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ON SCALAR WEAK M-POWER COMMUTATIVE ALGEBRAS

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Abstract:A right near – ring N is called weak commutative if xyz = xzy. A right near – ring N is called weak m – power commutative if $x\ y^m\ z = x\ z^m\ y$ for all x, y, z ϵ N, where m ≥ 1 is a fixed integer. An algebra A over a commutative ring R is called scalar weak commutative if for every x,y,z ϵ A there exists α = α(x,y,z) ϵ R depending on x,y,z such that xyz = α xzy. In this paper we combine the concept of scalar weak commutativity and weak m – power Commutativity as scalar weak m-power commutativity and prove many results.

Key words: weak commutative Near-Rings, Scalar weak commutative Near-Rings.

1.Introduction

Let A be an algebra(not necessarily associative) over a commutative ring R. A is called scalar commutative if for each x,y ϵ A,there exists αϵ R depending on x,y such that xy = α yx. Rich [13] proved that if A is scalar commutative over a field F,then A is either commutative or anti - commutative. Koh, Luh and Putcha [11] proved that if A is scalar commutative with 1 and if R is a principal ideal domain,then A is commutative. A near – ring N is said to be weak - commutative if xyz = xzy for all x,y,z ϵ N (Definition 9.4,p.289,pliz[12]). An algebra A over a commutative ring R is called scalar weak commutative, if for every x,y,zϵ A, there exists α = α(x,y,z) ϵ R depending on x,y,z such that xyz = α xzy [8].

In this paper we define scalar weak m-power commutativity and prove many interesting results analogous to our own results [8].

2. Preliminaries:

In this section we give some basic definitions and well known results which we use in the sequel.

Definition 2.1. *[12]*

Let N be a near – ring N is said to be weak commutative if xyz = xzy for all x,y,zϵ N.

Definition 2.2

Let N be a near – ring.N is said to be anti - weak commutative if xyz = - xzy for all x,y,zϵ N.

Definition 2.3. *[2]*

Let A be an algebra(not necessarily associative) over a commutative ring R.A is called scalar commutative if for each x,yϵ A, there exists $\alpha = \alpha(x,y) \epsilon$ R depending on x,y such that xy = αyx. A is called scalar anti - commutative if xy = - αyx.

Lemma 2.4. *[5]*

Let N be a distributive near- ring.If xyz = $\pm$ xzy for all x,y,z ϵN,then N is eitherweak commutative or weak anti-commutative.

3.Main Results:

Definition 3.1:

Let A be an algebra (not necessarily associative) over a commutative ring R. A is called scalar weak m - power commutative if for every x,y,z ϵ A ,there exists scalars $\alpha\epsilon$R depending on x,y,z such that $xy^mz = \alpha\ xz^my$.

Definition 3.2:

Let A be an algebra (not necessarily associative) over a commutative ring R .
A is called scalar weak m - power anti - commutative if for every x,y,z ϵ A, there exists scalar α ϵ R depending on x,y,z such that $xy^mz = -\alpha\ xz^my$.

Theorem 3.3:

Let A be an algebra (not necessarily associative) over a field F.Let m ϵ Z^+ . Let $(x+y)^m = x^m + y^m$ holds for all x,y ϵ A. Assume $\alpha^m = \alpha$ for all α ϵ R. If for each x,y,zϵ A, there exists scalar α ϵ F depending on x,y,z such that $xy^mz = \alpha\ xz^my$ then A is either weak m-power commutative or weak m - power anti - commutative.

Proof: Suppose $xy^mz = xz^my$ for all x,y,z ϵ A, there is nothing to prove. Suppose not, we shall prove that

$$xy^mz = -xz^my \text{ for all } x,y,z \ \epsilon\ A.$$

First we shall prove that if $xy^mz \neq xz^my$, then

$$xy^{m+1} = xz^{m+1} = 0.$$

So, assume

$$xy^mz \neq xz^my.$$

Since A is scalar weak m-power commutative,there exists $\alpha = \alpha(x,y,z)$ ϵ F such that

$$xy^mz = \alpha\ xz^my \qquad \rightarrow (1)$$

Also there exists a scalar $\gamma = \gamma(x,y+z,z)$ ϵ F such that

$$x(y+z)^mz = \gamma xz^m(y+z)$$

$$\text{i.e, } x(y^m + z^m)\ z = \gamma xz^m(y+z) \qquad \rightarrow (2)$$

(1) - (2) gives

$$xy^mz - xy^mz - xz^{m+1} = \alpha\ xz^my - \gamma xz^m(y+z)$$

$$= \alpha\ xz^my - \gamma xz^my - \gamma xz^{m+1}$$

$$-xz^{m+1} + \gamma xz^{m+1} = \alpha\ xz^my - \gamma xz^my$$

$$\Rightarrow (1-\gamma)xz^{m+1} = (\gamma - \alpha)\, xz^m y \quad \rightarrow (3)$$

Now, $xz^m y \neq 0$ for if $xz^m y = 0$ then from (1) we get $xy^m z = 0$ and so $xy^m z = xz^m y$, contradicting our assumption that $xy^m z \neq xz^m y$.

Also $\gamma \neq 1$, for if $\gamma = 1$, then from (3) we get $\alpha = \gamma = 1$.

Then from (1) we get $xy^m z = xz^m y$, again a contradiction.

Now,from (3) we get

$$xz^{m+1} = \frac{\gamma-\alpha}{1-\gamma}\, xz^m y$$

$$\text{i.e., } xz^{m+1} = \beta\, xz^m y \quad \text{for some } \beta \epsilon F. \quad \rightarrow (4)$$

Similarly, $xy^{m+1} = \delta\, xz^m y$ for some $\delta \epsilon$ F. $\rightarrow (5)$

Now corresponding to each choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in F,there is an $\eta \epsilon$ F such that

$$x(\alpha_1 y + \alpha_2 z)^m (\alpha_3 y + \alpha_4 z) = \eta\, (\alpha_3 y + \alpha_4 z)^m (\alpha_1 y + \alpha_2 z)$$

$$\text{i.e., } x(\alpha_1^m y^m + \alpha_2^m z^m)(\alpha_3 y + \alpha_4 z) = \eta\, (\alpha_3^m y^m + \alpha_4^m z^m)(\alpha_1 y + \alpha_2 z)$$

Since $\alpha^m = \alpha$ for all $\alpha \epsilon$ F,we get:

$$x(\alpha_1 y^m + \alpha_2 z^m)(\alpha_3 y + \alpha_4 z) = \eta\, (\alpha_3 y^m + \alpha_4 z^m)(\alpha_1 y + \alpha_2 z)$$

$$x(\alpha_1 \alpha_3 y^{m+1} + \alpha_1 \alpha_4 y^m z + \alpha_2 \alpha_3 z^m y + \alpha_2 \alpha_4 z^{m+1})$$

$$= \eta\, x\, (\alpha_1 \alpha_3 y^{m+1} + \alpha_2 \alpha_3 y^m z + \alpha_1 \alpha_4 z^m y + \alpha_2 \alpha_4 z^{m+1})$$

$$(\alpha_1 \alpha_3\, xy^{m+1} + \alpha_1 \alpha_4\, xy^m z + \alpha_2 \alpha_3\, xz^m y + \alpha_2 \alpha_4\, xz^{m+1})$$

$$= \eta(\alpha_1 \alpha_3\, xy^{m+1} + \alpha_2 \alpha_3\, xy^m z + \alpha_1 \alpha_4\, xz^m y + \alpha_2 \alpha_4\, xz^{m+1}) \quad \rightarrow (6)$$

Using (4) and (5), we get:

$$\alpha_1 \alpha_3 \delta\, xz^m y + \alpha_1 \alpha_4 xy^m z + \alpha_2 \alpha_3\, xz^m y + \alpha_2 \alpha_4 \beta xz^m y$$

$$= \eta(\alpha_1 \alpha_3\, \delta xz^m y + \alpha_2 \alpha_3\, xy^m z + \alpha_1 \alpha_4\, xz^m y + \alpha_2 \alpha_4\, \beta xz^m y)$$

$$\alpha_1 \alpha_3 \delta \alpha^{-1}\, xy^m z + \alpha_1 \alpha_4 xy^m z + \alpha_2 \alpha_3 \alpha^{-1}\, xy^m z + \alpha_2 \alpha_4 \beta \alpha^{-1}\, xy^m z$$

$$= \eta(\alpha_1 \alpha_3\, \delta xz^m y + \alpha_2 \alpha_3\, \alpha xz^m y + \alpha_1 \alpha_4\, xz^m y + \alpha_2 \alpha_4\, \beta xz^m y)$$

$$(\alpha_1 \alpha_3 \delta \alpha^{-1} + \alpha_1 \alpha_4 + \alpha_2 \alpha_3 \alpha^{-1} + \alpha_2 \alpha_4 \beta \alpha^{-1})\, xy^m z$$

$$= \eta(\alpha_1 \alpha_3\, \delta + \alpha_2 \alpha_3\, \alpha + \alpha_1 \alpha_4 + \alpha_2 \alpha_4\, \beta)\, xz^m y \quad \rightarrow (7)$$

In (7) we choose $\alpha_2 = 0$; $\alpha_3 = \alpha_1 = 1$; $\alpha_4 = -\delta$.

The right handside of (7) is zero where as the left hand side of (7) is

$$(\delta \alpha^{-1} - \delta)\, xy^m z = 0.$$

$$\delta\, (\alpha^{-1} - 1)\, xy^m z = 0.$$

Since $xy^m z \neq 0$ and $\alpha \neq 1$,we get $\delta = 0$.

Hence from (5) we get $xy^{m+1} = 0$.

Also if in (7) we choose $\alpha_3 = 0$; $\alpha_4 = \alpha_2 = 1$; $\alpha_4 = -\beta$, the right side of (7) is zero where as the left side of (7) is:

$$(\beta \alpha^{-1} - \beta)\, xy^m z = 0.$$

$$\beta(\alpha^{-1} - 1)\, xy^m z = 0.$$

Since $xy^m z \neq 0$ and $\alpha \neq 0$,we get $\beta = 0$.

Hence from (4) we get $xz^{m+1} = 0$.

Then from (6) becomes:

$$\alpha_1 \alpha_4 xy^m z + \alpha_2 \alpha_3\, xz^m y = \eta\, (\alpha_2 \alpha_3\, xy^m z + \alpha_1 \alpha_4 xz^m y)$$

$$\alpha_1 \alpha_4 xy^m z + \alpha_2 \alpha_3 \alpha^{-1}\, xy^m z = \eta\, (\alpha_2 \alpha_3\, xy^m z + \alpha_1 \alpha_4 \alpha^{-1}\, xy^m z)$$

$$(\alpha_1 \alpha_4 + \alpha_2 \alpha_3 \alpha^{-1})\, xy^m z = \eta\, (\alpha_2 \alpha_3 + \alpha_1 \alpha_4 \alpha^{-1})\, xy^m z$$

This is true for any choice of $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\alpha_2 = -\alpha^{-1}$,we get

$$(1 - (\alpha^{-1})^2)\, xy^m z = 0.$$

Since $xy^m z \neq 0$, $1 - (\alpha^{-1})^2 = 0$.

Hence $(\alpha^{-1})^2 = 1$, i.e., $\alpha = \pm 1$.

Since $\alpha \neq 1$, we get $\alpha = -1$.

i.e., $xy^m z = -\, xz^m y$ for all x,y,z ϵ A.

i.e, A is either weak m – power commutative or weak m – power anti – commutative.

Note: Taking m = 1,we get Theorem 3.2[8].

Lemma 3.4:

Let A be an algebra (not necessarily associative) over a commutative ring R.Let $m \epsilon Z^+$. Suppose A is scalar weak m-power commutative.Then for all $x,y,z \epsilon A$, $\alpha \epsilon R$, $\alpha xy^mz = 0$ iff $\alpha xz^my = 0$. Also $xy^mz = 0$ iff $xz^my = 0$.

Proof: Let $x,y,z \epsilon A$ and $\alpha \epsilon R$ such that $\alpha xy^mz = 0$.
Since A is scalar weak m- power commutative there exists $\beta = \beta(\alpha x,z,y) \epsilon R$ such that

$$\alpha xz^my = \beta(\alpha x) y^mz$$
$$\alpha xz^my = \beta(\alpha xy^mz) = 0.$$

Conversely assume $\alpha xz^my = 0$.Since A is scalar weak m - power commutative there exists $\gamma = \gamma(\alpha x,y,z) \epsilon R$ such that

$$\alpha xy^mz = \gamma\alpha xz^my$$
$$\text{i.e., } \alpha xy^mz = \gamma\alpha xz^my = 0.$$

Thus $\alpha xy^mz = 0$ iff $\alpha xz^my = 0$.
Now assume $xy^mz = 0$.Since A is scalar weak m - power commutative,there exists scalar $\delta(x,z,y) \epsilon R$ such that $xz^my = \delta xy^mz = 0$.
Conversely assume $xz^my = 0$. Then there exists scalar $\eta = \eta(x,y,z) \epsilon R$ such that $xy^mz = \eta xz^my = 0$.
Then $xy^mz = 0$ iff $xz^my = 0$.

Note: Taking m = 1,we get Lemma 3.3 [8].

Lemma 3.5.

Let A be an algebra(not necessarily associative) over a commutative ring R. Let $m \epsilon Z^+$. Suppose $(x+y)^m = x^m + y^m$ for all $x,y \epsilon A$ and every element of R is m - potent (i.e., $\alpha^m = \alpha$ for all $\alpha \epsilon R$).Let $x,y,z,u \epsilon A,\alpha,\beta \epsilon R$ such that $y^mu = u^my, xz^my = \alpha xy^mz$. $x(z+u)^m y = \beta xy^m(z+u)$,then $x(y^mu - \alpha y^mu - \beta y^mu + \alpha\beta y^mu) = 0$.

Proof: Given

$$x(z+u)^m y=\beta xy^m(z+u) \quad \rightarrow (8)$$
$$xz^my =\alpha xy^mz \quad \rightarrow (9)$$
$$y^mu = u^my \quad \rightarrow (10)$$

From (8) we get

$$x(z^m + u^m)y=\beta xy^mz + \beta xy^m u$$
$$xz^m y + x u^my = \beta xy^mz + \beta xy^m u \rightarrow (4)$$
$$\alpha xy^mz + x u^my = \beta xy^mz + \beta xy^m u \quad \text{(using(9))}$$
$$\alpha xy^mz + x y^mu - \beta xy^mz - \beta xy^m u = 0 \quad \text{(using(10))}$$
$$xy^m (\alpha z + u - \beta z - \beta u)= 0$$

By Lemma 3.4 , we get

$$x (\alpha z + u - \beta z - \beta u)^m y = 0$$
$$x (\alpha^m z^m + u^m - \beta^m z^m - \beta^m u^m)y = 0$$
$$x (\alpha z^m + u^m - \beta z^m - \beta u^m)y = 0$$
$$x (\alpha z^my + u^my - \beta z^m y - \beta u^m y) = 0$$
$$\alpha xz^my +x u^my - \beta xz^m y - \beta x u^m y = 0$$
$$\alpha xz^my + x u^my - \alpha\beta xy^m z - \beta x u^m y = 0 \quad \text{(using(9))} \quad \rightarrow (11)$$

Multiply (11) by α, we get

$$\alpha xz^my + \alpha x u^my = \alpha\beta xy^m z + \beta x y^m u = 0 \quad \rightarrow (12)$$

(11) – (12) gives

$$\alpha xz^my + x u^my - \alpha\beta xy^m z - \beta x u^m y - \alpha xz^my - \alpha x u^my + \alpha\beta xy^m z + \alpha\beta x y^m u = 0$$

$$(x u^m y - \alpha x u^m y - \beta x u^m y + \alpha\beta x y^m u) = 0$$
$$x(y^m u - \alpha y^m u - \beta y^m u + \alpha\beta y^m u) = 0$$

Corrollary 3.6

Taking u = y, we get:

$$x (y^{m+1} - \alpha y^{m+1} - \beta y^{m+1} + \alpha\beta y^{m+1}) = 0.$$
$$x (y^m - \alpha y^m) (y - \beta y) = 0.$$

Theorem 3.7

Let A be an algebra (not necessarily associative) over a commutative ring R. Let $m \epsilon Z^+$ .Suppose $(x+y)^m = x^m + y^m$ for all x,y ϵ A and that A has no zero divisors.Assume every element of R is m - potent. If A is scalar weak m - power commutative,then A is weak m - power commutative.

Proof: Let x,y,z ϵ A. Since A is scalar weak commutative,there exists $\alpha = \alpha(x,z,y) \epsilon$ R and $\beta = \beta(x,z,y) \epsilon$ R, such that:

$$x z^m (y + z) = \beta x(y+z)^m z \quad \rightarrow (13)$$
$$xz^m y = \alpha xy^m z \quad \rightarrow (14)$$

From (13) we get

$$xz^m y + xz^{m+1} - \beta xy^m z - \beta xz^{m+1} = 0 \quad \rightarrow (15)$$
$$\text{i.e., } \alpha xy^m z + xz^{m+1} - \beta xy^m z - \beta xz^{m+1} = 0 \quad \text{(using (14))}$$
$$x (\alpha y^m + z^m - \beta y^m - \beta z^m) z = 0$$
$$\text{i.e., } x (\alpha y + z - \beta y - \beta z)^m z = 0$$

By Lemma 2.3, we get

$$xz^m (\alpha y + z - \beta y - \beta z) = 0$$
$$\text{i.e., } \alpha xz^m y + xz^{m+1} - \beta xz^m y - \beta xz^m z = 0$$
$$\alpha xz^m y + xz^{m+1} - \alpha\beta xy^m z - \beta xz^{m+1} = 0 \quad \text{(using (14))} \quad \rightarrow (16)$$

Multiply (15) by α

$$\alpha xz^m y + \alpha xz^{m+1} - \alpha\beta xy^m z - \alpha\beta xz^{m+1} = 0 \quad \rightarrow (17)$$

(16) – (17) gives:

$$xz^{m+1} - \alpha xz^{m+1} - \beta xz^{m+1} + \alpha\beta xz^{m+1} = 0$$
$$x(z^2 - \alpha z^2 - \beta z^2 + \alpha\beta z^2) z^{m-1} = 0$$
$$x (z - \alpha z) (z - \beta z) z^{m-1} = 0$$

Since A has no zero divisors z = 0 or $z - \alpha z = 0$ or $z - \beta z = 0$.

If z = 0,then $xy^m z = xz^m y$.

If $z = \alpha z$,then from (2) ,we get

$\alpha xz^m y = \alpha xy^m z$, i.e., $\alpha (xz^m y - xy^m z) = 0$. Since $\alpha \neq 0$, $xz^m y - xy^m z = 0$.

If $z = \beta z$, then from (15) we get:

$$xz^m y + xz^{m+1} - xy^m z - xz^{m+1} = 0$$
$$\text{i.e., } xz^m y = xy^m z \quad (\text{since } \beta = \beta^m)$$

Thus A is scalar weak m power commutative.

Note: Taking m =1,we get Theorem 3.6[8].

Definition 3.8

Let R be any ring.Let m > 1 be a fixed integer .An element a ϵ R is said tobe m-potent if $a^m = a$.

Lemma 3.9:

Let A be an algebra with unity over a P.I.D R.Let $m \epsilon Z^+$. Assume $(x+y)^m = x^m + y^m$

for all x,y ϵ A and that every element of R is m-potent.If A is scalar weak m-power commutativez ϵ A such that $O(z^{m+1}) = 0$,then $xy^mz = xz^my$ for all x,y,z ϵ A.
Proof:

Let z ϵ A such that $O(z^{m+1}) = 0$. Let x,y,z ϵ A.
Then there exists scalars $\alpha = \alpha(x,z,y) \epsilon$ R and $\beta = \beta(x,z,y+z) \epsilon$ R such that

$$x z^m (y + z) = \beta x(y+z)^m z \quad \rightarrow (18)$$

and

$$xz^m y = \alpha xy^m z \quad \rightarrow (19)$$

From (19) we get:

$$x z^m (y + z) = \beta x(y+z)^m z$$
$$xz^m y + xz^{m+1} = \beta xy^m z + \beta xz^{m+1} \quad \rightarrow (20)$$
$$\alpha xy^m z + xz^{m+1} - \beta xy^m z - \beta xz^{m+1} = 0 \quad \text{(using (19))}$$
$$x (\alpha y^m + z^m - \beta y^m - \beta z^m) z = 0$$
$$x (\alpha y + z - \beta y - \beta z)^m z = 0 \quad \text{(since R is m-potent)}$$

By Lemma 3.5, we get:

$$x z^m (\alpha y + z - \beta y - \beta z) = 0$$
$$\alpha x z^m y + x z^{m+1} - \beta x z^m y - \beta xz^{m+1} = 0$$
$$\alpha x z^m y + x z^{m+1} - \alpha\beta x y^m z - \beta xz^{m+1} = 0 \quad \text{(using (19))} \quad \rightarrow (21)$$

Multiply (20) by α, we get

$$\alpha x z^m y + \alpha x z^{m+1} = \alpha\beta x y^m z + \alpha\beta xz^{m+1} \quad \rightarrow (22)$$

From (21) and (22) we get

$$\alpha xz^m y + xz^{m+1} - \alpha\beta xy^m z - \beta xz^{m+1} - \alpha xz^m y + \alpha xz^{m+1} = \alpha\beta xy^m z + \alpha\beta xz^{m+1} = 0$$
$$(\alpha\beta xz^{m+1} - \alpha x z^{m+1} - \beta xz^{m+1} + x z^{m+1}) = 0$$
$$x (1- \alpha- \beta + \alpha\beta) x z^{m+1} = 0$$
$$(1 - \alpha) (1 - \beta) x z^{m+1} = 0 \quad \rightarrow (23)$$

Thus for each z ϵ A, there exists scalars $\gamma \epsilon$ R and $\delta \epsilon$ R such that

$$\gamma x z^{m+1} = 0 \quad \rightarrow (24)$$

and

$$\delta (x+1)z^{m+1} = 0 \quad \rightarrow (25)$$

γ x (25) – δ x (24) gives
Therefore $\gamma\delta x z^{m+1} + \gamma\delta z^{m+1} - \gamma\delta x z^{m+1} = 0$

$$\gamma\delta z^{m+1} = 0$$

Since $O(z^{m+1}) = 0$,we get

$$\gamma = 0 \text{ (or) } \delta = 0$$

Hence from (23) we get $(1 - \alpha) (1 - \beta) = 0$, i.e., either $\alpha = 1$ (or) $\beta = 1$.
If $\alpha = 1$, from (18) we get
If $\beta = 1$, from (18) we get

$$x z^m (y + z) = x(y+z)^m z$$
$$x z^m y + x z^{m+1} = xy^m z + x z^{m+1}$$

i.e.,$x z^m y = xy^m z$.

Lemma 3.10

Let A be an algebra with identity over a P.I.D R.Let m ϵ Z^+ . Suppose that $(x+y)^m = x^m + y^m$ for all x,y ϵA and that every element of R is m-potent.Suppose that A is scalar weak m-power commutative.Assume further that there exists a prime pϵ R such thatp^m A = 0.Then A is weak m-power commutative.
Proof: Let x,y ϵ A such that $O(x^my) = p^k$ forsome k ϵ Z^+ . We prove by induction on k that $ux^my = uy^mx$ for all u ϵ A.

If $k = 0$,then $O(x^m y) = p^0 = 1$ and so $x^m y = 0$. So $ux^m y = 0$. By Lemma 3.5 $uy^m x$ for all $u \epsilon$ A. So assume that $k > 0$ and that the statements true for all $1 < k$.
If $ux^m y - uy^m x$ for all $u \epsilon A$,then there is nothing to prove. So, let $ux^m y - uy^m x \neq 0$.Since A is scalar weak m- power commutative ,there exists scalars $\alpha = \alpha(u,x,y) \epsilon$ R and $\beta = \beta(u,x+y,y) \epsilon$ R such that:

$$ux^m y = \alpha\, uy^m x \quad \rightarrow (26)$$

and

$$u\,(x+y)^m y = \beta u y^m (x+y) \quad \rightarrow (27)$$

From (27) we get:

$$u\,(x^m + y^m)\, y = \beta u y^m x + \beta u y^{m+1}$$

$$u\, x^m y + u\, y^{m+1} = \beta u y^m x + \beta u y^{m+1} \quad \rightarrow (28)$$

$$\alpha\, uy^m x + u\, y^{m+1} = \beta u y^m x + \beta u y^{m+1} \quad \text{(using (26))}$$

$$\alpha\, uy^m x - \beta u y^m x = \beta u y^{m+1} - u\, y^{m+1}$$

$$(\ -\beta\ )\, uy^m x = (\beta - 1)\, u\, y^{m+1} \quad \rightarrow (29)$$

If $(\alpha - \beta)\, uy^m x = 0$ then $(\beta - 1)\, u\, y^{m+1} = 0$.
Since $u\, y^{m+1} \neq 0$, $\beta = 1$.Hence from (28) we get

$ux^m y = uy^m x$, contradicting our assumption that $ux^m y \neq uy^m x$.

So $(\alpha\ -\ )\, uy^m x \neq 0$.In particular $\alpha - \beta \neq 0$. Let $\alpha - \beta = p^t \delta$
For some $t \epsilon Z^+$ and $\delta \epsilon$ R with $(\delta, p) = 1$.
If $t \geq k$,then since $O(x^m y) = p^k$ we would get

$(\alpha\ -\ )\, uy^m x = 0$, again a contradiction.

Hence $t < k$.. Since $p^k ux^m y = 0$, by Lemma 3.5, $p^k uy^m x = 0$.
From (29) we get:

$$p^{k-t} (\beta - 1)\, uy^{m+1} = p^{k-t} (\alpha - \beta) uy^m x$$
$$= p^{k-t} p^t \delta u y^m x$$
$$= p^{k-t} \delta u y^m x$$
$$= 0$$

Let $O(y^{m+1}u) = p^i$ .If $i < k$,then by induction hypothesis,$ux^m y = uy^m x$,a contradiction.
So, $i \geq k$.
Now, $\quad p^k | p^i | p^{k-t} (\beta - 1)$
And $p^t | (\beta - 1)$.
Let $\beta - 1 = p^t \gamma$ for some $\gamma \epsilon$ R.
Then from (29) weget

$$(\alpha\ -\ )\, uy^m x = (\beta - 1)\, uy^{m+1}$$
$$p^t \delta u y^m x = p^t \gamma u y^{m+1}$$
$$p^t (\delta x - \gamma y) u y^m = 0.$$

By Lemma 3.5, we get:

$$p^t u\, (\delta x - \gamma y)^m y = 0.$$

Hence by induction hypothesis,

$$(\delta x - \gamma y)^m (uy)\, w = (uy)^m (\delta x - \gamma y) w \text{ for all } w \epsilon A.$$

Taking u = 1,we get

$$(\delta x - \gamma y)^m (1.y)\, w = (1.y)^m (\delta x - \gamma y) w$$
$$(\delta^m x - \gamma^m y^m)\, yw = y^m (\delta x - \gamma y) w$$
$$\delta x^m y w - \gamma y^{m+1} w = \delta x y^m w - \gamma y^{m+1} w$$
$$\delta (x^m y w - x y^m w) = 0$$

Since, $(\ , p) = 1$,there exists $\mu, \delta \epsilon$ R such that

$$\mu p^m + \gamma \delta = 1.$$

$$\therefore \mu p^m (w x^m y - w y^m x) + \gamma \delta (w x^m y - w y^m x) = w x^m y - w y^m x$$

$$0 + 0 = wx^my - wy^mx \quad (\therefore p^mA \quad)$$

$\therefore \quad wx^my - wy^mx = 0$ for all $w \epsilon A$.

Hence the proof.

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