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ON COMMUTATIVITY OF PRIME NEAR-RINGS WITH GENERALIZED DERIVATIONS

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Abstract: Let R be a prime near-ring. The commutativity of R satisfying the conditions:

$$(i) D([x,y]) = \pm x^k [x^m, y] x^l$$
$$(ii) D(x \circ y) = \pm x^k (x^m \circ y) x^l$$

where $k \geq 0, l \geq 0, m \geq 1$, are fixed integers is studied. Further, some interesting relations between the prime graph and Zero-divisor graph of R are studied.

1.Introduction

Let R be a right near-ring. R is called zero-symmetric if $x \circ 0 = 0\ \forall x \in R$ (Recall that in a right near-ring R, $0.x = 0$ is true for all $x \in R$). A near ring R is said to be prime if

$$xRy = 0\ for x, y \in R\ implies\ x = 0\ (or) y = 0\ [Here : xRy = \{xry \backslash r \in R\}$$

An endomorphism d of R is called a derivation if

$$(i) d(x+y) = d(x) + d(y)\ and$$
$$(ii) d(xy) = xd(y) + d(x)y\ for\ all\ x, y \in R$$

Implies $x = 0\ (or) y = 0\ [Here: xRy = \{xry \backslash r \in R\}]$

An endomorphism D of R is called a generalized endomorphism associated with a non-zero derivation d of R, if

$$(i) D(x+y) = D(x) + D(y)\ and\ (ii) D(xy) = D(x)y + xd(y) \forall x, y \in R$$

Let $Z(R)$ denote the centre of R. For all $x, y \in R$, $let [x,y] = xy - yx$, called the commutator of x and y and $x \circ y = xy + yx$, called the anti-commutator of x and y. In [11] the authors showed that a prime ring R must be commutative if R admits a derivation d such that $d([x,y]) = [x,y]\ or$ $d([x,y]) = [x,y]\ or\ d([x,y]) = -[x,y]\ \ for\ all\ x, y \in I$ where I is non-zero ideal of R. In [15] Yilun shang proved that a prime near-ring R which admits a generalized derivation D associated with a non-zero derivation d satisfying either

$$(i) D([x,y]) = x^k [x,y] x^l\ for\ all\ x, y \in R \text{ (or)}$$
$$(ii) D([x,y]) = -x^k [x,y] x^l\ for\ all\ x, y \in R$$

Then R is a commutative ring

He also proved that if R is a prime near-ring which admits a generalized derivation D associated witha non-zero derivation d satisfying either

$$(i) D(x \circ y) = x^k (x \circ y) x^l \ for\ all\ x, y \in R$$
$$\text{or } (ii) D(x \circ y) = -x^k (x \circ y) x^l \ for\ all\ x, y \in R$$

then R is a commutative ring.

In this paper we investigate the commutativity of a prime near ring$R$ satisfying the following conditions.

$$(i) D([x, y]) = \pm x^k [x^m, y] x^l \ for\ all\ x, y \in R$$
$$(ii) D(x \circ y) = \pm x^k (x^m \circ y) x^l \ for\ all\ x, y \in R$$

Where $k \geq 0, l \geq 0, m \geq 1$are fixed integers.

Lemma 1.1 [6]

Let R be a prime near-ring. If R admits a non-zero derivation d for which$d(R) \subset z(R)$, then R is a commutative ring.

2. Main results

Through out this paper R denote a prime near-ring(right). $Z(R)$ denote the centre of R.Let$R^m = \{x^m | x \in R\}$

Theorem 2.1

Let R be a prime near-ring. If thereexists integers$k \geq 0, l \geq 0, m \geq 1$ such that R admits a generalized derivation D associated with a non-zero derivation d satisfying either

$(i) D([x, y]) = x^k [x^m, y] x^l forall x, y \in R$(or)

$(ii) D([x, y]) = -x^k [x^m, y] x^l forall x, y \in R$

then R is a commutative ring.

Proof: We first assume that (i) holds

(ie) $D([x, y]) = x^k [x^m, y] x^l \forall x, y \in R$……………………………………………….(1)

Replace $y by yx$ in(1)

$D([x, yx]) = x^k [x^m, yx] x^l \forall x, y \in R$……………………………………………(2)

Since $[x, yx] = [x, y]x \forall x, y \in R$, (2) becomes

$D([x, y]x) = x^k [x^m, y] x^{l+1} \forall x, y \in R$……………………………………………..(3)

By definition we have:

$$D([x, y]x = D([x, y])x + [x, y]d(x)$$
$$\text{(ie)} x^k [x^m, y] x^{l+1} = D([x, y])x + [x, y]d(x) \qquad \text{(using (3))}$$
$$x^k [x^m, y] x^{l+1} = (x^k [x^m, y] x^l)x + [x, y]\, d(x) \qquad \text{(using (1))}$$
$$x^k [x^m, y] x^{l+1} = x^k [x^m, y] x^{l+1} + [x, y]d(x)$$
$$\Rightarrow x^k [x^m, y] x^{l+1} - x^k [x^m, y] x^{l+1} = [x, y]d(x)$$

$\Rightarrow [x, y]d(x) = 0$…………………………………………………………..…………(4)

Replacing y by zy we have:

$$0 = [x, zy]d(x) = \{z[x, y] + [x, z]y\}d(x)$$
$$= z[x, y]d(x) + [x, z]yd(x)$$
$$= [x, z]yd(x) \text{ using (4) } \forall x, y, z \in R \text{………………………..(5)}$$

This implies: $[x, z]Rd(x) = 0 \ \ \forall x, z \in R$…………………………………………………(6)

Since R is prime (6) yields that for eachx $\in R$

$d(x) = 0 \quad (or)[x, z] = 0 \qquad \forall z \in R$

(ie) for each $x \in R, d(x) = 0 \ (or)\ x \in z(R)$……………………………………………..(7)

$If x \in z(R)$. then $xy = yx$ for all $y \in R$

Then $d(xy) = d(yx)$

$$d(x)y + xd(y) = d(y)x + yd(x)$$
$$d(x)y + xd(y) = xd(y) + yd(x) \because x \in z(R)$$

$d(x)y = yd(x)$ for all y $\in R$, $(x) \in z(R)$
Thus: $x \in z(R) => d(x) \in z(R)$……………………………………………………………(8)
So, by (7) and(8) we get that

$d(x) \in z(R), \forall x \in R$………………………………………………………………………(9)

(ie) $d(R) \subset z(R)$
Then by Lemma 1.1. , R is a commutative ring.
For (ii) we assumethat it holds:

$D([x,y]) = -x^k[x^m,y]x^l \forall x,y \in R$……………………………………….(10)

Replace y b yyx in (10)

$D([x,yx]) = -x^k[x^m,yx]x^l \forall x,y \in R$……………………………………........(11)

Since $[x,yx] = [x,y]x \forall x,y \in R$ (11) becomes

$D([x,y]x) = -x^k[x^m,y]x^{l+1} \forall x,y \in R$…………………………………………(12)

By definition, we have:

$$D([x,y]x) = D([x,y])x + [x,y]\, d(x)$$
$$-x^k[x^m,y]x^{l+1} = -x^k([x^m,y]x^l)x + [x,y]d(x) \qquad \text{(using (10))}$$
$$-x^k[x^m,y]x^{l+1} = -x^k[x^m,y]x^{l+1} + [x,y]d(x)$$
$$-x^k[x^m,y]x^{l+1} + x^k[x^m,y]x^{l+1} = [x,y]d(x)$$

$\Rightarrow [x,y]d(x) = 0$ $\forall x,y \in R$…………………………………………………..(13)

Replacing y b yzy , we have:

$$0 = [x,zy]d(x) = \{z[x,y] + [x,z]y\}d(x)$$
$$= z[x,y]d(x) + [x,z]yd(x)$$

$= [x,z]yd(x)$ using (13) $\forall x,y,z \in R$……………(14)

This implies $[x,z]Rd(x) = 0$ $\forall x,z \in R$.………………..*(15)*

Since R is prime (15) yields that for each $x \in Rd(x) = 0$ $(or)[x,z] = 0$ $\forall z \in R$

$(ie)$for each $x \in Rd(x) = 0$ $(or) x \in z(R)$…………………………………….(16)

Now $x \in z(R)$ then $d(x) \in z(R)$…………………………………………………(17)

So, by (16) and (17) we get that

$$d(x) \in z(R) \forall x \epsilon R$$
$$(ie) d(R) \subset z(R)$$

Then by Lemma1.1. R is a commutative ring.

Note: If $m = 1$, we get Theorem 1[15]

Definition 2.2. *[9]*

The prime graph of a near-ring R denoted by $G(R)$ is a graph with vertices as the set of elements of R and edges as the set of vertex pair $\{x,y\}$ such that $xRy = 0$ $oryRx = 0$,It is easy to check that R is prime if and only if $G(R)$ is a star graph.

Definition 2.3 *[9]*

The Zero-divisor graph of a commutative ring $R$is a graph with the set of non-zero zero divisors of R as the vertices and any two vertices x,y are adjacent if and only if$x \neq y and xy = 0$

Corrollary 2.4

Let R be a prime near-ring. If the prime graph $G(R)$ is a star and there exist $k,l,m \epsilon N$ such that R admits a generalized derivation d satisfying either (i) (or) (ii) of Theorem 2.1 then the zero divisor graph of R is a sub graph of $G(R)$

Remark 2.5 : The condition R is prime in Theorem 2.1 is necessary even in the case of arbitrary rings as seen in the following example.

Example 2.6.

Let R be a Commutative ring.

$Let R^* = \left\{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in R.\right\}$ Then R^* is a ring with respect to usual matrix addition and matrix multiplication.
Define d:$R^* \to R^* by$

$d\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, clearly d is an on − zero derivationon $R^*, If A = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ where $y \neq 0$,

then $AR^*A = 0$, which provesR^* is not Prime. Define$D: R^* \to R^* as D\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & y+z \\ 0 & 0 \end{pmatrix}$

We shall show that D is a generalized derivation on R^* with an associated derivation d on R^*.

Let $A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R^*$, $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R^*$. Then $AB = \begin{pmatrix} xa & xb+yc \\ 0 & zc \end{pmatrix}$ and

$$D(AB) = \begin{pmatrix} 0 & xb+yc+zc \\ 0 & 0 \end{pmatrix}$$

Also, $D(A)B + Ad(B) = \begin{pmatrix} 0 & y+z \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & yc+zc+xb \\ 0 & 0 \end{pmatrix}$. Hence $D(AB) = D(A)B + Ad(B)$. Then $D([A,B]) = [A,B]$ for all $A, B \in R^*$. But R^* is a non-commutative ring.

Theorem 2.7

Let R be a prime near ring. If there exist integers $k \geq 0, l \geq 0, m \geq 1$, such that R admits a generalized derivation D associated with a non-zero derivation d satisfying either

$$(i) D(x \circ y) = x^k (x^m \circ y) x^l \text{ for all } x, y \in R (or)$$
$$(ii) D(x \circ y) = -x^k (x^m \circ y) x^l \text{ for all } x, y \in R,$$

Then R is a commutative ring.

Proof: We first assume that (i) holds.

$$D(x \circ y) = x^k (x^m \circ y) x^l \forall x, y \in R \ldots\ldots(18)$$

Replace y by yx in (18)

$$D(x \circ yx) = x^k (x^m \circ yx) x^l \forall x, y \in R \ldots\ldots(19)$$

Since $(x \circ yx) = (x \circ y)x, \forall x, y \in R$, (19) becomes

$$D(x \circ y)x = x^k (x^m \circ y) x^{l+1} \forall x, y \in R \ldots\ldots(20)$$

By definition we have:

$$D(x \circ y)x = D(x \circ y)x + (x \circ y)d(x)$$
$$\text{(ie) } x^k (x^m \circ y) x^{l+1} = D(x \circ y)x + (x \circ y)d(x) \quad (\text{using } (20))$$
$$x^k (x^m \circ y) x^{l+1} = x^k (x^m \circ y) x^{l+1} + (x \circ y)d(x) \text{ using } (18)$$
$$x^k (x^m \circ y) x^{l+1} = x^k (x^m \circ y) x^{l+1} + (x \circ y)d(x)$$
$$\Rightarrow x^k (x^m \circ y) x^{l+1} - x^k (x^m \circ y) x^{l+1} = (x \circ y)d(x)$$
$$\Rightarrow (x \circ y)d(x) = 0 \ldots\ldots(21)$$

Replacing y by zy, we have:

$$0 = (x \circ zy)\, d(x) = \{z(x \circ y) + (x \circ z)y\}\, d(x)$$
$$= z(x \circ y)\, d(x) + (x \circ z)yd(x)$$
$$= (x \circ z)yd(x) \text{ using } (21) \quad \forall x, y, z \in R \text{------------------}(22)$$

This implies: $(x \circ z)Rd(x) = 0 \quad \forall x, z \in R \ldots\ldots(23)$

Since R is prime(23)yields that for each $x \in R\ d(x) = 0\ (or)(x \circ z) = 0 \quad \forall z \in R$

$(ie)$for each $x \in d(x) = 0$ (or) $x \in z(R)$ ……(24)

Here $z(R)$ is the centre of R. f $x \in z(R)$ then $xy = yx$, then $d(xy) = d(yx)$,

$$d(x)y + xd(y) = d(y)x + yd(x)$$
$$(ie) d(x)y + xd(y) = xd(y) + yd(x) \because x \in z(R)$$
$$d(x)y = yd(x) \text{for all } y \in R$$
$$d(x) \in z(R)$$

Thus: $x \in z(R) \Rightarrow d(x) \in z(R) \ldots\ldots(25)$

So, by (24) and (25) we get that:

$$d(x) \in z(R) \forall x \in R \ldots\ldots(26)$$
$$(ie)\ d(R) \subset z(R)$$

Then by Lemma 1.1 . R is a commutative ring

For (ii) we assume that it holds.

$D(x \circ yx) = -x^k(x^m \circ y)x^l \forall x, y \in R$……………………………………………..(27)

Replace y by yx in (27)

$D(x \circ yx) = -x^k(x^m \circ yx)x^l \forall x, y \in R$……………………………………………..(28)

Since $(x \circ yx) = (x \circ y)x, \forall x, y \in R$, (28) becomes

$D((x \circ y)x) = -x^k(x^m \circ y)x^{l+1} \forall x, y \in R$……………………………………………(29)

By definition we have:

$$D((x \circ y)x) = D(x \circ y)x + (x \circ y)d(x)$$

$$-x^k(x^m \circ y)x^{l+1} = -x^k\left((x^m \circ y)x^l\right)x + (x \circ y)d(x) \text{ using (27)}$$

$$-x^k(x^m \circ y)x^{l+1} = -x^k(x^m \circ y)x^{l+1} + (x \circ y)d(x)$$

$$-x^k(x^m \circ y)x^{l+1} + x^k(x^m \circ y)x^{l+1} = (x \circ y)d(x)$$

$\Rightarrow (x \circ y)d(x) = 0 \quad \forall x, y \in R$…………………………………………………………..(30)

Replacing y by zy we have:

$$0 = (x \circ zy)d(x) = \{z(x \circ y) + (x \circ y)y\}d(x)$$

$$= z(x \circ y)d(x) + (x \circ z)yd(x)$$

$= (x \circ z)yd(x)$ using (30) $\forall x, y, z \in R$……………………………………………(31)

This implies $(x \circ z)Rd(x) = 0 \quad \forall x, z \in R$………………………………………………..(32)

Since R is prime (32) yields that for each:

$$x \in Rd(x) = 0\ (or)(x, z) = 0 \qquad \forall z \in R$$

(ie for each $x \in Rd(x) = 0\ \ (or)\ x\epsilon z(R)$………………………………………………..(33)

Now $x \in z(R)$ then $d(x) \in z(R)$……………………………………………………..(34)

So, by (33) and (34) we get:

$$d(x) \in z(R) \forall x \in R, (ie)\ d(R) \subset z(R)$$

Then by Lemma 1.1. R is a commutative ring.

Remark 2.8 If $m = 1$, we get Theorem 2[15]

Corollary 2.9

Let R be a prime near-ring. If the prime graph $G(R)$ is a star and there exist $k, l, m \in N$ such that N admits a generalized derivation D associated with a non-zero derivation d satisfying either (i) (or) (ii) in Theorm 2.8, then the zero divisor graph of R is sub graph of $G(R)$.

Remark 2.10The condition R is prime in Theorem 2.7 is necessary even in the case of arbitrary rings as seen in the following example.

Example 2.11.

Let S be a non-commutative ring in which the square of each elements is zero.

Let $R = \left\{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| x, y, z \in S\right\}$. Defined: $R \to R$ as: $d\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$

Then d is a derivation on R. Define $D\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & y+z \\ 0 & 0 \end{pmatrix}$.

Then D is a generalized derivation with association derivation d. As already stated$R$ is not prime. For any $x, y \in S$,we have

$$0 = (x + y)^2 = x^2 + xy + yx + y^2 = xy + yx$$

So, $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \circ \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} 0 & xv + yw + uy + vz \\ 0 & 0 \end{pmatrix}$ for all $x, y, z, u, v, w \in S$

Consequently $D(A \circ B) = (A \circ B)$ for all $A, B \in R$. But R is non - commutative ring.

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