leq a \leq b \leq 1$ and the binary operations $" \rightarrow ", " \rightsquigarrow "$ and quasi complements $" ^{-} ", " ^{\sim} "$ given by the following tables 3.1.1., 3.1.2., 3.1.3., and 3.1.4.

| $x$ | $x^{-}$ |
|---|---|
| 0 | 1 |
| $a$ | $b$ |
| $b$ | $a$ |
| 1 | 0 |

Table:3.1.1.
Complement

| $\rightarrow$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | 1 | 1 |
| $b$ | $a$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | 1 |

Table:3.1.2.
Implication

| $x$ | $x^{\sim}$ |
|---|---|
| 0 | 1 |
| $a$ | $b$ |
| $b$ | $a$ |
| 1 | 0 |

Table:3.1.3.
Complement

| $\rightsquigarrow$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | 1 | 1 |
| $b$ | $a$ | $a$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | 1 |

Table:3.1.4.
Implication

Then, $A = (A, \wedge, \vee, \rightarrow, \rightsquigarrow, 0, 1)$ is lattice pseudo-Wajsberg algebra and consider the subset $F_1 = \{0,1\}$, then easily verify that $F_1$is a pseudo-Boolean implicative filter of $A$. But $F_2 = \{a, b\}$ is not a pseudo-Boolean implicative filter of $A$.
Since, $0 \vee 1 = (0 \rightarrow 1) \rightsquigarrow 1 = (1 \rightarrow 0) \rightsquigarrow 0 = (0 \rightsquigarrow 1) \rightarrow 1 = (1 \rightsquigarrow 0) \rightarrow 0 = 1$;
But, $(b \rightsquigarrow a) \rightarrow a = 1 \notin F_2$.

***Proposition 3.1.3.*** Let $A$ be lattice pseudo-Wajsberg algebra. If an implicative filter $F$ of $A$ is a pseudo-Boolean then which satisfies the following inequalities (i) $x \rightarrow x^{-} \in F$ imply $x \in F$
(ii) $x \rightsquigarrow x^{\sim} \in F$ imply $x \in F$ for all $x \in A$.

**Proof.** (i)Let $F$ be a pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra $A$.
Then, $x \vee x^{-} \in F$ for all $x \in A$. [From the definition of 3.1.1]
Consider,$(x \vee x^{-}) \rightarrow x = (x \rightarrow x) \wedge (x \rightarrow x^{-})$ [From (xii) of proposition 2.4]
$= 1 \wedge (x \rightarrow x^{-}) = x \rightarrow x^{-} \in F$ [From (i) of proposition 2.4]
Therefore, $(x \vee x^{-}) \rightarrow x \in F$. Thus, $x \in F$ . [From (ii) of definition 2.5]
(ii) Let $F$ be a pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra $A$.
Then, $x \vee x^{\sim} \in F$for all $x \in A$. [From the definition of 3.1.1]
Now,$(x \vee x^{\sim}) \rightsquigarrow x = (x \rightsquigarrow x) \wedge (x \rightsquigarrow x^{\sim})$ [From (xiii) of proposition 2.4]
$(x \vee x^{\sim}) \rightsquigarrow x = 1 \wedge (x \rightsquigarrow x^{\sim}) = x \rightarrow x^{\sim} \in F$ [From (i) of proposition 2.4]
Therefore, $(x \vee x^{\sim}) \rightsquigarrow x \in F$.Thus, $x \in F$. ■ [From (iii) of definition 2.5]

***Proposition 3.1.4.*** Let $A$ be lattice pseudo-Wajsberg algebra. If an implicative filter $F$ of $A$ is a pseudo-Boolean if and only if it satisfies (i) $(x \rightarrow y) \rightsquigarrow x \in F$ imply $x \in F$ (ii) $(x \rightsquigarrow y) \rightarrow x \in F$ imply $x \in F$ for all $x, y \in A$.

**Proof.** Let $F$ be a pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra $A$.
Let $(x \rightarrow y) \rightsquigarrow x \in F$, $(x \rightsquigarrow y) \rightarrow x \in F$
Then, we have $x^{-} \leq x \rightarrow y$ and $x^{\sim} \leq x \rightsquigarrow y$
So,$(x \rightarrow y) \rightsquigarrow x \leq x^{-} \rightsquigarrow x$ and $(x \rightsquigarrow y) \rightarrow x \leq x^{\sim} \rightarrow x$ [From (iv) of proposition 2.4]
$x^{-} \rightarrow x \in F$and $x^{-} \rightsquigarrow x \in F$ [From the proposition 2.7]
(i) Now,$(x \vee x^{-}) \rightarrow x = (x \rightarrow x) \wedge (x \rightarrow x^{-})$ [From (xii) of proposition 2.4]
$= 1 \wedge (x \rightarrow x^{-}) = x \rightarrow x^{-} \in F$ [From (i) of proposition 2.4]
Therefore, $(x \vee x^{-}) \rightarrow x \in F$

Thus, $x \in F$. [From (ii) of definition 2.5]

(ii) Consider $(x \vee x^{\sim}) \rightsquigarrow x = (x \rightsquigarrow x) \wedge (x \rightsquigarrow x^{\sim})$ [From (xiii) of proposition 2.4]

$= 1 \wedge (x \rightsquigarrow x^{-}) = x \rightsquigarrow x^{\sim} \in F$ [From (i) of proposition 2.4]

Therefore, $(x \vee x^{\sim}) \rightsquigarrow x \in F$

Thus, $x \in F$. [From (iii) of definition 2.5]

Conversely, to prove $F$ be a pseudo-Boolean implicative filter of $A$

That is to prove that $x \vee x^{-} \in F$ and $x \vee x^{\sim} \in F$ for all $x \in A$

Let $\big((x \rightsquigarrow y) \to x\big) \rightsquigarrow x \in F$, $\big((x \to y) \rightsquigarrow x\big) \to x \in F$ and put $y = 0$

Then, we have $\big((x \rightsquigarrow 0) \to x\big) \rightsquigarrow x \in F$ and $\big((x \to 0) \rightsquigarrow x\big) \to x \in F$

Hence, $(x^{\sim} \to x) \rightsquigarrow x \in F$ and $(x^{-} \rightsquigarrow x) \to x \in F$

We have $x \vee x^{-} = (x \to x^{-}) \rightsquigarrow x^{-} = (x^{-} \to x) \rightsquigarrow x = (x \rightsquigarrow x^{-}) \to x^{-} = (x^{-} \rightsquigarrow x) \to x$

[From (ii) of definition 2.3]

Thus, $x \vee x^{-} \in F$. Similarly, we prove $x \vee x^{\sim} \in F$ for all $x \in A$. ■

***Proposition 3.1.5.*** Let $A$ be a lattice pseudo-Wajsberg algebra. If an implicative filter $F$ of $A$ is a pseudo-Boolean**,** then which satisfies the following inequalities

(i) $(x^{-} \to x) \rightsquigarrow x \in F$and $(x^{\sim} \to x) \rightsquigarrow x \in F$

(ii) $(x^{-} \rightsquigarrow x) \to x \in F$and $(x^{\sim} \rightsquigarrow x) \to x \in F$

(iii) If $(x \to y) \rightsquigarrow x \in F$ then $x \in F$ and if $(x \rightsquigarrow y) \to x \in F$ then $x \in F$

(iv) If $(x \to y) \rightsquigarrow y \in F$ imply $(y \rightsquigarrow x) \to x \in F$ and $(x \rightsquigarrow y) \to y \in F$ imply $(y \to x) \rightsquigarrow x \in F$

(v) If $(x \to y) \to y \in F$ imply $(y \to x) \rightsquigarrow x \in F$ and $(x \rightsquigarrow y) \rightsquigarrow y \in F$ imply $(y \rightsquigarrow x) \to x \in F$ for all $x, y \in A$.

**Proof.** (i) Let $F$ be a pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra $A$.

Since $x \leq (x^{-} \to x) \rightsquigarrow x$ and$x^{-} \leq (x^{-} \to x) \rightsquigarrow x$

We have $x \vee x^{-} \leq (x^{-} \to x) \rightsquigarrow x$ and hence $(x^{-} \to x) \rightsquigarrow x \in F$

Similarly, we prove $(x^{\sim} \to x) \rightsquigarrow x \in F$ for all $x, y \in A$.

(ii) Similarly we prove $(x^{-} \rightsquigarrow x) \to x \in F$ and $(x^{\sim} \rightsquigarrow x) \to x \in F$ for all $x, y \in A$.

(iii) Let $(x \to y) \rightsquigarrow x \in F$.

Then, from (i)(a) of definition 2.2, we have $1 \to \big((x \to y) \rightsquigarrow x\big) = (x \to y) \rightsquigarrow x \in F$

Since 1$\in F$ and $F$ is an implicative filter

Thus, $x \in F$ [From (iii) of definition 2.5]

Similarly to prove, if $(x \rightsquigarrow y) \to x \in F$, then $x \in F$

(iv) Let $(x \to y) \rightsquigarrow y \in F$.

Then, we have $(x \to y) \rightsquigarrow y \leq (y \to x) \rightsquigarrow \big((x \rightsquigarrow y) \to x\big)$ [From (vi) of proposition 2.4]

$= (x \rightsquigarrow y) \to \big((y \to x) \rightsquigarrow x\big)$ [From (viii) of proposition 2.4]

We have $x \leq (y \to x) \rightsquigarrow x$

$\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow y \leq x \rightsquigarrow y$ [From (iv) of proposition 2.4]

And $(x \rightsquigarrow y) \to \big((y \to x) \rightsquigarrow x\big) \leq \Big(\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow y\Big) \to \big((y \to x) \rightsquigarrow x\big)$

[From (vi) of proposition 2.4]

Then, $(x \rightsquigarrow y) \to y \leq \Big(\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow y\Big) \to \big((y \to x) \rightsquigarrow x\big)$

Thus, $\Big(\big((y \to x) \rightsquigarrow x\big) \rightsquigarrow y\Big) \to \big((y \to x) \rightsquigarrow x\big) \in F$ [From the proposition 2.7]

From (iii) we get $(y \to x) \rightsquigarrow x \in F$ for all $x, y \in A$. [From (ii) of definition 2.5]

(v) Similarly, we prove $(y \rightsquigarrow x) \to x \in F$ for all $x, y \in A$. ■

***Proposition 3.1.6.*** Let $A$ be a lattice pseudo-Wajsberg algebra. If an implicative filter $F$ of $A$ is a pseudo-Boolean if and only if $(x^{-} \to x) \rightsquigarrow x \in F$ and $(x^{\sim} \to x) \rightsquigarrow x \in F$ for all $x \in F$.

**Proof.** Let $F$ be a pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra $A$.

Since $x \leq (x^{-} \to x) \rightsquigarrow x$ and $x^{-} \leq (x^{-} \to x) \rightsquigarrow x$

We have $x \vee x^{-} \leq (x^{-} \to x) \rightsquigarrow x$ and hence $(x^{-} \to x) \rightsquigarrow x \in F$ [From (ii) of definition 2.3]

Similarly, we prove that $(x^{\sim} \to x) \rightsquigarrow x \in F$ for all $x, y \in A$.

Conversely, To prove an implicative filter $F$ be a pseudo-Boolean of $A$.
We have $\left((x \vee x^-)^- \rightsquigarrow (x \vee x^-)\right) \to x \vee x^- \in F$
Since $(x \vee x^-)^- = x^- \wedge x^{--}$ [From (xiv) of proposition 2.4]
$\leq x^- \leq x \vee x^-$
We have $(x \vee x^-)^- \rightsquigarrow (x \vee x^-) = 1$ [From (i) of proposition 2.2]
Thus, $x \vee x^- \in F$ for all $x, y \in A$.
Similarly, we prove $x \vee x^\sim \in F$ for all $x, y \in A$. ■

***Proposition 3.1.7.*** Let $A$ be lattice pseudo-Wajsberg algebra. If an implicative filter $F$ of $A$ is a pseudo-Boolean, then which satisfies the following inequalities
(i) $x \to (x \rightsquigarrow y) \in F$ implies $x \rightsquigarrow y \in F$ and
(ii) $x \rightsquigarrow (x \to y) \in F$ implies $x \to y \in F$ for all $x, y \in A$.
**Proof.** (i) Let $F$ be a pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra $A$.
Let $x \to (x \rightsquigarrow y) \in F$.
Then from (v) of proposition 2.4, we have $x \to (x \rightsquigarrow y) \leq \left((x \rightsquigarrow y) \to y\right) \to (x \rightsquigarrow y)$
$\left((x \rightsquigarrow y) \to y\right) \to (x \rightsquigarrow y) \in F$ [From the proposition 2.7]
Thus, $(x \rightsquigarrow y) \in F$ for all $x, y \in A$. [From (ii) of definition 2.5]
Similarly, we prove (ii). ■

### 3.2. Fuzzy pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra

In this section, we investigate property of fuzzy implicative filter of lattice pseudo-Wajsberg algebra and we introduce the notion of fuzzy pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra.

***Proposition 3.2.1.*** Any fuzzy subset $\mu$ of lattice pseudo-Wajsberg algebra $A$ is a fuzzy implicative filter of $A$ if and only if for each $t \in [0, 1]$, $\mu^t$ is either empty or an implicative filter of $A$.
***Proof.*** Let $\mu$ be a fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$ and $\mu^t \neq \emptyset$ for all $t \in [0, 1]$.
If $1 \in \mu^t$ then $\mu(1) \geq t$.
If $x \in \mu^t$, $x \to y \in \mu^t$ and $x \rightsquigarrow y \in \mu^t$ then we have $\mu(x \to y) \geq t$ and $\mu(x \rightsquigarrow y) \geq t$.
Thus, from (ii) and (iii) of the definition 2.5, we have $\mu(y) \geq \min\{(\mu(x \to y), \mu(x))\} \geq t$ and $\mu(y) \geq \min\{(\mu(x \rightsquigarrow y), \mu(x))\} \geq t$. Then, $y \in \mu^t$.
Hence $\mu^t$ is an implicative filter of $A$.
Conversely, if for each $t \in [0, 1]$, $\mu^t$ is either empty or an implicative filter of $A$.
To prove that $\mu$ is a fuzzy implicative filter of $A$.
**Case (i)** For any $x \in A$, let $\mu(x) = t$ then, we have $x \in \mu^t$.
Since $\mu^t \neq \emptyset$ is an implicative filter of $A$, $1 \in \mu^t$.
So $\mu(1) \geq t = \mu(x)$.
Thus, $\mu(1) \geq \mu(x)$ for all $x \in A$.
**Case (ii)** We prove that $\mu(y) \geq \min\{(\mu(x \to y), \mu(x))\}$. If not, there exist $x^-, y^- \in A$ such that $\mu(y^-) \geq \min\{(\mu(x^- \to y^-), \mu(x^-))\}$. Let $t_1 = \frac{1}{2}\left[\mu(y^-) + \min\{(\mu(x^- \to y^-), \mu(x^-))\}\right]$.
Then, we have $\mu(x^-) > t_1 > \min\{(\mu(x^- \to y^-), \mu(x^-))\}$.
Hence, we have $x^- \notin \mu_{t_1}$ and $x^- \to y^- \in \mu_{t_1}$ is not an implicative filter of $A$. Which is a contradiction to implicative filter of $A$.
**Case (iii)** We prove that $\mu(y) \geq \min\{(\mu(x \rightsquigarrow y), \mu(x))\}$. If not, there exist $x^\sim, y^\sim \in A$ such that $\mu(y^\sim) \geq \min\{(\mu(x^\sim \rightsquigarrow y^\sim), \mu(x^\sim))\}$. Let $t_2 = \frac{1}{2}\left[\mu(y^\sim) + \min\{(\mu(x^\sim \rightsquigarrow y^\sim), \mu(x^\sim))\}\right]$. Then, we have $\mu(x^\sim) > t_2 > \min\{(\mu(x^\sim \rightsquigarrow y^\sim), \mu(x^\sim))\}$. Hence, we have $x^\sim \notin \mu_{t_1}$ and $x^\sim \rightsquigarrow y^\sim \in \mu_{t_2}$ is not an implicative filter of $A$. Which is a contradiction to implicative filter of $A$. Therefore, $\mu$ is a fuzzy implicative filter of $A$. ■

***Definition 3.2.2.*** Let $A$ be lattice pseudo-Wajsberg algebra. A non-empty fuzzy implicative filter $\mu$ of lattice pseudo-Wajsberg algebra $A$ is said to be pseudo-Boolean if it satisfies the following axioms

(i) $\mu(x \vee x^{-}) = \mu(1)$

(ii) $\mu(x \vee x^{\sim}) = \mu(1)$ for all $x \in A$ .

***Example 3.2.3.*** Consider a set $A = \{0, a, b, c, 1\}$. Define a partial ordering "≤" on $A$, such that $0 \leq a \leq b$ , $c \leq 1$ and the binary operations " $\rightarrow$ " , " $\rightsquigarrow$ " and quasi complements " $^{-}$ " , " $^{\sim}$ " given by the following tables 3.2.1, 3.2.2, 3.2.3 and 3.2.4.

| $x$ | $x^{-}$ |
|---|---|
| 0 | 1 |
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $a$ |
| 1 | 0 |

Table:3.2.1
Complement

| $\rightarrow$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | 1 | 1 | 1 |
| $b$ | $c$ | $c$ | 1 | $c$ | 1 |
| $c$ | $a$ | $b$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

Table:3.2.2
Implication

| $x$ | $x^{\sim}$ |
|---|---|
| 0 | 1 |
| $a$ | $b$ |
| $b$ | $a$ |
| $c$ | $b$ |
| 1 | 0 |

Table:3.2.3
Complement

| $\rightsquigarrow$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | 1 | 1 | 1 |
| $b$ | $a$ | $c$ | 1 | $c$ | 1 |
| $c$ | $b$ | $b$ | $c$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

Table:3.2.4
Implication

Consider the fuzzy subset $\mu$ on $A$ as, $\mu(x) = \begin{cases} 0.8 & if \quad x = 0,1 \\ 0.2 & \text{Otherwise} \end{cases}$ for all $x \in A$. Then, we have $\mu$ is fuzzy pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra $A$.

In the same Example 3.2.3, let us consider a fuzzy pseudo-Boolean implicative filter on $A$ as,
$\mu(x) = \begin{cases} 0.5 & if \quad x = 0,1 \\ 0.3 & Otherwise \end{cases}$ for all $x \in A$.
Since $\mu(b \vee a) = \mu\big((b \rightarrow a) \rightsquigarrow a\big) = \mu\big((a \rightarrow b) \rightsquigarrow b\big) = \mu\big((a \rightsquigarrow b) \rightarrow b\big) = \mu((b \rightsquigarrow a) \rightarrow a)$

$$\mu(b) = \mu(b) = \mu(a) = \mu(b) \neq \mu(1)$$

0.3=0.3=0.3 =0.3≠ 0.5

Then, $\mu$ is not a fuzzy pseudo-Boolean implicative filter of $A$.

***Proposition 3.2.4.*** Let $A$ be lattice pseudo-Wajsberg algebra. If an implicative filter $\mu$ of $A$ is a fuzzy pseudo-Boolean if and only if for each $t \in [0, 1]$, $\mu_t$ is either empty or an implicative filter of $A$.

**Proof.** Let $\mu$ be a fuzzy pseudo-Boolean implicative filter of $A$ and for each $t \in [0, 1]$, $\mu_t \neq \emptyset$.

Then, $\mu_t$ is an implicative filter of $A$. [From the proposition3.2.1]

Thus $1 \in \mu_t$, since $\mu_t(1) \geq t$

$\mu(x \vee x^{-}) = \mu(x \vee x^{\sim}) = \mu(1) \geq t$ for all $x \in A$ [From the definition 3.2.2]

That is, $x \vee x^{-} \in \mu_t$ and $x \vee x^{\sim} \in \mu_t$

Therefore $\mu_t$ is a pseudo-Boolean implicative filter of $A$.

Conversely, suppose that $\mu$ is a fuzzy implicative filter of $A$ and for each $t \in [0, 1]$, $\mu_t$ is either empty or an implicative filter of $A$.

Since $1 \in \mu_{\mu(1)}$, this means that $\mu_{\mu(1)}$ is pseudo-Boolean implicative filter of $A$.

Thus $x \vee x^{-} \in \mu_{\mu(1)}$ and $x \vee x^{\sim} \in \mu_{\mu(1)}$ for all $x \in A$

That is, $\mu(x \vee x^{-}) = \mu(1)$ and $\mu(x \vee x^{\sim}) = \mu(1)$ for all $x \in A$

From the definition 3.2.2, we have $\mu$ is a fuzzy pseudo-Boolean implicative filter of $A$.■

***Proposition 3.2.5.*** Let $A$ be lattice pseudo-Wajsberg algebra. Let $\mu, \vartheta$ be two fuzzy implicative filters in $A$ which satisfy $\mu \leq \vartheta$, $\mu(1) = \vartheta(1)$. If $\mu$ is a fuzzy pseudo-Boolean implicative filter of $A$, so is $\vartheta$.

**Proof.** Let $A$ be a lattice pseudo-Wajsberg algebra and let $\vartheta$ fuzzy implicative filter of $A$.

Let $\mu$ is a fuzzy pseudo-Boolean implicative filter of $A$.

From the definition of 3.2.2, we have $\mu(x \vee x^{-}) = \mu(1)$ and $\mu(x \vee x^{\sim}) = \mu(1)$ for all $x \in A$

From $\mu \leq \vartheta$and $\mu(1) = \vartheta(1)$, it follows that $\vartheta(x \vee x^{-}) \geq \vartheta(1)$ and $\vartheta(x \vee x^{\sim}) \geq \vartheta(1)$.

From (i) of the definition 2.9, we have $\vartheta(x \vee x^{-}) = \vartheta(1)$ and $\vartheta(x \vee x^{\sim}) = \vartheta(1)$ for all $x \in A$. Thus $\vartheta$ is a fuzzy pseudo-Boolean implicative filter of $A$.■

***Proposition 3.2.6.*** Let $A$ be a lattice pseudo-Wajsberg algebra. If $\mu$ is a fuzzy pseudo-Boolean implicative filter of $A$ then which satisfies the following inequalities

(i) $\mu(x \to z) \geq \min\{\mu(x \rightsquigarrow (z^{-} \to y)), \mu(y \to z)\}$

(ii) $\mu(x \rightsquigarrow z) \geq \min\{\mu(x \to (z^{\sim} \rightsquigarrow y)), \mu(y \rightsquigarrow z)\}$ for all $x, y, z \in A$.

**Proof.** Let $A$ be a lattice pseudo-Wajsberg algebra and

Let $y \to z \leq (z^{-} \to y) \rightsquigarrow (z^{-} \to z)$ [From (vi) of the proposition 2.4]

From (vii) of the proposition 2.4,we have

$$\leq (x \rightsquigarrow (z^{-} \to y)) \rightsquigarrow (x \rightsquigarrow (z^{-} \to z)) \qquad \textbf{(3.2.1)}$$

We have $\mu(y \to z) \leq \mu\left((x \rightsquigarrow (z^{-} \to y)) \rightsquigarrow (x \rightsquigarrow (z^{-} \to z))\right)$ [From the proposition 2.8]

And let $\mu(x \rightsquigarrow (z^{-} \to z)) \geq \min\left\{\mu(x \rightsquigarrow (z^{-} \to y)), \mu\left((x \rightsquigarrow (z^{-} \to y)) \rightsquigarrow (x \rightsquigarrow (z^{-} \to z))\right)\right\}$

[From (iii) of the definition 2.9]

From the equation **3.2.1**, we have $\mu(x \rightsquigarrow (z^{-} \to z)) \geq \min\{\mu(x \rightsquigarrow (z^{-} \to y)), \mu(y \to z)\}$ **(3.2.2)**

Since $z \vee z^{-} = ((z^{-} \to z) \rightsquigarrow z) \wedge ((z \to z^{-}) \rightsquigarrow z^{-}) \leq (z^{-} \to z) \rightsquigarrow z$

From the proposition 2.8 and (i) of the definition 3.2.2, we have

$$\mu((z^{-} \to z) \rightsquigarrow z) \geq \mu(z \vee z^{-}) = \mu(1)$$

Again $x \rightsquigarrow (z^{-} \to z) \leq \left(((z^{-} \to z) \rightsquigarrow z) \to (x \rightsquigarrow z)\right)$ [From (vi) ofthe proposition 2.4]

$\mu(x \rightsquigarrow (z^{-} \to z)) \leq \mu\left(\left(((z^{-} \to z) \rightsquigarrow z) \to (x \rightsquigarrow z)\right)\right)$ [From the proposition 2.8]

Now, $\mu(x \to z) \geq \min\{\mu((z^{-} \to z) \rightsquigarrow z), \mu((z^{-} \to z) \rightsquigarrow z) \to (x \rightsquigarrow z)\}$

[From (ii) of the definition 2.9]

From (xx), (i) and (vi) of the proposition 2.4, we have

$$\geq \min\{\mu(1), \mu(x \rightsquigarrow (z^{-} \to z))\} = \mu(x \rightsquigarrow (z^{-} \to z))$$

From the equation 3.2.2**,** we have $\mu(x \to z) \geq \min\{\mu(x \rightsquigarrow (z^{-} \to y)), \mu(y \to z)\}$

Hence $\mu(x \to z) \geq \min\{\mu(x \rightsquigarrow (z^{-} \to y)), \mu(y \to z)\}$ for all $x, y \in A$.

Similarly we prove $\mu(x \rightsquigarrow z) \geq \min\{\mu(x \to (z^{\sim} \rightsquigarrow y)), \mu(y \rightsquigarrow z)\}$ for all $x, y \in A$. ■

***Proposition 3.2.7.*** Let $A$ be a lattice pseudo-Wajsberg algebra. If $\mu$ satisfy the following inequalities $\mu(x) \geq \mu((x^{-} \rightsquigarrow x) \to x)$and $\mu(x) \geq \mu((x^{\sim} \to x) \rightsquigarrow x)$ for all $x, y \in A$. Then $\mu$ is a fuzzy pseudo-Boolean implicative filter of $A$.

**Proof.** Let $A$ be a lattice pseudo-Wajsberg algebra.

Let $1 = x \rightsquigarrow ((x^{-} \rightsquigarrow x) \to x)$ [From (ii) of proposition 2.4]

$\leq ((x^{-} \rightsquigarrow x) \to x)^{-} \to x^{-}$

$\leq (x^{-} \rightsquigarrow x) \rightsquigarrow (((x^{-} \rightsquigarrow x) \to x)^{-} \to x)$ [From (vi) of proposition 2.4]

$= ((x^{-} \rightsquigarrow x) \to x)^{-} \rightsquigarrow ((x^{-} \to x) \to x)$ [From (viii) of proposition 2.4]

$= (((x^{-} \rightsquigarrow x) \to x) \to 0) \rightsquigarrow ((x^{-} \to x) \to x)$ [From (xviii) of proposition 2.4]

From the proposition 2.8,(i) of the definition 2.9 and $\mu(x) \geq \mu((x^{-} \rightsquigarrow x) \to x)$, we have

$$\mu((x^{-} \rightsquigarrow x) \to x) \geq \mu\left((((x^{-} \rightsquigarrow x) \to x) \to 0) \rightsquigarrow ((x^{-} \rightsquigarrow x) \to x)\right) = \mu(1)$$

$$\mu((x^{-} \rightsquigarrow x) \to x) = \mu(1) \qquad \textbf{(3.2.3)}$$

Now,$((x^{-} \rightsquigarrow x) \to x) \leq ((x^{-} \rightsquigarrow x) \to x) \vee ((x^{-} \rightsquigarrow x) \to x^{-})$ [From (ix) of proposition 2.4]

$= (x^{-} \rightsquigarrow x) \to (x \vee x^{-})$ [From (ix) of proposition 2.4]

$= (1 \wedge (x^{-} \rightsquigarrow x)) \to (x \vee x^{-})$

$= ((x \rightsquigarrow x) \wedge (x^{-} \rightsquigarrow x)) \to (x \vee x^{-})$ [From (i) of proposition 2.4]

$= ((x \vee x^{-}) \rightsquigarrow x) \to (x \vee x^{-})$ [From (xiii) of proposition 2.4]

From the proposition 2.8, we have

$\mu((x^{-} \rightsquigarrow x) \to x) \leq \mu\left(((x \vee x^{-}) \rightsquigarrow x) \to (x \vee x^{-})\right) \leq \mu(x \vee x^{-})$ **(3.2.4)**

From the equation **3.2.3** and **3.2.4**, we get $\mu(x \vee x^{-}) = \mu(1)$ for all $x \in A$.
Similarly, we prove that $\mu(x \vee x^{\sim}) = \mu(1)$ for all $x, x^{-}, x^{\sim} \in A$
Therefore, $\mu$ is a fuzzy pseudo-Boolean implicative filter of $A$.■

***Proposition 3.2.8.*** Let $A$ be a lattice pseudo-Wajsberg algebra. If $\mu$ is a fuzzy pseudo-Boolean implicative filter of $A$ then $\mu$satisfies the following inequalities

(i) $\mu(x \to z) \geq \min\{\mu(x \rightsquigarrow (z^{-} \to y)), \mu(y \to z)\}$implies $\mu(x) \geq \mu((x^{-} \rightsquigarrow x) \to x)$
(ii)$\mu(x \rightsquigarrow z) \geq \min\{\mu(x \to (z^{\sim} \rightsquigarrow y)), \mu(y \rightsquigarrow z)\}$ implies $\mu(x) \geq \mu((x^{\sim} \to x) \rightsquigarrow x)$
for all $x, y, z \in A$.

**Proof.** Let $A$ be a lattice pseudo-Wajsberg algebra.
Let $(x^{-} \rightsquigarrow x) \to x \leq x^{-} \to x$
From the proposition 2.8 and (i)(a) of the definition 2.2,
We have $\mu(x) = \mu(1 \to x)$

$\geq \min\{\mu(1 \to (x^{-} \rightsquigarrow x^{-})), \mu(x^{-} \to x)\}$ [From (ii) of the definition 2.9]

$\geq \min\{\mu(1), \mu((x^{-} \rightsquigarrow x) \to x)\}$ [From (i) and (xx) of proposition 2.4]

$= \mu((x^{-} \rightsquigarrow x) \to x)$

Hence $\mu(x) \geq \mu((x^{-} \rightsquigarrow x) \to x)$ for all $x, y, z \in A$.
Similarly, we prove (ii). ■

***Proposition 3.2.9.*** Let $A$ be lattice pseudo-Wajsberg algebra. Every fuzzy pseudo-Boolean implicative filter $\mu$ of $A$ satisfies the following inequalities

(i) $\mu(x \to z) \geq \min\{\mu(x \rightsquigarrow (y \to z)), \mu(x \to y)\}$
(ii) $\mu(x \rightsquigarrow z) \geq \min\{\mu(x \to (y \rightsquigarrow z)), \mu(x \rightsquigarrow y)\}$ for all $x, y, z \in A$.

**Proof.** Let $A$ be a lattice pseudo-Wajsberg algebraand let $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$
[From (viii) of proposition 2.4]

$\leq (x \to y) \rightsquigarrow (x \to (x \rightsquigarrow z))$ [From (vii) of proposition 2.4] **(3.2.5)**

Now, $x \rightsquigarrow (x \to z) \leq x \rightsquigarrow \left(((x \to z) \to z) \to z\right)$ [From (i) of proposition 2.10]

$= ((x \to z) \to z) \to (x \rightsquigarrow z)$ [From (viii) of proposition 2.4] **(3.2.6)**

From the proposition 2.8, (ii) of the definition 2.9 and $\mu(x) \geq \mu((x^{-} \rightsquigarrow x) \to x)$ ,
we have $\mu(x \to z) \geq \mu\left(((x \to z) \to z) \to (x \rightsquigarrow z)\right) \geq \mu(x \rightsquigarrow (x \to z))$ [From the equation 3.2.6]

$\geq \min\left\{\mu\left((x \to y) \rightsquigarrow (x \to (x \rightsquigarrow z))\right), \mu(x \to y)\right\}$ [ From (iii) of the definition 2.9]

$\mu(x \to z) \geq \min\{\mu(x \rightsquigarrow (y \to z)), \mu(x \to y)\}$ [From the equation 3.2.5]

Similarly, we prove $\mu(x \rightsquigarrow z) \geq \min\{\mu(x \to (y \rightsquigarrow z)), \mu(x \rightsquigarrow y)\}$ for all $x, y, z \in A$. ■

## 4. Conclusion

In this paper, we have introduced the notion of pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra and discussed some properties with illustrations. Further, we have introduced fuzzy pseudo-Boolean implicative filter of lattice pseudo-Wajsberg algebra. Then, we have obtained the concepts of pseudo-Boolean implicative filter, fuzzy pseudo-Boolean implicative filters of lattice pseudo-Wajsberg algebra.

## References


[1] Basheer Ahamed, M., and Ibrahim, A., *Fuzzy implicative filters of lattice Wajsberg Algebras*, Advances in Fuzzy Mathematics 6 (2) (2011), 235-243.

[2] Ceterchi Rodica, *The Lattice Structure of Pseudo-Wajsberg Algebras*, Journal of universal Computer Science, 6 (2000), 22-38.
[3] Font, J. M., Rodriguez, A. J., and Torrens, A., *Wajsberg algebras*, Stochastica, 8 (1984) 5-31.
[4] Ibrahim, A., and Jeya Lekshmi, K., *Implicative Filters of lattice Pseudo-Wajsberg Algebras*, Global journal of Pure and Applied Mathematics, 14 (2018), 1-15.
[5] Ibrahim, A., and Jeya Lekshmi, K., *Fuzzy Implicative Filters of lattice Pseudo-Wajsberg Algebras*, International Journal of mathematical Archive, 5 (2018), 42-48.
[6] Wajsberg, M., *Beiträge zum Metaaussagenkalkül I*, Monat. Mat. Phys. 42, (1935), 221-242.
[7] Xiao-zhang Zhang, Weihua Li, *On pseudo-BL algebras and BCC-algebras*, Soft Computer 10(2006), 941-952.
[8] Young Bae Jun, Yang Xu, Xiao Hong Zhang, *Fuzzy Filters of MTL-algebras*, Information Science an international journal, 175 (2005) 120-138.
[9] Zadeh, L. A., *Fuzzy sets*, Information Control 8 (1965), 338-353.