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ON CYCLIC COMMUTATIVITY NEAR – RINGS III

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AbstractA right near – ring N is called weak commutative if $xyz = xzy$ for every $x, y, z \in N$. A right near – ring N is called pseudo commutative, if $xyz = zyx$ for all $x, y, z \in N$. A right near – ring N is called quasi weak commutative, if $xyz = yxz$ for every$x, y, z \in N$. It is quite natural to investigate the properties of a right near – ring N satisfying $xyz = yzx$ for every$x, y, z \in N$. We call such a near –ring as cyclic commutative near – ring. We obtain some interesting results on cyclic commutative near – rings.

1. Introduction

Through out this paper N denotes a right – near ring with at least two elements. For any non –empty subset A of N, we denote A – {0} = A*.

Definition 1.1 An element $a \in \mathbf{N}$ is said to be

1. Idempotent if $a^2 = a$

2. Nilpotent, if there exists a positive integer k such that ak = 0

Definition 1.2 A near – ring N is said to be regular if for each $a \in N$,there exists $b \in N$ such that:

$$a = aba.$$

Result: (Theorem 1.62 Pilz [9]) Each Near – ring N is isomorphic to subdirect product of subdirectly irreducible near – ring.

Definition 1.3 A near – ring N is said to be zero commutative if ab = 0 implies ba = 0, where $a, b \in N$.

Result: If N is zero symmetric, then every left ideal A of N is an N – subgroup of N. Every ideal I of N satisfies the condition $NIN \subseteq I$. (i.e) every ideal is an N – subgroup $N^*I^*N^* \subseteq I^*$

Result: Let N be a near – ring. Then the following are true.

- If A is an ideal of N and B is any subset of N, then (A:B) = {n ∈ N such that $nB \subseteq A$} is always a left ideal.
- If A is an ideal of N and B is an N – subgroup , then (A:B) is an ideal.
- In particular if A and B are ideals of a zero – symmetric near – ring, then (A:B) is an ideal.

Result:

1. Let N be a regular near – ring, a∈ N and a = axa, then ax, xa are idempotents and so the set of idempotent elements of N is non – empty.
2. axN = aN and Nxa = Na.
3. N is S and $\mathbf{S}'$ near – rings.

Result: Let N be a zero – symmetric reduced near – ring. For any a, b ∈ N and for any idempotent element e ∈ N, abe = aeb.

Result: (Gatzer [6] and Fain [3]) A near –ring N is subdirectly irreducible if and only if the intersection of all non – zero ideals of N is not zero.

Result: (Gatzer [6]) Each simple near –ring issubdirectly irreducible.

Result: (Pilz [9]) Any non - zero symmetric near – ring N has IFP if and only if (o:s) is an ideal for any subset S of N.

Result: (Oswald [8]) An N – subgroup A of N is essential if $\mathbf{A} \cap \mathbf{B} = \{0\}$, where B is any N subgroup of N implies B = {0}.

Definition1.13

A near – ring N is said to be reduced if N has no non – zero nilpotent elements.

Definition 1.4
A near – ring N is said to be integral near –ring, if N has no non – zero divisors.

Lemma 1.5
Let N be a near – ring. If for all a∈ N, $a^2 = 0$ implies a = 0, then has no non – zero nilpotent elements.

Definition 1.6
Let N be a near – ring. N is said to satisfy intersection of factors property(IFP) if ab = 0, anb = 0 for all n ∈ N, where a,b ∈ N

Definition 1.7
1. An ideal I of N is called a prime ideal if for all A,B of N, AB is a subset of I ⇒ A is subset of I or B is subset of I.
2. I is called semi – prime ideal if for all ideals A of N, A^2 is subset of I implies A is subset of I.
3. I is called a completely semi – prime ideal if for any x ∈ N, $x^2 \in I \Rightarrow x \in I$
4. A completely prime ideal if for any x, y ∈ I ⇒x∈I or y ∈ I
5. N is said to have strong IFP, if for all ideals I of N, ab∈ I implies anb ∈I for all n ∈N.

Result: Let N be a Pseudo commutative near – ring. Then every idempotent element is central.

Result: Let N be a regular quasi weak commutative near – ring. Then:

1. A = $\sqrt{\mathbf{A}}$, for every N subgroup A of N
2. N is reduced
3. N has IFP

Result: Let N be a regular quasi weak commutative near – ring. Then every N sub group is an ideal N = Na = Na^2 = aN = aNa for all a ∈ N

Result: Let N be a quasi weak commutative near – ring. For every ideal I of N, (I :S) is an ideal of N where S is any subset of N.

Result: Every quasi weak commutative near – ring N is isomorphic to a sub – direct product of sub – directly irreducible quasi weak commutative near – rings.

2. Main Results

Theorem 2.1. Let N be a regular cyclic commutative near – ring. Then

i) $P \cap Q = PQ$ for any two N – subgroups P,Q of N

ii) $P = P^2$ for every N subgroup (Ideal) P of N

iii) If P is a proper N – subgroup of N, then each element if P is a zero divisor.

iv) $N_a N_b = N_a \cap N_b = N_{ab}$ forall $a, b \in N$

v) Every N- subgroup of N is essential if N is integral

Proof:

(i) Let P and Q be two subgroup of N. Then by [2] they are ideals. Hence PQ ⊆ Q. So PQ ⊆ P∩Q.

Let a ∈ P ∩ Q. Since N is regular, there exists b∈ N, such that a = aba = (ab), a∈ (PN)Q ⊆ PQ. Hence $P \cap Q = PQ$. This Completes (i)

(ii) Taking Q = P in (i) we get $P = P^2$

(iii) Let P be a proper N- subgroup of N

Let $0 \neq a \in P$. Then by (ii) $Na = (N_a)^2 = N_a N_a$

If for every $n \in N$, there exists $x, y \in N$ such that $n_a = x_a y_a$

So, $(n - xay)a = 0$. If a is not a zero divisor, then n – xay = 0.

That is, n = xay∈ NPN⊆ P.

Hence N = P, contradicting P is a proper ideal of N. So a is a zero divisor of N.

This proves (iii)

(iv) Since N_a and N_b are N – subgroups, $N_a \cap N_b = N_a N_b$

Since $N_a \subseteq N, N_a \cap N = N_a = N_a N_a \subseteq N_a N$ .... (1)

and N_a is an ideal

This implies that, $N_a N = (N_a)N \subseteq N_a = N_a \cap N$................. (2)

Hence $N_a = N_a \cap N = N_a N$

This implies $N_{ab} = (N_a)b = (N_a N)b = N_a N_b = N_a \cap N_b$

This proves (iv)

v) Let P be a non –zero N – subgroups of N.

Suppose there exists an N – subgroup Q of N such that $P \cap Q = \{0\}$.

Then by (i)PQ = {0} and since N is an integral near – ring, Q = {0}.

This proves (v).

Theorem 2.2

Let N be a regular Cyclic Commutative near –ring and P be a proper N subgroup of N. Then the following are equivalent

(i) P is a prime ideal

(ii) P is completely prime ideal

(iii) P is a primary ideal

(iv) P is a maximal ideal

Proof: Let P be a proper N – subgroup of N.

(i) $\Longrightarrow$ (ii)

Assume P is prime. Let ab ∈ P. By Theorem 2.1 (iv), $N_a N_b = N_{ab} \subseteq NP \subseteq P$. Also by [2], N_a and N_b are ideals of N. Since P is prime, $N_a N_b \subseteq P$ implies $N_a \subseteq P$ or $N_b \subseteq P$

Since N is regular, there exists $x, y \in N$ such that a = axa and b = byb.

If $N_b \subseteq P$, then b = byb ∈ $N_b \subseteq P$, that is, either a∈ P or b∈ P. Hence P is completely prime.

(ii)$\Longrightarrow$ (i) is obvious.

(ii) $\Longrightarrow$ (iii)

Let a, b, c ∈ N. By Theorem 2.1 (iv) Nab = NaNb. Since $N_a \cap N_b = N_b \cap N_a$, we have Nab = Nba for all a,b∈N. Then Nabc = Nacb = Nbca. Suppose abc∈ P and ab ∉ P by (ii) c ∈ P . Again suppose abc∈ P and ac ∉ P . Since N is regular, acb∈ Nacb⊆NP ⊂ P. Thus acb = (ac)b∈ P implies b ∈ P (by (ii))

Continuing in the same way, we can early prove that if abc∈ P and if the product of any two of a,b,c does not belong to P, then the third belongs to P . This Proves (iii)

(iii) ⇒ (ii)

Let abc∈ P and a ∉ P . Since N is regular, a = axa for some x ∈ N. We shall now prove that xa∉ P. Suppose xa∈ P, then a = axa = a (ax) ∈ NP ⊆P. Which is a contradiction. Therefore xa∉ P. Also x(ab) ∈ NP ⊆P. Thus xab ∈ P and xa ∉P . As P is a prime ideal of N, bk∈ P for some integer.

Hence b ∈ $\sqrt{P}$. But by Theorem 1.8 (a) [4], $\sqrt{P}$ = P. So b ∈ P.

(i) $\Longrightarrow$ (iv)

Let J be an ideal of N such that $P \subseteq J \subseteq N$. Suppose P = J, there is nothing to prove.

Assume P does not contains J. We shall prove that J = N. Let $a \in J/P$. Since N is regular, there exists $x \in N$ such that a = axa . Then a =axa= xaa (Cyclic Commutative) = $xa^{2.}$ So for all $n \in N$, na = nxa2 and this implies (n – nxa)a = 0. Since N has IFP, we get (n – nxa)ya = 0 for all $y \in N$. That is, (n – nxa) Na = 0 . Consequently N (n – nxa) Na = N0 = {0 }. If b = n – nxa, then NbNa = {0} $\subseteq$ P. Since P is prime ideal and Na and Nb are ideals in N, either Na$\subseteq$ P or $N_b \subseteq P$. If Na$\subseteq$ P, then a = axa$\in$ P, which is a contradiction. Hence $N_b \subseteq P \subseteq J$.

Since N is regular, there exists $g \in N$ such that b = byb. That is, b = (by)b$\in$ Nb$\subseteq$ J. That is, b = n- nx$\in$ J. Since a$\in$ J, nxa $\in$nJ$\subseteq$ J. Therefore n$\in$J.Hence J = N. So P is maximal (iv) $\Rightarrow$ (i) is obvious.

This completes the proof of the theorem.

Theorem2.3

Any Cyclic Commutative near – ring N with left identity is commutative

Proof: Let a,b$\in$N and $c \in N$ be the left identity. Then ab = abe = bea = (be)a = ba. Hence N is commutative.

Theorem 2.4

Let N be a sub directly irreducible cyclic commutative near – ring. Then N is either simple with each non – zero idempotent element in an identity or the intersection of non – zero ideals of N has no non – zero idempotents.

Proof: Let N be a sub directly irreducible cyclic commutative near – ring. Suppose that N is simple. Let $e \in N$ be non – zero idempotent element. Then by result 8 [4], N has IFP by Theorem 1.00 [4], (o : e) is an ideal. Since $e \notin$ (o : e) and N is simple, we get (o :e) ={ 0 }. Hence (ene – en)e = ene^2 –ene = ene – ene , for all $n \in N$. This implies ene – en = 0

That is: ene = en ......(3)

Also since N is cyclic commutative

$$ene = nee = ne^2 = ne \quad ...... (4)$$

(3) and (4) gives ne = en , for all $n \in N$ ......(5)

Also, (ne – n)e = ne^2 – ne = nee – ne = 0 for all $n \in N$.

This implies: ne = n for all $n \in N$ ......(6)

(5) and (6) gives: ne = en = n ......(7)

Hence e is an identity.

Suppose N is not simple.

Let I be the intersection of non-zero ideal of N. Since N is subdirectly irreducible, we have I ≠ {0} Suppose that I contains a non-zero idempotent e. We claim that e is a right identity. If not, there exists n ∈ N such that ne ≠ n. Hence ne – n ≠ o. Since (ne – n)e = 0, we have ne – n ∈ (o : e) and hence (o: e) is a non-zero ideal of N. Therefore I ⊆ (o : e). Hence e ∈ I ⊂ (o : e), that is e ∈(o : e)

This contradiction leads to conclude that e is a right identity of N. Hence for all n ∈ N, n = ne ∈ NI⊂ I. This implies that I = N, again a contradiction. Hence the intersection of the non-zero ideal of N has non-zero idempotents. This proves the theorems.

Theorem 2.5

Let N be a regular cyclic commutative near-ring. Then the following are equivalent

(i) N is sub directly irreducible.

(ii) Non-zero idempotents of N are not zero divisors.

(iii) N is simple.

Proof: (i) ⇒ (ii)

Let J be the set of all non-zero idempotents of N, which are zero divisors too. We shall prove that J is empty. Suppose J ≠ P. Let I = ⋂{ (o:e) / e ∈ J }. Since N is sub directly irreducible, I ≠ {0}, by Result 9 ([6] [3]). Let a∈ I be a non-zero element.

Since N is regular, there exists b ∈N, such that a = aba ......(8)

Also ab, ba are idempotents. Since

s a = 0. This is a contradiction as a ≠ 0. Hence J is empty.

(ii)⟹(iii)

Let I be a non –zero ideal of N and 00 ≠ a∈ I, ae = 0, for all e ∈ J.......(9)

Then (ae)b = 0. Since N is zero symmetric. b(ae) = 0. That is, (ba)e = 0. Hence ba is a zero divisor and so ba∈ J. So by (9), a(ba) = 0. That i ≠ x = xyx......(10)

Also yx is an idempotent element of N. Therefore for every n ∈ N, nx = nxyx . That is (n – nxy) x = Since N has IFP (n – nyx)yx = 0, by (ii) n – nxy = 0.That is, for every n ∈ N, n = nxy ∈ NIN ⊂ I

Thus N ⊂ I.This prove that N has no non-zero trivial ideal of N. So N is simple.

(iii) ⟹ (i)

This follows from the result.

Corollary 2.6

Let N be a regular cyclic commutative near – ring. Then N is subdirectly irreducible if and only if N is a field.

Proof: By theorem 2.4 and 2.5, every non-zero idempotents is an identity. Let $a \neq 0 \in N$. Since N is regular, a = aba for some $b \in N$. That is, a = (ab)a. That is inverse exists for every $a \in N$. Hence N is a field. The converse is obvious.

Theorem 2.7

Let N be a regular cyclic commutative near-ring. Then N is isomorphic to a sub direct product of fields.

Proof: By result 1.22[4], N is isomorphic to sub direct product of sub directly irreducible near-rings $N\alpha$'s, where each $N\alpha$ is regular weak commutative.

Corollary 2.8

Let N be a regular cyclic commutative near-ring. Then N has no non-zero divisors if and only if N is a field.

Proof: Proof follows from the theorem.

References

[1] Foster .A.L., The theory if Boolean like rings. Amer. Math. Soc, Vol.59,1946.

[2] Gopalakrishnamoorthy . G and Geetha.S, on Quasi weak commutative near-ring, Int.Jour of Math. Research, Vol.5, No 5,2013, 431 – 440.

[3] Gopalakrishnamoorthy . G and Anitha.S, On Commutative Property of Q k,n,Qk,m, Pk,n,Pmand Qm Rings, jour. OF Inst. Of Mathematicsand Computer Sciences, Vol.23, No.2 (2010) 63- 70.

[4] KetselaHailu, BerhanuBekeleBelayneh, ZelabesmTeshome and K. Venkateswarlu, Boolean like semi ring of fractions, Inter. Jour.Of Math.Archieve, Vol.3, No.4, 2012, 1554 – 1560.

[5] Pilz.Gunter, Near – Rings, The theory and its applications, North Holland, 1983.

[6] Swaminathan.V, Boolean – like rings, Ph.D.dissertation, Andhra University, India , 1982.

[7] Swaminathan.V, On Fosters Boolean – like rings, Math. Seminar Notes, Kobe University, Japan, Vol.8, 1980, 347 -367.

[8] Uma.S, Balakrishnan.R and TamizhChelvam.T, Pseudo Commutative near –rings, Scientia Magna, Vol.6,No.2, 2010, 75 -85.

[9] Yaqub.A, A generalization of certain Rings of A.L. Foster, University of California, 1962.