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BATCH ARRIVAL MULTI-STAGE RETRIAL G-QUEUE WITH FLUCTUATING MODES OF SERVICES AND FEEDBACK

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Abstract: In this paper, a single server batch arrival multi-stage retrial G-queue is considered. The server renders essential service followed by 'M' stages of additional services. Each stage consists of multi-optional fluctuating modes of services. The system consists of positive and negative customers. Positive customer arrives in batches according to the Poisson process. If the server is idle upon the arrival of a batch, then one of the customers in the batch begins his essential service and the rest join the orbit. Otherwise, all the arriving customers join the orbit. After completion of essential service the customer may opt any one of first stage services, feedback to the orbit or leave the system. After first stage service completion, the customer may move to second stage, join the orbit or leave the system and so on. The arrival of the negative customer makes the server breakdown with the removal of customer being in service from the system. Using the supplementary variable technique, performance measures and reliability indices are derived. Stochastic decomposition law is verified.

1. Introduction

Retrial queueing system with multi stages plays a major role in computer networks, communication, bank and manufacturing systems. Ebenesar and Udaya Chandrika (2015) analyzed a single server bulk arrival multi-stage retrial queue with vacation. Radha et al. (2017) studied a group arrival retrial G - queue with multi optional stages of service, orbital search and server breakdown. Queue with negative arrivals was first introduced by Gelenbe (1989) and named as G-queue. Gao and Wang (2014), Sumitha and Udaya Chandrika (2015), Rajadurai et al. (2015) and Varalakshmi et al. (2017) discussed retrial queue by including the concept of G-queue.

In most of the queueing models it is assumed that the server is providing service to all customers at the same service rate. But it is not always true in real life situations. Sometimes, there could be variation in mean service rate due to several reasons. This kind of service is considered as fluctuating modes of service. Baruah et al. (2014) studied a batch arrival single server queue with server providing general service in two fluctuating modes. Yamuni et al. (2016) derived bulk arrival retrial queue with fluctuating modes of service, immediate feedback, server vacation and orbital search. Dshalalo and Merie (2018) investigated a fluctuation analysis in queues with several operational modes and priority customers. Ayyappan and Supraja (2018) studied an $M^x/G(a,b)/1$ queueing system with two fluctuating modes of service under bernoulli vacation schedule for unreliable server and delaying repair.

Our present investigation deals with a single server batch arrival multi-stage retrial G-queue with fluctuating modes of services and feedback.

2. Model Description

A single server multi-stage retrial G-queue with fluctuating modes of service is considered. The system consists of positive and negative customers. Positive customers arrive in batches according to Poisson process with rate λ^+. The batch size Y is a random variable with distribution function P(Y=k) = C_k , k=1,2,3,..... and probability generating function C(z) having the first two moments m_1 and m_2. If an arriving batch finds the server idle, then a random customer from the batch receives the service and the rest join the orbit. Otherwise, all the arriving customers join the orbit. The inter-retrial times form an arbitrary distribution $A(x)$ with Laplace-Stieljes transform $A^*(s)$ and the hazard rate function $\eta(x) = \dfrac{d\,A(x)}{1-A(x)}$.

The server renders first essential service to all the arriving positive customers. The essential service time is generally distributed with distribution function $B_0(x)$, Laplace-Stieltjes transform $B_0^*(s)$, n^{th} (n=1,2,3,...) factorial moments $\mu_0^{(n)}$ and the conditional completion rate $\mu_0(x) = \dfrac{d\,B_0(x)}{1-B_0(x)}$.

The system consists of M stages of services followed by essential service. In each stage there are multi heterogeneous optional services. Stage i (i=1,2,….,M) consists k_i optional services. After completion of essential service, the customer moves to first stage and opts j_1^{th} (j_1=1,2,....,k_1) mode of service with probability p_{j_1}, departs the system with probability q_0 or joins the orbit as a feedback customer with probability δ_0. In similar way the customer after first stage service completion moves to second stage, joins the orbit or leaves the system. In general, the customer from $(i-1)^{th}$ stage proceeds to i^{th} stage and opts j_i^{th} (j_i =1,2,....,k_i) mode of service with probability p_{j_i}, departs from the system with probability q_i or joins the orbit with probability δ_i. After the completion of last stage, the customer either leave the system with probability q_M or join the orbit with probability δ_M such that $\delta_M + q_M = 1$. The service time of i^{th} (i=1,2,.....,M) stage j_i^{th} (j_i =1,2,......,k_i) mode of service is generally distributed with distribution function $B_{i,j_i}(x)$, Laplace-Stieltjes transform $B_{i,j_i}^*(s)$, n^{th} factorial moments $\mu_{i,j_i}^{(n)}$ and the conditional completion rate $\mu_{i,j_i}(x) = \dfrac{d\,B_{i,j_i}(x)}{1-B_{i,j_i}(x)}$, $i = 1,2,3,.....,M$, $j_i = 1,2,3,......,k_i$.

Negative customers arrive according to the Poisson process with rate λ^-. The arrival of negative customers removes the positive customer being in service from the system and breakdowns the server. As soon as the server fails, it stops rendering service and is sent for repair immediately. The repair time of the server failed during essential service is generally distributed with distribution function $R_0(x)$, Laplace-Stieltjes transform $R_0^*(s)$, n^{th} factorial moments $\beta_0^{(n)}$, and the conditional completion rate $\beta_0(x) = \dfrac{d\,R_0(x)}{1-R_0(x)}$. The repair time of the server failed during i^{th} (i=1,2,.....,M) stage

j_i^{th} (j_i =1,2,......,k_i) mode of service follows general distribution with distribution function $R_{i,j_i}(x)$, Laplace-Stieltjes transform $R^*_{i,j_i}(s)$, n^{th} factorial moments $\beta^{(n)}_{i,j_i}$ and the conditional completion rate $\beta_{i,j_i}(x) = \dfrac{d\,R_{i,j_i}(x)}{1 - R_{i,j_i}(x)}$, $i = 1,2,3,.....,M$, $j_i = 1,2,3,.....,k_i$

3. Stability Condition

Let $\{t_n\ ,\ n \in N\}$ be the epoch of the completion of service or repair period. The sequence of random variables $Y_n = \{C(t_n^+), X(t_n^+)\}$ forms an embedded Markov chain which is ergodic iff

$$m_1(1 - A^*(\lambda^+)) + T_1'(1) > 0$$

where

$$\begin{aligned} T_1'(1) = {} & \sum_{i=0}^{M} \delta_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) + (\delta_0 + q_0) f_0^{(1)} + \lambda^+ m_1 \sum_{i=1}^{M} (\delta_i + q_i)[M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}] - f_0^{(1)} \\ & + (\lambda^+ m_1 / \lambda^-)[1 - B_0^*(\lambda^-) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) B_0^*(\lambda^-)] + \lambda^+ m_1[(1 - B_0^*(\lambda^-))\beta_0^{(1)} \\ & + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)} B_0^*(\lambda^-)] + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1 - B_{i,j_i}^*(\lambda^-)) B_0^*(\lambda^-) \\ & - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} B_0^*(\lambda^-) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) f_0^{(1)} \end{aligned}$$

$$\Lambda_0^* = 1\ , \qquad \Lambda_i^* = \prod_{l=1}^{i} \sum_{j_l=1}^{k_l} p_{j_l} B_{l,j_l}^*$$

$$f_0^{(1)} = \lambda^+ m_1 \int_0^\infty x e^{-\lambda^- x} b_0(x)\,dx \qquad , \qquad f_{i,j_i}^{(1)} = \lambda^+ m_1 \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)\,dx$$

$$M_i^{(1)} = \lim_{z \to 1} \Lambda_i^{*\prime}(g(z)) \qquad , \qquad M_i^{(2)} = \lim_{z \to 1} \Lambda_i^{*\prime\prime}(g(z))$$

$$g(z) = \lambda^+(1 - C(z)) + \lambda^-$$

4. Steady State Distribution

Define the state of the server as

$$C(t) = \begin{cases} 0\ , & server\ is\ idle \\ 1, & server\ is\ busy\ at\ essential\ service \\ 2, & server\ is\ busy\ at\ i^{th}\ stage,\ j_i^{th}\ \mathrm{mod}e\ service \\ 3, & server\ on\ essential\ service\ is\ under\ repair \\ 4, & server\ on\ i^{th}\ stage\ \ j_i^{th}\ \mathrm{mod}e\ \ is\ under\ repair \end{cases}$$

For the process, $\{N(t)\ ;\ t \geq 0\}$, define the following probabilities

$$I_0(t) = P\{C(t) = 0,\ \ N(t) = 0\}$$

$$I_n(x,t) = P\{C(t) = 0,\ \ N(t) = n\ ,\ \ x < \xi(t) \leq x + dx\},\quad n \geq 1$$

$$P_{0,n}(x,t)dx = P\{C(t) = 1,\ \ N(t) = n,\ \ x < \xi(t) \leq x + dx\},\quad n \geq 0$$

$$P_{i,j_i,n}(x,t)dx \;=\; P\{C(t)=2,\; N(t)=n,\; x<\xi(t)\le x+dx\},\; n\ge 0,\; i=1,2,....,M,\; j_i=1,2,....,k_i$$
$$R_{0,n}(x,t)dx \;=\; P\{C(t)=3\;,\; N(t)=n\;,\; x<\xi(t)\le x+dx\}\;,\quad n\ge 0$$
$$R_{i,j_i,n}(x,t)dx \;=\; P\{C(t)=4,\; N(t)=n,\; x<\xi(t)\le x+dx\},\quad n\ge 0,\; i=1,2,....,M,\; j_i=1,2,....,k_i$$

The steady state equations governing the model under consideration are

$$\lambda^+ I_0 = q_0\int_0^\infty P_{0,0}(x)\mu_0(x)dx+\sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,0}(x)\mu_{i,j_i}(x)dx+\int_0^\infty R_{0,0}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,0}(x)\beta_{i,j_i}(x)dx \quad (1)$$

$$\frac{d}{dx}I_n(x) \;=\; -(\lambda^+ +\eta(x))\; I_n(x) \quad ,\quad n\ge 1 \quad (2)$$

$$\frac{d}{dx}P_{0,n}(x) \;=-(\lambda^+ +\lambda^- +\mu_0(x))\; P_{0,n}(x) \;+\; \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k P_{0,n-k}(x)\;\; ,n\ge 0 \quad (3)$$

$$\frac{d}{dx}P_{i,j_i,n}(x)=-(\lambda^+ +\lambda^- +\mu_{i,j_i}(x))\; P_{i,j_i,n}(x)+\lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k P_{i,j_i,n-k}(x)\;\; ,n\ge 0,\quad i=1,2,....,M,\;\; j_i=1,2,....,k_i \quad (4)$$

$$\frac{d}{dx}R_{0,n}(x) \;=-(\lambda^+ +\beta_0(x))\; R_{0,n}(x) \;+\; \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k R_{0,n-k}(x)\;\; ,n\ge 0 \quad (5)$$

$$\frac{d}{dx}R_{i,j_i,n}(x)=-(\lambda^+ +\beta_{i,j_i}(x))\; R_{i,j_i,n}(x)+\lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k R_{i,j_i,n-k}(x)\;\; ,n\ge 0,\quad i=1,2,....,M,\;\; j_i=1,2,....,k_i \quad (6)$$

with boundary conditions

$$I_n(0)=\delta_0\int_0^\infty P_{0,n-1}(x)\mu_0(x)dx+\sum_{i=1}^{M}\delta_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-1}(x)\mu_{i,j_i}(x)dx+q_0\int_0^\infty P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx+\int_0^\infty R_{0,n}(x)\beta_0(x)dx+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,n}(x)\beta_{i,j_i}(x)dx,\quad n\ge 0 \quad (7)$$

$$P_{0,0}(0) \;=\; \lambda^+ C_1 I_0 \;+\; \int_0^\infty I_1(x)\eta(x)dx \quad (8)$$

$$P_{0,n}(0)=\lambda^+ C_{n+1}I_0 \;+\; \lambda^+\sum_{k=1}^{n}C_k\int_0^\infty I_{n-k+1}(x)dx+\int_0^\infty I_{n+1}(x)\eta(x)dx,\qquad n\ge 1 \quad (9)$$

$$P_{1,j_1,n}(0)=\; p_{j_1}\int_0^\infty P_{0,n}(x)\mu_0(x)dx,\qquad n\ge 0,\quad j_1=1,2,....,k_1 \quad (10)$$

$$P_{i,j_i,n}(0)=p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1},n}(x)\mu_{i,j_i}(x)dx,\qquad n\ge 0,\; i=2,3,......,M,\;\; j_i=1,2,....,k_i \quad (11)$$

$$R_{0,n}(0)=\lambda^-\int_0^\infty P_{0,n}(x)dx\;,\qquad n\ge 0 \quad (12)$$

$$R_{i,j_i,n}(0)=\lambda^-\int_0^\infty P_{i,j_i,n}(x)dx\;,\qquad n\ge 0,\; i=1,2,3,......,M,\;\; j_i=1,2,....,k_i \quad (13)$$

5. Probability Generating Functions

Define the following probability generating functions

$$\left.\begin{aligned} I(x,z) &= \sum_{n=1}^{\infty} I_n(x) z^n \quad ; \quad P_0(x,z) = \sum_{n=0}^{\infty} P_{0,n}(x) z^n \\ P_{i,j_i}(x,z) &= \sum_{n=0}^{\infty} P_{i,j_i,n}(x) z^n \quad ; \quad R_0(x,z) = \sum_{n=0}^{\infty} R_{0,n}(x) z^n \\ \text{and} \quad R_{i,j_i}(x,z) &= \sum_{n=0}^{\infty} R_{i,j_i,n}(x) z^n \end{aligned}\right\} \tag{14}$$

Multiplying equations (2) to (6) by z^n and summing over n, we get

$$\left(\frac{d}{dx} + \lambda^+ + \eta(x)\right) I(x,z) = 0 \tag{15}$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \lambda^- + \mu_0(x)\right) P_0(x,z) = 0 \tag{16}$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \lambda^- + \mu_{i,j_i}(x)\right) P_{i,j_i}(x,z) = 0, \quad i = 1,2,.....,M, \quad j_i = 1,2,.......,k_i \tag{17}$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \beta_0(x)\right) R_0(x,z) = 0 \tag{18}$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \beta_{i,j_i}(x)\right) R_{i,j_i}(x,z) = 0, \quad i = 1,2,.....,M, \quad j_i = 1,2,.........,k_i \tag{19}$$

Solving the partial differential equations (15) - (19), we obtain

$$I(x,z) = I(0,z) e^{-\lambda^+ x}(1 - A(x)) \tag{20}$$

$$P_0(x,z) = P_0(0,z) e^{-(\lambda^+ + \lambda^- - \lambda^+ C(z))x}(1 - B_0(x)) \tag{21}$$

$$P_{i,j_i}(x,z) = P_{i,j_i}(0,z) e^{-(\lambda^+ + \lambda^- - \lambda^+ C(z))x}(1 - B_{i,j_i}(x)), \quad i = 1,2,......,M, \quad j_i = 1,2,.....,k_i \tag{22}$$

$$R_0(x,z) = R_0(0,z) e^{-(\lambda^+ - \lambda^+ C(z))x}(1 - R_0(x)) \tag{23}$$

$$R_{i,j_i}(x,z) = R_{i,j_i}(0,z) e^{-(\lambda^+ - \lambda^+ C(z))x}(1 - R_{i,j_i}(x)), \quad i = 1,2,......,M, \quad j_i = 1,2,.....,k_i \tag{24}$$

Multiplying equations (7)-(13) by z^n and summing over n, we have

$$\begin{aligned} I(0,z) = {} & (q_0 + \delta_0 z)\int_0^{\infty} P_0(x,z)\,\mu_0(x)dx + \sum_{i=1}^{M} (q_i + \delta_i z)\sum_{j_i=1}^{k_i}\int_0^{\infty} P_{i,j_i}(x,z)\,\mu_{i,j_i}(x)dx \\ & + \int_0^{\infty} R_0(x,z)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^{\infty} R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx - \lambda^+ I_0 \end{aligned} \tag{25}$$

$$P_0(0,z) = \frac{1}{z}[\lambda^+ C(z) I_0 + \int_0^{\infty} I(x,z)\eta(x)dx + \lambda^+ C(z)\int_0^{\infty} I(x,z)dx] \tag{26}$$

$$P_{1,j_1}(0,z) = p_{j_1}\int_0^{\infty} P_0(x,z)\mu_0(x)dx, \qquad j_1 = 1,2,......,k_1 \tag{27}$$

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^{\infty} P_{i-1,j_{i-1}}(x,z)\mu_{i-1,j_{i-1}}(x)dx, \qquad i = 2,3,......,M, \quad j_i = 1,2,......,k_i \tag{28}$$

$$R_0(0,z) = \lambda^- \int_0^{\infty} P_0(x,z)dx \tag{29}$$

$$R_{i,j_i}(0,z)=\lambda^{-}\int_0^{\infty} P_{i,j_i}(x,z)dx, \quad i=1,2,3,......,M, \quad j_i=1,2,......,k_i \tag{30}$$

Using the expressions in equations (20) – (24) in the equations (25) – (30) and solving, we obtain

$$I(0,z)=\lambda^{+}I_0[C(z)T_1(z)-z]\big/D(z) \tag{31}$$

$$P_0(0,z)=\lambda^{+}I_0A^{*}(\lambda^{+})(C(z)-1)\big/D(z) \tag{32}$$

$$P_{1,j_1}(0,z)=p_{j_1}\lambda^{+}I_0A^{*}(\lambda^{+})(C(z)-1)B_0^{*}(g(z))\big/D(z), \qquad j_1=1,2,......,k_1 \tag{33}$$

$$P_{i,j_i}(0,z)=p_{j_i}\lambda^{+}I_0A^{*}(\lambda^{+})(C(z)-1)\Lambda_{i-1}^{*}(g(z))B_0^{*}(g(z))\big/D(z), i=2,3,......,M, \ j_i=1,2,......,k_i \tag{34}$$

$$R_0(0,z)=\lambda^{-}\lambda^{+}I_0A^{*}(\lambda^{+})(C(z)-1)((1-B_0^{*}(g(z)))/g(z))\big/D(z) \tag{35}$$

$$R_{i,j_i}(0,z)=p_{j_i}\lambda^{-}\lambda^{+}I_0A^{*}(\lambda^{+})(C(z)-1)\Lambda_{i-1}^{*}(g(z))((1-B_{i,j_i}^{*}(g(z)))/g(z))B_0^{*}(g(z))\big/D(z), \quad i=1,2,3,......,M, \ j_i=1,2,......,k_i \tag{36}$$

where $D(z)=z-[A^{*}(\lambda^{+})+C(z)(1-A^{*}(\lambda^{+}))]T_1(z)$

$$T_1(z)=(\mathrm{q}_0+\delta_0 z)\mathrm{B}_0^{*}(g(z))+\sum_{i=1}^{M}(\mathrm{q}_\mathrm{i}+\delta_\mathrm{i} z)\Lambda_\mathrm{i}^{*}(g(z))B_0^{*}(g(z))+\lambda^{-}((1-B_0^{*}(g(z)))/g(z))R_0^{*}(h(z))$$

$$+\lambda^{-}\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{\mathrm{i-1}}^{*}(g(z))((1-B_{i,j_i}^{*}(g(z)))/g(z))R_{i,j_i}^{*}(h(z))B_0^{*}(g(z))$$

$$g(z)=\lambda^{+}(1-C(z))+\lambda^{-} \qquad and \qquad h(z)=\lambda^{+}(1-C(z))$$

The probability generating function of the orbit size when the server is idle is given by

$$I(z)=\int_0^{\infty} I(x,z)dx$$
$$=I_0(1-A^{*}(\lambda^{+}))\big[C(z)T_1(z)-z\big]\big/D(z) \tag{37}$$

The probability generating function of the orbit size when the server is busy in providing essential service is given by

$$P_0(z)=\int_0^{\infty} P_0(x,z)dx$$
$$=\lambda^{+}I_0A^{*}(\lambda^{+})(C(z)-1)((1-B_0^{*}(g(z)))/g(z))\big/D(z) \tag{38}$$

The probability generating function of the orbit size when the server is busy in providing i^{th} (i=1,2,.....,M) stage j_i^{th} (j_i =1,2,......,k_i) mode of service is given by

$$P_{i,j_i}(z)=\int_0^{\infty} P_{i,j_i}(x,z)dx$$
$$=\lambda^{+}I_0A^{*}(\lambda^{+})(C(z)-1)B_0^{*}(g(z))p_{j_i}\Lambda_{i-1}^{*}(g(z))((1-B_{i,j_i}^{*}(g(z)))/g(z))\big/D(z) \tag{39}$$

The probability generating function of the orbit size when the server is under repair at essential stage is given by

$$R_0(z)=\int_0^{\infty} R_0(x,z)dx$$
$$=-\lambda^{-}I_0A^{*}(\lambda^{+})((1-B_0^{*}(g(z)))/g(z))(1-R_0^{*}(h(z)))\big/D(z) \tag{40}$$

The probability generating function of the orbit size when the server is under repair at i^th^ (i=1,2,.....,M) stage j_i^{th} (j_i =1,2,......,k_i) mode of service is given by

$$R_{i,j_i}(z) = \int_0^{\infty} R_{i,j_i}(x,z)dx$$

$$= -\lambda^- I_0 A^*(\lambda^+) p_{j_i} B_0^*(g(z)) \Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z)))/g(z))(1-R_{i,j_i}^*(h(z)))\big/D(z) \quad (41)$$

Using the normalizing condition

$I_0 + \lim_{z\to1} I(z) + \lim_{z\to1} P_0(z) + \lim_{z\to1} \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + \lim_{z\to1} R_0(z) + \lim_{z\to1} \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) = 1$, we obtain

$$I_0 = \frac{1 - m_1(1 - A^*(\lambda^+)) - T_1'(1)}{\begin{array}{l} A^*(\lambda^+)\{1 - \sum_{i=0}^{M} \delta_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) - (\delta_0 + q_0) f_0^{(1)} - \sum_{i=1}^{M} (\delta_i + q_i)[M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}] \\ + f_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}[\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} - M_{i-1}^{(1)}(1 - B_{i,j_i}^*)] B_0^*(\lambda^-) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) f_0^{(1)}\} \end{array}} \quad (42)$$

The probability generating functions of the orbit size and system size are given by

$$\begin{aligned} P_q(z) &= I_0 + I(z) + P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) \\ &= I_0 A^*(\lambda^+)[z - \sum_{i=0}^{M} (\delta_i z + q_i) \Lambda_i^*(g(z)) B_0^*(g(z)) + \lambda^+(C(z)-1)((1 - B_0^*(g(z))/g(z)) \\ &\quad + \lambda^+(C(z)-1) B_0^*(g(z)) \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z))(1 - B_{i,j_i}^*(g(z)))/g(z) \\ &\quad - \lambda^-((1 - B_0^*(g(z))/g(z)) - \lambda^- B_0^*(g(z)) \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z))(1 - B_{i,j_i}^*(g(z)))/g(z)]/D(z) \end{aligned} \quad (43)$$

$$\begin{aligned} P_s(z) &= I_0 + I(z) + zP_0(z) + z\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) \\ &= I_0 A^*(\lambda^+)[z - \sum_{i=0}^{M} (\delta_i z + q_i) \Lambda_i^*(g(z)) B_0^*(g(z)) + z\lambda^+(C(z)-1)((1 - B_0^*(g(z))/g(z)) \\ &\quad + z\lambda^+(C(z)-1) B_0^*(g(z)) \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z))(1 - B_{i,j_i}^*(g(z)))/g(z)] \\ &\quad - \lambda^-((1 - B_0^*(g(z))/g(z)) - \lambda^- B_0^*(g(z)) \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z))(1 - B_{i,j_i}^*(g(z)))/g(z)]/D(z) \end{aligned} \quad (44)$$

6. Performance Measures

- The probability that the server is idle is
  $I = \lim_{z\to1} I(z)$

$$= \frac{I_0(1-A^*(\lambda^+))\left(m_1+T_1'(1)-1\right)}{1-m_1(1-A^*(\lambda^+))-T_1'(1)} \tag{45}$$

- Mean number of customers in the orbit when the server is idle is

$$L_I = \lim_{z\to 1}\frac{d}{dz}I(z)$$
$$= I_0(1-A^*(\lambda^+))\left\{T_2\left[m_2+2m_1T_1'(1)+T_1''(1)\right]+T_3\left[m_1+T_1'(1)-1\right]\right\}\Big/2T_2^2 \tag{46}$$

- The probability that the server is busy in providing essential service is

$$P_0 \;=\; \lim_{z\to 1} P_0(z)$$
$$= \frac{I_0(\lambda^+ m_1/\lambda^-)A^*(\lambda^+)(1-B_0^*(\lambda^-))}{1-m_1(1-A^*(\lambda^+))-T_1'(1)} \tag{47}$$

- Mean number of customers in the orbit when the server is busy in providing essential service is

$$L_{P_0} = \lim_{z\to 1}\frac{d}{dz}P_0(z)$$
$$= I_0A^*(\lambda^+)\{T_2[(\lambda^+m_2/\lambda^-)(1-B_0^*(\lambda^-))-2(\lambda^+m_1/\lambda^-)\,f_0^{(1)}$$
$$+(\lambda^+m_1/\lambda^-)^2(1-B_0^*(\lambda^-))]+T_3[(\lambda^+m_1/\lambda^-)(1-B_0^*(\lambda^-))]/2T_2^2 \tag{48}$$

- The probability that the server is busy in providing additional services is

$$P \;=\; \lim_{z\to 1}\sum_{i=1}^{M-1}\sum_{j_i=1}^{k_i}P_{i,j_i}(z)$$
$$= \frac{I_0(\lambda^+m_1/\lambda^-)A^*(\lambda^+)B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))}{1-m_1(1-A^*(\lambda^+))-T_1'(1)} \tag{49}$$

- Mean number of customers in the orbit when the server is providing additional services is

$$L_P = \lim_{z\to 1}\frac{d}{dz}P(z)$$
$$= I_0A^*(\lambda^+)\{T_2[(\lambda^+m_2/\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))$$
$$+2(\lambda^+m_1/\lambda^-)[\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\,f_0^{(1)}+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)$$
$$-\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-)\,f_{i,j_i}^{(1)}]+2(\lambda^+m_1/\lambda^-)^2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)]$$
$$+T_3[(\lambda^+m_1/\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)]/2T_2^2$$
$$\tag{50}$$

- The probability that the server is under repair during essential service is

$$R_0 = \lim_{z\to 1} R_0(z)$$
$$= \frac{I_0 \lambda^+ m_1 A^*(\lambda^+)(1-B_0^*(\lambda^-))\beta_0^{(1)}}{1-m_1(1-A^*(\lambda^+))-T_1'(1)} \tag{51}$$

- Mean number of customers in the orbit when the server is under repair during essential service is

$$L_{R_0} = \lim_{z\to 1}\frac{d}{dz}R_0(z)$$
$$= I_0 A^*(\lambda^+)\{T_2[((\lambda^+ m_1)^2 \beta_0^{(2)} + \lambda^+ m_2 \beta_0^{(1)})(1-B_0^*(\lambda^-)) - 2\lambda^+ m_1 f_0^{(1)}\beta_0^{(1)}$$
$$+ 2((\lambda^+ m_1)^2/\lambda^-)(1-B_0^*(\lambda^-))\beta_0^{(1)}] + T_3[(\lambda^+ m_1/\lambda^-)(1-B_0^*(\lambda^-))\beta_0^{(1)}]/2T_2^2 \tag{52}$$

- The probability that the server is under repair during additional services is

$$R = \lim_{z\to 1}\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$
$$= \frac{I_0 \lambda^+ m_1 A^*(\lambda^+)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\Lambda_{i-1}^*(\lambda^-)\beta_{i,j_i}^{(1)}}{1-m_1(1-A^*(\lambda^+))-T_1'(1)} \tag{53}$$

- Mean number of customers in the orbit when the server is under repair during additional services is

$$L_R = \lim_{z\to 1}\frac{d}{dz}R(z)$$
$$= I_0 A^*(\lambda^+)\{ T_2[\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))((\lambda^+ m_1)^2 \beta_{i,j_i}^{(2)} + \lambda^+ m_2 \beta_{i,j_i}^{(1)})$$
$$+ 2\lambda^+ m_1[\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-)) f_0^{(1)}\beta_{i,j_i}^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)} B_0^*(\lambda^-)$$
$$- \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-)) f_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)}] + 2((\lambda^+ m_1)^2/\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)\beta_{i,j_i}^{(1)}]$$
$$+ T_3[\lambda^+ m_1 \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)\beta_{i,j_i}^{(1)}]/2T_2^2 \tag{54}$$

- Expected orbit size is given by

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z)$$

$$
\begin{aligned}
&= I_0 A^*(\lambda^+)\{2T_2[-2\delta_0 f_0^{(1)} - (\delta_0+q_0) f_0^{(2)} - 2\sum_{i=1}^{M}\delta_i[M_i^{(1)}B_0^*(\lambda^-)+\Lambda_i^*(\lambda^-) f_0^{(1)}] - \sum_{i=1}^{M}(\delta_i+q_i)[M_i^{(2)}B_0^*(\lambda^-) \\
&+2M_{i-1}^{(1)} f_0^{(1)} + \Lambda_i^*(\lambda^-) f_0^{(2)}] + f_0^{(2)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}(1-B^*_{i,j_i}(\lambda^-))M_{i-1}^{(2)}B_0^*(\lambda^-) - 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}M_{i-1}^{(1)}(1-B^*_{i,j_i}(\lambda^-)) f_0^{(1)} \\
&+2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} f_{i,j_i}^{(1)} M_{i-1}^{(1)}B_0^*(\lambda^-) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_0^{(2)}(1-B^*_{i,j_i}(\lambda^-)) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_0^{(1)} f_{i,j_i}^{(1)} \\
&+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} f_{i,j_i}^{(2)}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-)] + T_4T_3\}/2T_2^2
\end{aligned}
\tag{55}
$$

where

$$
\begin{aligned}
T_1''(1) &= 2[\sum_{i=0}^{M}\delta_i\Lambda_i^*(\lambda^-) f_0^{(1)} + \sum_{i=1}^{M}\delta_i M_i^{(1)}B_0^*(\lambda^-)] + (\delta_0+q_0) f_0^{(2)} + \sum_{i=1}^{M}(\delta_i+q_i)[M_i^{(2)}B_0^*(\lambda^-) + M_i^{(1)}\mu_0^{(1)} + \Lambda_i^*(\lambda^-) f_0^{(2)}] \\
&- f_0^{(2)} - 2(\lambda^+ m_1/\lambda^-)[f_0^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}M_{i-1}^{(1)}(1-B^*_{i,j_i}(\lambda^-))B_0^*(\lambda^-) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)}B_0^*(\lambda^-) \\
&- \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B^*_{i,j_i}(\lambda^-)) f_0^{(1)}] - 2\lambda^+ m_1[f_0^{(1)}\beta_0^{(1)} - B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}M_{i-1}^{(1)}(1-B^*_{i,j_i}(\lambda^-))\beta_{i,j_i}^{(1)} \\
&+ B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B^*_{i,j_i}(\lambda^-))\beta_{i,j_i}^{(1)} f_0^{(1)}] + (\lambda^+ m_2/\lambda^-)[1-B_0^*(\lambda^-)) \\
&+ B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B^*_{i,j_i}(\lambda^-))] + 2(\lambda^+ m_1/\lambda^-)^2[1-B_0^*(\lambda^-)) + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B^*_{i,j_i}(\lambda^-))] \\
&+ 2((\lambda^+ m_1)^2/\lambda^-)[(1-B_0^*(\lambda^-)))\beta_0^{(1)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B^*_{i,j_i}(\lambda^-))\beta_{i,j_i}^{(1)}] + (\lambda^+ m_2/\lambda^-)[(1-B_0^*(\lambda^-)))\beta_0^{(1)} \\
&+ B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B^*_{i,j_i}(\lambda^-))\beta_{i,j_i}^{(1)}] + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}M_{i-1}^{(2)}(1-B^*_{i,j_i}(\lambda^-))B_0^*(\lambda^-) - B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}M_{i-1}^{(1)} f_{i,j_i}^{(1)} \\
&+ 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}M_{i-1}^{(1)}(1-B^*_{i,j_i}(\lambda^-)) f_0^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(2)}B_0^*(\lambda^-) - 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} f_0^{(1)} \\
&+ (\lambda^+ m_1)^2[(1-B_0^*(\lambda^-)))\beta_0^{(2)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B^*_{i,j_i}(\lambda^-))\beta_{i,j_i}^{(2)}]
\end{aligned}
$$

$$T_2 = 1 - m_1(1-A^*(\lambda^+)) - T_1'(1)$$

$$T_3 = T_1''(1) + (1-A^*(\lambda^+))\left(m_2 + 2m_1T_1'(1)\right)$$

$$
\begin{aligned}
T_4 &= 1 - \sum_{i=0}^{M}\delta_i\Lambda_i^*(\lambda^-)B_0^*(\lambda^-) - (\delta_0+q_0) f_0^{(1)} - \sum_{i=1}^{M}(\delta_i+q_i)[M_i^{(1)}B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}] \\
&+ f_0^{(1)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}[\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} - M_{i-1}^{(1)}(1-B^*_{i,j_i}(\lambda^-))] - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_0^{(1)}(1-B^*_{i,j_i}(\lambda^-))
\end{aligned}
$$

$$f_0^{(2)} = (\lambda^+ m_1)^2\int_0^\infty x^2 e^{-\lambda^- x} b_0(x)dx + \lambda^+ m_2\int_0^\infty x e^{-\lambda^- x} b_0(x)dx \quad ,$$

$$f_{i,j_i}^{(2)} = (\lambda^+ m_1)^2\int_0^\infty x^2 e^{-\lambda^- x} b_{i,j_i}(x)dx + \lambda^+ m_2\int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

- Expected system size is given by

$$L_s = L_q + P_0 + P \qquad (56)$$

7. Stochastic Decomposition

Average system size can be expressed as the sum of the mean number of customers in the classical batch arrival G-queue with multi-stage fluctuating modes of services and feedback (L_ϕ) and the mean number of customers in the orbit when the server is idle (L_ψ).

The probability generating function of the system size of the classical batch arrival G-queue with multi-stage fluctuating modes of services and feedback is given by

$$\phi(z) = \lim_{A^*(\lambda^+)\to 1} P_s(z)$$

$$= \frac{\begin{array}{l}[1-T_1'(1)][z-\sum_{i=0}^{M}(\delta_i z+q_i)\Lambda_i^*(g(z))B_0^*(g(z))+z\lambda^+(C(z)-1)((1-B_0^*(g(z))/g(z)) \\ +z\lambda^+(C(z)-1)B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(g(z))(1-B_{i,j_i}^*(g(z)))/g(z)] \\ -\lambda^-((1-B_0^*(g(z))/g(z))-\lambda^- B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(g(z))(1-B_{i,j_i}^*(g(z)))/g(z)]\end{array}}{T_4\,[z-T_1(z)]} \qquad (57)$$

The probability generating function of the number of customers in the orbit when the server is idle is given by

$$\psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

$$= \frac{[z-T_1(z)][1-m_1(1-A^*(\lambda^+))-T_1'(1)]}{[1-T_1'(1)]\,D(z)} \qquad (58)$$

By the equations (44) , (57) and (58), we obtain

$$P_s(z) = \phi(z)\,\psi(z)$$

Mean number of customers in the classical batch arrival G-queue with multi-stage fluctuating modes of services and feedback is given by

$$L_\phi = \lim_{z\to 1}\frac{d}{dz}\phi(z)$$

$$
= \frac{\begin{array}{l}[1-T_1'(1)]\{\lambda^+ m_1[\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} f_{i,j_i}^{(1)} M_{i-1}^{(1)} B_0^*(\lambda^-) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} f_0^{(1)} \\ -\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-)) f_0^{(1)} - \sum_{i=1}^{M}(\delta_i + q_i) M_i^{(1)} f_0^{(1)}] \\ +(\lambda^+ m_1/\lambda^-)[(1-B_0^*(\lambda^-)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)] \\ -\sum_{i=1}^{M}\delta_i M_i^{(1)} B_0^*(\lambda^-) + \sum_{i=0}^{M}\delta_i \Lambda_i^*(\lambda^-) f_0^{(1)} - \sum_{i=1}^{M}(\delta_i+q_i) M_i^{(2)} B_0^*(\lambda^-) \\ +\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}(1-B_{i,j_i}^*(\lambda^-)) M_{i-1}^{(2)} B_0^*(\lambda^-) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} f_{i,j_i}^{(2)} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-)]\} + T_4 T_1''(1)\end{array}}{2T_4T_2} \tag{59}
$$

Mean number of customers in the orbit when the server is idle is given by

$$
L_\psi = \lim_{z\to 1}\frac{d}{dz}\psi(z)
$$

$$
= \frac{T_3[1-T_1'(1)] - T_2 T_1''(1)}{2T_2} \tag{60}
$$

From the results in equations (56) , (59) and (60), we get

$$
L_s = L_\phi + L_\psi
$$

8. Reliability Indexes

The system availability A(t) at time t is the probability that the server is either working for a customer or in an idle period

The steady state availability of the server A is

$$
A = 1 - R_0 - R
$$

$$
= 1 - \frac{\lambda^+ m_1\left[(1-B_0^*(\lambda^-))\beta_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\Lambda_{i-1}^*(\lambda^-)\beta_{i,j_i}^{(1)}\right]}{1-m_1(1-A^*(\lambda^+)) - T_1'(1)}
$$

The steady state failure frequency of the server F is

$$
F = \lambda^- \left(P_0 + P\right)
$$

$$
= \frac{\lambda^+ m_1\left[(1-B_0^*(\lambda^-)) + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\right]}{1-m_1(1-A^*(\lambda^+)) - T_1'(1)}
$$

References

[1] Ayyappan G., and Supraja R. (2018), An M^x/G(a,b)/1 Queueing System with Two Fluctuating Modes Of Service Under Bernoulli Vacation Schedule For Unreliable Server and Delaying Repair, International Journal of Pure and Applied Mathematics, Volume 120, No. 8, 49-57.

[2] Ebenesar Anna Bagyam J., and Udaya Chandrika K. (2015), Bulk Arrival Multi-Stage Retrial Queue with Vacation, Applied Mathematical Sciences, Vol. 9, no. 54, 2691 - 2706.

[3] Gao S., and Wang J. (2014), Performance and Reliability Analysis of an M/G/1-G Retrial Queue with Orbital Search and Non-Persistent Customers, European Journal of Operational Research, 236 (2), 561-572.
[4] Gelenbe E. (1989), Random Neural Networks with Negative and Positive Signals and Product Form Solution, Neural Computation, 1, 502-510.
[5] Jewgeni H., Dshalalo and Merie (2018), Fluctuation analysis in queues with several operational modes and priority customers, An Official Journal of the Spanish Society of Statistics and Operations Research, Volume 26, Issue 2, 309–333.
[6] Monita Baruah, Kailash C. Madan and Tillal Eldabi (2014), A Batch Arrival Single Server Queue with Server Providing General Service in Two Fluctuating Modes and Reneging during Vacation and Breakdowns, Journal of Probability and Statistics Volume 2, 1-12.
[7] Radha J., Indhira K., and Chandrasekaran V. M. (2017), A group arrival retrial G - queue with multi optional stages of service, orbital search and server breakdown , IOP Conf. Series: Materials Science and Engineering 263.
[8] Rajadurai P., Saravanarajan M.C., and Chandrasekaran V.M. (2015). Analysis of Repairable M[X]/(G1, G2)/1-Feedback Retrial G-Queue with Balking and Starting Failures under at most J Vacations, Journal of Applications and Applied Mathematics, 10 (1), 13-39.
[9] Rajadurai P., Venkatesh S., and Parameswari K. (2018), An M/G/1 Retrial Queue with Fluctuating Modes of Services, International Journal of Engineering & Technology, 7 , 762-765.
[10] Sumitha D., and Udaya Chandrika K., (2015), Batch Arrival Retrial Queue with Positive and Negative Customers, Priority or Collisions, Delayed Repair and Orbital Search, Elixir International Journal : Applied Mathematics, 88, 36182-36189.
[11] Varalakshmi M., Chandrasekaran V.M., and Saravanarajan M.C. (2017), A study on M/G/1 retrial G - queue with two phases of service, immediate feedback and working vacation, IOP Conf. Series: Materials Science and Engineering 263, 1-15.
[12] Yamuni A., Kirupa K., and Udaya Chandrika K. (2016), Bulk Arrival Retrial Queue with Fluctuating Modes Of Service, Immediate Feedback, Server Vacation And Orbital Search, International Journal Of Engineering Sciences & Research Technology, Volume 9, Issue 5, 760-768.