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ON QUASI – WEAK COMMUTATIVE NEAR – RINGS III

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Abstract. A left near – ring N is called weak commutative if xyz = xzy for every x,y,z ϵ N. A left near – ring N is called pseudo commutative if xyz = zyx for every x,y,z ϵ N. A left near – ring N is called quasi weak commutative near- ring if xyz = yxz for every x,y,z ϵ N. In [3,5], we have obtained many interesting results on quasi – weak commutative near – rings (right). In this paper we obtain some more results of quasi-weak commutative near-rings (left).

Key Words: weak commutative, pseudo commutative, quasi - weak commutative, Boolean – like near – rings.

1. Introduction

Throughout this paper, N denotes a left near – ring (N,+,.) with at least two elements. For every non – empty subset A of N, we denote A - {0} = A^* . The following definition and results are well known.

Example 1.1. (of Left near – ring)

Let (G,+) be any group. Define a.b = b ∀a,b ϵ G. Then (G,+,.) is a Quasi weak commutative near – ring (left).

For '.' is associative: (a.b) . c = b.c = c; also a.(b.c) = a.c = c; therefore (a.b) . c = a . (b.c) ∀ a,b,c ϵ G.

Also a.(b+c) = b+c; and a.b + a.c = b+c ; so, a.(b+c) = a.b + a.c
Also: abc = c = bac ; Hence (G,+,.) is a quasi weak commutative near ring (left).

Definition 1.2:
An element a ϵ N is said to be
(i) Idempotent if $a^2 = a$.
(ii) Nilpotent, if there exists a positive integer k such that $a^k = 0$.

Result: Each near – ring N is isomorphic to a sub direct product of sub directly irreducible near – rings.

Definition 1.4:
A near – ring N is said to be zero commutative if ab = 0 implies ba = 0, where a,b ϵ N.

Definition 1.5:
A near – ring (left) N is said to be Boolean if $a^2 = a$ ∀ a ϵ N. It is said to be anti – Boolean if $a^2 = -a$ ∀ a ϵ N.

Result: If N is zero symmetric, then
(i) Every left ideal A of N is an N-subgroup of N.
(ii) Every ideal I of N satisfies the condition NIN $\subseteq$ I. (ie) every ideal is an N – Subgroup.
(iii) $N^* I^* N^* \subseteq I^*$.

Result: Let N be a near – ring. Then the following are true.
(i) If A is an ideal of N and B is any subset of N, then (A:B) = { nϵN such that n B $\subseteq$ A }is always a left ideal.
(ii) If A is an ideal of N and B is an N-subgroup, then (A:B) is an ideal. In particular if A and B are ideals of a zero – symmetric near – ring, then (A:B) is an ideal.

Result:
1. Let N be a regular near – ring, a ϵ N and a = axa, then ax, xa are idempotents.
2. axN = aN and Nxa = Na.
3. N is S and S' near – rings.

Result: (Lemma 4 Dheena [2]) Let N be a zero – symmetric reduced near – ring. For any a,b ϵ N and for any idempotent element eϵ N, abe = aeb.

Result: A near – ring N is sub –directly irreducible if and only if the intersection of all non – zero ideals of N is not zero.

Result *(Gratzer [7]* Each simple near – ring is sub directly irreducible.

Result *(Pliz [10])* A non – zero symmetric near – ring N has Intersection of factors Property (IFP) if and only if (O:S) is an ideal for any subset S of N.

Result *(Oswald [9])* An N – subgroup A of N is essential if A∩B = {0}, where B is any N subgroup of N, implies B = {0}

.
Definition 1.6:
A near – ring is said to be reduced if N has no non – zero nilpotent elements.

Definition 1.7
A near – ring N is said to be an integral near – ring if N has no non – zero divisors.

Lemma 1.8

Let N be a near – ring if for all a ϵ N, $a^2 = 0 \Rightarrow a = 0$, then N has no non – zero nilpotent elements.

Definition 1.9

Let N be a near – ring. N is said to satisfy intersection of factors property (IFP) if ab = 0 => anb = 0 ∀n ϵ N, where a,b ϵ N.

Definition 1.10

(1) An ideal I of N is called a prime ideal if for all ideals A, B of N, AB is subset of I => A is subset of I or B is subset of I.

(2) I is called a semi – prime ideal if all ideals A of N, A^2 is subset of I implies A is subset of I.

(3) I is called a completely semi – prime ideal, if for any $x \epsilon N$, $x^2 \epsilon I \Rightarrow x \epsilon I$.

(4) A completely prime ideal, if for any x,y ϵ N, xy ϵ I => x ϵ I or y ϵI.

(5) N is said to have strong IFP, if for all ideals I of N, ab ϵ I implies anb ϵ I.

Result:(Proposition 2.4[11]) Let N be a pseudo commutative near – ring. Then every idempotent element is central.

Result: *[3]* Let N be a regular quasi weak commutative near – ring. Then

(i) $A = \sqrt{A}$, for every N – subgroup A of N.

(ii) N is reduced.

(iii) N has (IFP).

Result: *[3]* Let N be a regular quasi weak commutative near – ring. Then every N sub group is an ideal

$$N = Na = Na^2 = aN = aNa \text{ for all } a \in N.$$

Result: *[3]* Let N be a quasi weak commutative near – ring. For every ideal I of N, (I:S) is an ideal of N where S is any subset of N.

Result: *[3]* Every quasi weak commutative near – ring N is isomorphic to a sub – direct product of sub – directly irreducible quasi weak commutative near – rings.

2. Main Results

Lemma 2.1:

If N is a Boolean (left) near – ring, then for any a,b ϵ N, ab = 0 => ba = 0.a.

Proof: Let a,b ϵN.

$$ba = (ba)^2 = ba\,ba = b(ab)a = b0a = 0a.$$

Lemma 2.2:

If N is a Boolean (left) near – ring, then for any x,y ϵ N, xyx = yx.

Proof: Let x,y ϵ N.

Now,

$$yx\,(xyx - yx) = yx^2yx - (yx)^2$$
$$= yx\,yx - (yx)^2 = 0$$

So, by Lemma 2.1, we have: $(xyx - yx)\,yx = 0yx$ ……………….(1)

Also, $xyx\,(xyx - yx) = xyx^2yx - xyxyx$

$= xyxyx - xyxyx = 0$

So, by Lemma 2.1, $(xyx - yx)\,xyx = 0xyx$ …………………….(2)

Now, $xyx - yx = (xyx - yx)^2$

$= (xyx - yx)(xyx - yx)$

$= (xyx - yx)\,xyx - (xyx - yx)\,yx$

$= 0xyx - 0yx$

$xyx - yx = 0\,(xyx - yx)$ ………………………….(3)

Now

$0 = x\,(xyx - yx)$

$= x \,.\, 0\,(xyx - yx)$ (using (3))

$= 0\,(xyx - yx)$

$= xyx - yx$ (using (3))

Hence $xyx = yx$ $\forall\, x,y \in N$.

Theorem 2.3: Every Boolean (left) near – ring is Quasi – weak commutative. That is, if N is a Boolean (left) near – ring then $xyz = yxz$ for all $x,y,z \in N$.

Proof: Let $x,y,z \in N$, Now

$x\,(z - xz)\,y = (xz - x^2z)\,y = (xz - xz)\,y$

$= 0\,y$

$\Rightarrow$ $x\,(z - xz)\,y\,(z - xz) = 0\,y\,(z - xz)$

$\Rightarrow$ $xy\,(z - xz) = 0\,y\,(z - xz)$ (by Lemma 2.2)

$\Rightarrow$ $xyz - (xyx)\,z = 0yz - 0yxz$

$\Rightarrow$ $xyz - yxz = 0yz - 0yxz$ (by Lemma 2.2) → (4)

Also,

$x\,(z - xz)\,x = (xz - x^2z)\,x = (xz - xz)\,x = 0\,x$

$\Rightarrow$ $(z - xz)\,x = 0\,x$ (by Lemma 2.2)

$\Rightarrow$ $(z - xz)\,xy = 0\,xy$ → (5)

Now: $xy\,(z - xz)\,xy = (z - xz)xy$ (by Lemma 2.2)

$= 0\,xy$ (using (2))

$xy\,(z - xz)\,xy\,(z - xz) = 0\,xy\,(z - xz)$

$(xy\,(z - xz))^2 = 0\,xy\,(z - xz)$

$xy\,(z - xz) = 0\,xy\,(z - xz)$

i.e, $xyz - xyxz = 0xyz - 0xyxz$

$= 0\,xyz - 0\,yxz$ (using Lemma 2.2) → (6)

From (4) and (5) we get

$0\,yz - 0\,yxz = 0\,xyz - 0\,yxz$

$0\,yz = 0\,xyz$ for all $x,y,z \in N$ → (7)

Replacing x by y and y by z in (7) we get

$0\,zz = 0\,yzz$

(i.e) $0\,z^2 = 0\,yz^2$

$\Rightarrow \quad 0z = 0yz$ for all $y,z \in N$ → (8)

From (4), we get

$$xyz - yxz = 0yz - 0yxz$$
$$= 0z - 0xz \quad \text{(using (8))}$$
$$= 0z - 0z \quad \text{(using (8))}$$
$$= 0.$$

Hence $xyz = yz$ for all $x,y,z \in N$.

Note: If N is anti – Boolean near ring then $2x = 0 \quad \forall x \in N$.

Proof: Let $x \in N$. Then $(x + x)^2 = -(x + x)$

$$(x + x)(x + x) = -(x + x)$$
$$(x + x)x + (x + x)x = -(x + x)$$
$$2x \cdot x + 2x \cdot x = -(x + x)$$
$$2x^2 + 2x^2 = -x - x$$
$$-2x - 2x = -2x$$
$$-2x = 0$$
$$2x = 0 \quad \forall x \in N.$$

Lemma 2.4

If N is anti – Boolean left near – ring then for any $a,b \in N$, $ab = 0 \Rightarrow ba = -0a$.

Proof: Let $a,b \in N$. $-ba = (ba)^2 = ba\,ba = b(ab)a = b0a = 0a \Rightarrow ba = -0a$.

Lemma 2.5

If N is anti Boolean (left) near ring then for any $x, y \in N$, $xyx = -yx$.

Proof: Let $x, y \in N$.

$$yx(xyx + yx) = yx^2yx + (yx)^2$$
$$= -yx\,yx + (yx)^2 = 0$$

By Lemma 2.4

$$(xyx + yx)yx = -0yx \qquad \to (9)$$

Also

$$xyx(xyx + yx) = xyx^2yx + xyxyx$$
$$= -xyxyx + xyxyx = 0$$

By Lemma 2.4,

$$(xyx + yx)xyx = -0xyx \qquad \to (10)$$

Now,

$$-(xyx + yx) = (xyx + yx)^2$$
$$= (xyx + yx)(xyx + yx)$$
$$= (xyx + yx)xyx + (xyx + yx)yx$$
$$= -0xyx - 0yx$$
$$\Rightarrow (xyx + yx) = 0(xyx + yx) \qquad \to (11)$$

Now

$$0 = x(xyx + yx)$$
$$= x0(xyx + yx) \quad \text{(using (11))}$$
$$= 0(xyx + yx)$$
$$= xyx + yx \quad \text{(using (11))}$$

Hence $xyx = -yx$ for all $x,y \in N$.

Theorem 2.6

Every anti – Boolean (left) near – ring is anti – quasi weak commutative. That is, if N is anti – Boolean near – ring (in which $x^2 = -x \ \forall x \in N$), then $xyz = -yxz$ for all $x,y,z \in N$.

Proof: Let $x,y,z \in N$

Now $-x(z - xz)y = -(xz - x^2z)y$

$$= -(xz + xz)y$$

$= - (2xz) y$
$= - 0 y.$

Therefore

$-x (z – xz) y (z – xz) = - 0 y (z – xz)$
$-x (-y (z – xz)) = - 0 y (z – xz)$ (using Lemma 2.4)
$xy (z – xz) = - 0y (z – xz)$
$xyz – xyxz = - 0yz + 0 yxz$
$xyz – (- yxz) = - 0yz + 0 yxz$
$xyz + yxz = - 0yz + 0 yxz$ → (12)

Also

$-x (z – xz) x = - (xz – x^2z) x$
$= - (xz + xz) x$
$= - (2xz) x$
$= - 0 x.$

Using Lemma 2.4,we get

$-(- (z – xz) x) = - 0 x.$
$(z – xz) x = - 0 x.$
$(z – xz) xy = - 0. xy$ → (13)

Now by Lemma 2.4,

$xy (z – xz) xy = - (z – xz) xy$
$= - (- 0xy)$ (using(13))
$= 0xy$

$\therefore xy (z – xz) xy (z – xz) = 0 xy (z – xz)$
$(xy (z – xz))^2 = 0 xy (z – xz)$
$-xy (z – xz) = 0 xyz – 0 xyxz$
$-xyz + xyxz = 0 xyz – 0 xyxz$
$-xyz – yxz = 0 xyz + 0 yxz$ (using Lemma 2.4)
(i.e) $xyz + yxz = - 0 xyz – 0 yxz$ → (14)

From (12) and (14) we get,

$-0 yz + 0 yxz = 0 xyz + o yxz$
$\Rightarrow - 0 yz = 0 xyz$ → (15)

Replacing x by y and y by z in (15) we get,

$-0 z^2 = 0 yz^2$
$0 z = - 0 yz \ \forall y, z \in N$ → (16)

From (12) we get,

$xyz + yxz = -0 yz + 0 yxz$
$= 0 z – 0x z$ (using (16))
$= 0 z + 0 z$ (using (16))
$= 2 0 z$
$= 0$
$xyz + yxz = 0$
$\Rightarrow xyz = - yxz \quad \forall x,y,z \in N$

Hence N is anti – Quasi – weak Commutative.

Lemma 2.7

If N is a Boolean (left) near – ring then for any $x,y \in N$, $x^my^nx^m = y^nx^m$ where $m\geq1,n\geq1$ are fixed integers.

Proof: Let $x,y \in N$

$y^nx^m (x^my^nx^m – y^nx^m) = y^nx^{2m}y^nx^m - y^nx^my^nx^m$
$= y^nx^my^nx^m - y^nx^my^nx^m$
$= 0$

By Lemma 2.1,

$(x^my^nx^m – y^nx^m) y^nx^m = 0 y^nx^m$ → (17)

Also

$x^my^nx^m (x^my^nx^m - y^nx^m) = x^my^nx^{2m}y^nx^m - x^my^nx^my^nx^m$

$= x^my^nx^my^nx^m - x^my^nx^my^nx^m$

$= 0$

So, by Lemma 2.1,

$(x^my^nx^m - y^nx^m) x^my^nx^m = 0 x^my^nx^m$ → (18)

Now,

$x^my^nx^m - y^nx^m = (x^my^nx^m - y^nx^m)^2$

$= (x^my^nx^m - y^nx^m) (x^my^nx^m - y^nx^m)$

$= (x^my^nx^m - y^nx^m) x^my^nx^m - (x^my^nx^m - y^nx^m) y^nx^m$

$= 0 x^my^nx^m – 0 y^nx^m$

$x^my^nx^m - y^nx^m = 0 (x^my^nx^m - y^nx^m)$ → (19)

Now,

$0 = x^m (x^my^nx^m - y^nx^m)$

$= x^m 0 (x^my^nx^m - y^nx^m)$ (using (19))

$= 0 (x^my^nx^m - y^nx^m)$

$= (x^my^nx^m - y^nx^m)$ (using (19))

Hence $x^my^nx^m = y^nx^m \quad \forall x, y \in N$.

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