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# CONNECTEDNESS ON FUZZY BITOPOLOGICAL SPACES

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**Abstract:** In this paper we introduce the concept of connectedness in fuzzy bitopological spaces. We define (i-j) fuzzy connected set and study some of its properties. We also investigate the idea of connectedness in fuzzy bitopological spaces with respect to fuzzy ideals by an example.

**Keywords** : fuzzy bitopological spaces , (i-j) fuzzy connected, (i-j) fuzzy $\mathcal{J}-$ connected.

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## 1. Introduction

In this paper we follow the definition of fuzzy bitopological spaces in Lowen's [1] sense. Various results in fuzzy topological spaces with respect to fuzzy ideals are found in [7]. Based on the results given in [7],we are going to investigate the concept of Connectedness in fuzzy bitopological spaces in our present work.

## 2. Preliminaries

***Definition 2.1.*** [7] Let A and B are two fuzzy sets of X. Then A intersection B is defined as:

$$(A \cap B)(x) = \max\{0, A(x) + B(x) - 1\} \text{ for all } x \in X.$$

***Definition 2.2.*** [7] Let $(X,\tau)$ be a fuzzy topological space. The cl(A), the closure of a fuzzy set A is a fuzzy set defined by $cl(A)(x) = \vee\{\lambda / B \in \tau, B(x) > 1 - \lambda \Rightarrow A \cap B \neq \overline{0}\}$ for all $x \in X$.

***Definition 2.3.*** [7] A non-empty collection $\mathcal{J}$ of fuzzy sets of $X$ is said to be a fuzzy ideal on X, if $A, B \in \mathcal{J} \Rightarrow A \vee B \in \mathcal{J}$ and $A \in \mathcal{J}$, $B \leq A \Rightarrow B \in \mathcal{J}$.

***Definition 2.4.***[7] If A is a fuzzy set of X, then the support of A is defined as $S(A) = \{x \in X / A(x) > 0\}$. Let $X$ be a non-empty set. A fuzzy set B is said to be finite fuzzy set of X if and only if $S(B)$ is a finite set.

***Definition 2.5.***[7] Let (X,τ) be a fuzzy topological space. The interior $A^0$ of a fuzzy set A of X is defined as $A^0 = \vee\{B: B \leq A, B \in \tau\}$.

***Lemma 2.6.*** [7] If $g: X \to Y$ and $A, B \in I^Y$ then $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$.

**Lemma 2.7.**[7] Let (X,τ) and $(Y, \sigma)$ be two fuzzy topological spaces. Let $g: X \to Y$ be a fuzzy continuous function and $A: Y \to [0,1]$ be a fuzzy set of Y. Then $cl\left(g^{-1}(A)\right)(x) \leq g^{-1}\left(cl(A)\right)(x)$ for all $x \in X$.

***Definition 2.8.*** [3] Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces. Then a function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called pairwise continuous if $f^{-1}(U)$ is $\tau_i -$ open set in $(X, \tau_1, \tau_2)$ for each $\sigma_i -$ open set U of $(Y, \sigma_1, \sigma_2)$for $i = 1,2$.
We may also give similar definition of fuzzy pairwise continuous function in fuzzy bitopological spaces.

***Definition 2.9.*** Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two fuzzy bitopological spaces. Then a function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called fuzzy Pairwise continuous if $f^{-1}(U)$is $\tau_i -$ open set in $(X, \tau_1, \tau_2)$ for each $\sigma_i -$ open set U of $(Y, \sigma_1, \sigma_2)$for $i = 1,2$.

## 3. (i-j) fuzzy connected spaces

***Definition 3.1*** Let$(X, \tau_1, \tau_2)$ be a fuzzy bitopological space. A fuzzy set C is called (i-j) fuzzy disconnected if there exists two fuzzy sets $B_1$ and $B_2$ such that

(i) $C = B_1 \vee B_2$
(ii) There exists $x_1 \neq y_1 \in X$ such that $B_1(x_1) = C(x_1) \neq 0$ and $B_2(y_1) = C(y_1) \neq 0$.
(iii) $cl(B_1)_i \cap B_2 = \bar{0} = B_1 \cap cl(B_2)_j$ , where $i, j \in \{1,2\}$ and $i \neq j$.

Cis said to be (i-j) fuzzy connected iff C is not (i-j) fuzzy disconnected.

***Example 3.2*** Consider the two fuzzy topologies$\tau_1 = \{\bar{\alpha}/0 \leq \alpha \leq 1\}$ and $\tau_2 = \{f/f: X \to [0,1]\}$. We know that $(X, \tau_1)$ is fuzzy connected and $(X, \tau_2)$ is not fuzzy connected. We shall show that $(X, \tau_1, \tau_2)$ is (i-j) fuzzy connected.
Suppose there exists two fuzzy sets $B_1$ and $B_2$ such that i) $1_X = B_1 \vee B_2$. (ii) there exists$x_1 \neq y_1 \in X$ with $B_1(x_1) = 1_X(x_1) \neq 0$ and $B_2(y_1) = 1_X(y_1) \neq 0$.
Then$B_1 \cap cl(B_2) = \bar{0}$, where $cl(B_2)$ is $\tau_2$ −closed. Also $cl(B_1) \cap B_2 \neq \bar{0}$, where $cl(B_1)$ is $\tau_1$ −closed. That is $cl(B_1) \cap B_2 \neq \bar{0} = B_1 \cap cl(B_2)$. This implies that $(X, \tau_1, \tau_2)$ is not (i-j) fuzzy disconnected. Hence $(X, \tau_1, \tau_2)$ is (i-j) fuzzy connected.

***Theorem 3.3.*** Let$(X, \tau_1, \tau_2)$ be a fuzzy bitopological space. Let A be a (i-j) fuzzy connected set of X and B is $\tau_i$-fuzzy connected set of X, where $i \in \{1,2\}$ with$A \leq B$. Suppose $B = B_1 \vee B_2$ such that with $B_1(x_0) = B(x_0) \neq 0$ and $B_2(y_0) = B(y_0) \neq 0$ for some $x_0 \neq y_0 \in X$ and $cl(B_1)_i \cap B_2 = \bar{0} = B_1 \cap cl(B_2)_i$ then either $A \leq B_1$ (or) $A \leq B_2$.
**Proof:** As $cl(B_1)_i \cap (B_2) = \bar{0} = B_1 \cap cl(B_2)_i$ we get $cl(B_1 \wedge A)_i \cap (B_2 \wedge A) = \bar{0} = (B_1 \wedge A) \cap cl(B_2 \wedge A)_i$ where $i \in \{1,2\}$. Since $A \leq B_1 \vee B_2$, select $z_0 \in X$ such that $A(z_0) \neq 0$. Therefore $A(z_0) = (B_1 \wedge A)(z_0)$(or) $A(z_0) = (B_2 \wedge A)(z_0)$. Since B is $\tau_i$- fuzzy connected there is no $y \in X$ such that

$(y) = (B_2 \wedge A)(y) \neq 0$ . If $A(y) \neq 0$ then $B_2(y) < A(y)$. Therefore $A(y) = (B_1 \wedge A)(y)$ for all $y \in X$. Hence $A \leq B_1$. Similarly we can show that $A \leq B_2$.

***Theorem 3.4.*** The image of a (i-j) fuzzy connected set under a fuzzy pairwise continuous map is (i-j) fuzzy connected.

**Proof:** Let $(X, \tau_1, \tau_2)$and $(Y, \sigma_1, \sigma_2)$be two fuzzy bitopological spaces and $g: X \to Y$ be a fuzzy pairwise continuous map.

Let u be a (i-j) fuzzy connected set on X. We prove that $g(u)$ is a (i-j) fuzzy connected set on Y. Suppose $g(u)$ is (i-j) fuzzy disconnected. Then there exists two fuzzy sets A and B such that (i). $g(u) = A \vee B$. (ii). There exists $z_1 \neq z_2 \in Y$ such that $A(z_1) = g(u)(z_1) \neq 0$ and $B(z_2) = g(u)(z_2) \neq 0$ and (iii). $cl(A)_i \cap B = \bar{0} = A \cap cl(B)_j$, where $i, j \in \{1,2\}$ with $i \neq j$, by lemma (1.6) and lemma (1.7) $cl\left(g^{-1}(A)\right)_i \cap g^{-1}(B) = \bar{0} = g^{-1}(A) \cap cl\left(g^{-1}(B)\right)_j$.

As $g(u)(z_1) \neq 0$, select $x_1 \in X$ such that $g(x_1) = z_1$ and $u(x_1) > 0$. Similarly select $x_2 \in X$ such that $g(x_2) = z_2$and $u(x_2) > 0$. Therefore $g^{-1}(A)(x_1) = A(z_1) \neq 0$. Now $(g^{-1}(A) \wedge u)(x_1) = A\left(g(x_1)\right) \wedge u(x_1) = u(x_1) \neq 0$.

Similarly $(g^{-1}(B) \wedge u)(x_2) \neq 0$. Therefore $u = (g^{-1}(A) \wedge u) \vee (g^{-1}(B) \wedge u)$, which is a contradiction to our assumption that uis (i-j) fuzzy connected. Hence $g(u)$ is also (i-j) fuzzy connected.

***Definition 3.5.*** Let$(X, \tau_1, \tau_2)$ be a fuzzy bitopological space with a fuzzy ideal $\mathcal{J}$on X. A fuzzy set D is said to be (i-j) fuzzy $\mathcal{J} -$ connected if there do not exist two fuzzy sets A and B which are not in $\mathcal{J}$ such that i)$D = A \vee B$ (ii) there exists$x_0 \neq y_0 \in X$ such that $A(x_0) = D(x_0) \neq 0$ and $B(y_0) = D(y_0) \neq 0$. (iii) $cl(A)_i \cap B = \bar{0} = A \cap cl(B)_j$ where $i, j \in \{1,2\}$ with $i \neq j$.

The next example justifies the relation between the (i-j) fuzzy $\mathcal{J} -$ connected and (i-j) fuzzy connected.3

***Example 3.6.*** Consider the fuzzy bitopological space $(\mathbb{R}, \tau_1, \tau_2)$ where $\mathbb{R}$ denotes the reals $\tau_1 = \{$the discrete fuzzy topology on $\mathbb{R}\}$ and $\tau_2 = \{g: [0,1]/S(g)$ is an open set in the standard topology of $\mathbb{R}\}$. Let $\mathcal{J}$ denotes the fuzzy ideal of all fuzzy sets of $\mathbb{R}$ with finite support.

$$\text{Let } B(x) = \begin{cases} 1 & \text{if } x \in [0,3] \\ \frac{1}{x} & \text{if } x \in \{4,5,6\} \\ 0 & \text{otherewise} \end{cases}$$

Then B is (i-j) fuzzy $\mathcal{J} -$ connected, but not (i-j) fuzzy connected where $i, j \in \{1,2\}$ .

Let $B = C \vee D$ with $C \cap cl(D)_i = \bar{0} = cl(C)_j \cap D$ and there exists$x_0 \neq y_0$ such that $C(x_0) = B(x_0) \neq 0$ and $D(y_0) = B(y_0) \neq 0$. Then $S(C) \cup S(D) \subseteq [0,3] \cup \{4,5,6\}$. As $B(x) = C(x) \vee D(x) = 1$ for all $x \in [0,3]$ and as $C \cap D = 0$, we get $(1 - C) = D$ on $[0,3]$, $D \leq cl(D)_1 \leq 1 - C = D$ on $[0,3]$, $C(x) = 1 - cl(D)_1(x)$ for all $[0,3]$. Similarly $C(x) = 1 - cl(D)_2(x)$ for all $x \in [0,3]$. Also $S(C) \cap [0,3] = [0,3] \setminus S(1 - cl(D)_i)$where $i \in \{1,2\}$. Therefore $S(C) \cap [0,3]$ and $S(D) \cap [0,3]$ are disjoint relative open sets of $[0,3]$ in the relative topology. As $[0,3]$ is connected with respect to the standard topology on $\mathbb{R}$, we have either $S(C) \cap [0,3] = \emptyset$ (or) $S(D) \cap [0,3] = \emptyset$. Assume that $S(D) \cap [0,3] = \emptyset$. Then $S(D) \subseteq \{4,5,6\}$. So $D \in \mathcal{J}$. Therefore B is not (i-j) fuzzy $\mathcal{J} -$ disconnected. That is B is (i-j) fuzzy $\mathcal{J} -$ connected.

Next we show that B is not (i-j) fuzzy connected.

Consider $B = f \vee g$ where$f(x) = \begin{cases} 1 \text{ if } x \in [0,3] \\ 0 \quad \text{otherwise} \end{cases}$,

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \{4,5,6\} \\ 0 & \text{otherwise} \end{cases}.$$

Then there exists $x_1 \in [0,3]$ and $y_1 \in \{4,5,6\}$ with $f(x_1) = B(x_1) \neq 0$and $g(y_1) = B(y_1) \neq 0$.

Consider the following two fuzzy sets

$$A_1(x) = \begin{cases} 1 \; if \; x \in (-\infty, 0) \cup (3, \infty) \\ 0 \quad \text{otherwise} \end{cases}$$

$$A_2(x) = \begin{cases} 1 \; if \; x \in (-\infty, 4) \cup (4,5) \cup (5,6) \cup (6, \infty) \\ 0 \quad \text{otherwise} \end{cases}.$$

As $S(A_1)$ and $S(A_2)$are open sets in the standard topology of $\mathbb{R}$, $we$ get $A_1, A_2 \in \tau_2$. Clearly $A_1, A_2$are in$\tau_1$ also.

$$\text{Now } (1 - A_1)(x) = \begin{cases} 1 \; if \; x \in [0,3] \\ 0 \;\; \text{otherwise} \end{cases}$$

$$(1 - A_2)(x) = \begin{cases} 1 \; if \; x \in \{4,5,6\} \\ 0 \quad \text{otherwise} \end{cases}.$$

Since $f = (1 - A_1)$ , which is a $\tau_i$-fuzzy closed set, where $i \in \{1,2\}, cl(f)_i = f$. Note that $g \leq (1 - A_2)$. As $(1 - A_2)$ is a $\tau_i$-fuzzy closed set containing g, $\mathrm{cl(g)_i} \leq (1 - \mathrm{A_2})$. Now $\mathrm{cl(f)_i} \cap \mathrm{g} = \mathrm{f} \cap \mathrm{g} = \bar{0}$ and $\mathrm{f} \cap \mathrm{cl(g)_j} \leq \mathrm{f} \cap (1 - \mathrm{A_2}) = \bar{0}$ where $\mathrm{i}, \mathrm{j} \in \{1,2\}$and $\mathrm{i} \neq \mathrm{j}$. That is$\mathrm{f} \cap \mathrm{cl(g)_j} = \bar{0}$. Hence B is not (i-j) fuzzy connected. Thus every (i-j) fuzzy connected set is (i-j) fuzzy $\mathcal{J} -$ connected but the converse need not be true.

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