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FIXED POINT THEOREMS IN GENERALIZED b-FUZZY METRIC SPACES FOR PROXIMAL CONTRACTION

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Abstract: In this paper, we prove the existence of some fixed point in Generalized b – Fuzzy Metric Spaces for mapping satisfying proximal contractive conditions.

Keywords:t- norm, Generalized b – Fuzzy Metric Spaces, F-bounded, Fuzzy Approximately Compact, Proximal Contraction.

AMS Subject Classification: 47H10, 47H04.

1. Introduction

Fuzzy set was defined by Zadeh [5] in 1965 which is a mathematical frame to vagueness or uncertainity in daily life. Kramosil and Michalek [4] introduced fuzzy metric spaces and this concept was modified by George and Veeramani in 1994[1]. In 2006, S. Sedghi and N. Shobe proved common fixed point theorem in $\mathcal{M}$ – fuzzy metric spaces.On the other hand, Bakhtin [11] introduced the notion b – metric spaces. Sedghi and Shobe [10] combined the concepts of fuzzy set and b – metric space to introduce a b – fuzzy metric space. Hussain, Salimi and Parvaneh [6] derived any new fixed point results of mappings defined on triangular partially ordered fuzzy b – metric spaces. In 2016, S. Nadban [9] studied the concepts of fuzzy quasi b – metric and fuzzy quasi pseudo b – metric spaces. In 2007, Dosenovic, Javaheri and Shobe [12] proved coupled coincidence fixed point theorems in complete b – fuzzy metric spaces.

2. Preliminaries

Definition 2.1

A binary operation $*$: [0, 1] X [0, 1] $\rightarrow$ [0, 1]is said to be continuous t-norm if for any $a, b, c, d \in [0,1]$, the following conditions hold:

i. $*$ is associative and commutative,
ii. $*$ is continuous,
iii. $a * 1 = a$,
iv. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Three examples of a continuous t-norm are $a * b = ab$, $a * b = \min\{a, b\}$ and$a * b = \max\{a + b - 1, 0\}$.

Definition 2.2

A quadruple (X, $\mathcal{M}$,* , b) is called a generalized b – fuzzy metric spaces with $b \geq 1$ if X is an arbitrary non-empty set, * is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,

i. $\mathcal{M}(x, y, z, t) > 0$,
ii. $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,
iii. $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,
iv. $\mathcal{M}(x, y, z, t + s) \geq \mathcal{M}\left(x, y, a, \frac{t}{b}\right) * \mathrm{M}\left(a, z, z, \frac{s}{b}\right)$,
v. $\mathcal{M}(x, y, z, \ \cdot) : (0, \infty) \rightarrow [0,1]$ is continuous.

Note that generalized b – fuzzy metric spaces are a generalized fuzzy metric spaces if b = 1, but the converse does not hold in general.

Definition 2.3

Let (X, $\mathcal{M}$,* , b) be a generalized b – fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}$.

Definition 2.4

Let (X, $\mathcal{M}$,* , b) be a generalized b – fuzzy metric space, then

i. A sequence $\{x_n\}$ in X is said to be convergent to x if for each $t > 0$, $\mathcal{M}(x, x, x_n, t) \rightarrow 1$ as $n \rightarrow \infty$.
ii. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $0 < \mathcal{E} < 1$ and t>0, there exist $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_n, x_m, t) > 1 - \mathcal{E}$ for each $n, m \geq n_0$.
iii. A generalized b – fuzzy metric spaces are said to be complete if every Cauchy sequence is convergent.

Definition 2.5

Let A and B be two nonempty subsets of a generalized b – fuzzy metric space(X, $\mathcal{M}$,* , b). Define$A_0(t) = \{x \in A : \mathcal{M}(x, y, y, t) = \mathcal{M}(A, B, B, t)$ for some $y \in B\}$ and
$B_0(t) = \{y \in B : \mathcal{M}(x, y, y, t) = \mathcal{M}(A, B, B, t)$for some $x \in A\}$,
where, $\mathcal{M}(A, B, B, t) = \sup\{ \mathcal{M}(a, b, b, t) : a \in A, b \in B\}$.A distance of a point $x \in X$ from a non-empty set A is defined by $\mathcal{M}(x, A, A, t) = sup\mathcal{M}(x, a, a, t)$, where $t > 0$.

Definition 2.6

A point x in A is said to be the optimal coincidence point of a pair (g, T) of mappings $T : A \to B$ and $g : A \to A$ if $\mathcal{M}(gx, Tx, Tx, t) = \mathcal{M}(A, B, B, t)$ holds.

Definition 2.7

Let A, B be nonempty subsets of a generalized b - fuzzy metric space (X, $\mathcal{M}$,$*$, b). A set B is said to be fuzzy approximately compact with respect to A if for every sequence $\{y_n\}$ in B, if $\mathcal{M}(x, y_n, y_n, t) \to \mathcal{M}(x, B, B, t)$ for some $x \in A$, then $x \in A_0(t)$.

Let Ω be the set of all mappings $\eta : (0,1] \to [0, \infty)$ which satisfy the following properties:

i. η is continuous and decreasing,
ii. $\eta(t) = 0$ iff $t = 1$,
iii. For any $r, s \in (0,1]$ with $r * s > 0$, wehave $\eta(r * s) \leq \eta(r) + \eta(s)$, where $*$ is any continuous t-norm.

Definition 2.8

A mapping $T : A \to B$ is said to be generalized b – fuzzy proximal contraction of type –I with respect to$\eta \in \Omega$, if there exist $k \in (0,1)$ such that for any $x, y, u, v \in A$ and $t > 0$, we have$\mathcal{M}(u, Tx, Tx, t) = \mathcal{M}(A, B, B, t)$and $\mathcal{M}(v, Ty, Ty, t) = \mathcal{M}(A, B, B, t)$which implies that $\eta[\mathcal{M}(u, v, v, t)] \leq k\eta[\mathcal{M}(x, y, y, t)]$.

3. Main Result

Theorem 3.1

Let (X, $\mathcal{M}$,$*$, b) be a complete generalized b – fuzzy metric space and $T : A \to B$ be a generalized b – fuzzy proximal contraction of type – I with $T(A_0(t)) \subseteq B_0(t)$ for each $t > 0$. If B is fuzzy approximately compact with respect to a non-empty closed subset A in X. Then there exists an element $x^* \in A$ such that $\mathcal{M}(x^*, Tx^*, Tx^*, t) = \mathcal{M}(A, B, B, t)$.

Proof:

Let x_0 be an arbitrary element in $A_0(t)$. As $T(A_0(t)) \subseteq B_0(t)$, we may choose an element $x_1 \in A_0(t)$ such that $\mathcal{M}(x_1, Tx_0, Tx_0, t) = \mathcal{M}(A, B, B, t)$.
Also, since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, it follows that there exists an element $x_2 \in A_0(t)$ such that the following equation holds: $\mathcal{M}(x_2, Tx_1, Tx_1, t) = \mathcal{M}(A, B, B, t)$.
Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0(t)$ such that it satisfies:
$\mathcal{M}(x_n, Tx_{n-1}, Tx_{n-1}, t) = \mathcal{M}(A, B, B, t)$ and $\mathcal{M}(x_{n+1}, Tx_n, Tx_n, t) = \mathcal{M}(A, B, B, t)$
for each positive integer n and for $k \in (0,1)$.
As T is generalized b – fuzzy proximal contraction of type –I with respect to $\eta \in \Omega$, we have
$\eta[\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)] \leq k\eta[\mathcal{M}(x_{n-1}, x_n, x_n, t)]$, for all $n \geq 0$.
As η is a decreasing mapping on $[0, \infty)$, we obtain that

$$\begin{aligned}\eta[\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) &\leq k\eta[\mathcal{M}(x_{n-1}, x_n, x_n, t)] \\ &\leq k^2\eta[\mathcal{M}(x_{n-2}, x_{n-1}, x_{n-1}, t)] \\ &\vdots \\ &\leq k^n\eta[\mathcal{M}(x_0, x_1, x_1, t)], \text{ for each } t > 0.\end{aligned}$$

On taking limit as $n \to \infty$ on both sides of the above inequality,
we have $\lim_{n\to\infty} \eta[\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)] = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence.
Suppose that there exists some $n_0 \in \mathbb{N}$ with $m > n > n_0$ such that,

$$\eta[\mathcal{M}(x_n, x_m, x_m, t)] \leq \eta[\mathcal{M}\Big(x_n, x_n, x_n, \frac{t}{b} - \sum_{i=n}^{m-1} \frac{a_i t}{b}\Big) * \mathcal{M}\Big(x_n, x_m, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\Big)]$$
$$= \eta[1 * \mathcal{M}\Big(x_n, x_m, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\Big)]$$
$$\leq \eta(1) + \eta[\mathcal{M}\Big(x_n, x_m, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\Big)]$$
$$= \eta[\mathcal{M}\Big(x_n, x_m, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\Big)]$$

Where $\{a_i\}$ is a decreasing sequence of positive numbers satisfying $\sum_{i=n}^{m-1} a_i = 1$.
Moreover, we obtain that

$$\eta[\mathcal{M}(x_n, x_m, x_m, t)] \leq \eta[\mathcal{M}\Big(x_n, x_m, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\Big)]$$
$$\leq \eta[\mathcal{M}\Big(x_n, x_{n+1}, x_{n+1}, \frac{a_n t}{b^2}\Big) * \mathcal{M}\Big(x_{n+1}, x_{n+2}, x_{n+2}, \frac{a_{n+1} t}{b^2}\Big) * \cdots * \mathcal{M}\Big(x_{m-1}, x_m, x_m, \frac{a_{m-1} t}{b^2}\Big)]$$
$$\leq \eta[\mathcal{M}\Big(x_n, x_{n+1}, x_{n+1}, \frac{a_n t}{b^2}\Big) + \eta[\mathcal{M}\Big(x_{n+1}, x_{n+2}, x_{n+2}, \frac{a_{n+1} t}{b^2}\Big) + \cdots + \eta[\mathcal{M}\Big(x_{m-1}, x_m, x_m, \frac{a_{m-1} t}{b^2}\Big)]$$
$$\leq k^n \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_n t}{b^2}\Big) + k^{n+1} \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_{n+1} t}{b^2}\Big)] \cdots + k^{m-1} \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_{m-1} t}{b^2}\Big)].$$

Hence,

$$\eta[\mathcal{M}(x_n, x_m, x_m, t)] \leq k^n \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_n t}{b^2}\Big) + k^{n+1} \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_{n+1} t}{b^2}\Big)] \cdots + k^{m-1} \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_{m-1} t}{b^2}\Big)]\,. \tag{3.1.1}$$

Assume that,

$$\eta[\mathcal{M}(x_0, x_1, x_1, qt)] = \max\{\eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_n t}{b^2}\Big)]\,, \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_{n+1} t}{b^2}\Big)]\,, \cdots, \eta[\mathcal{M}\Big(x_0, x_1, x_1, \frac{a_{m-1} t}{b^2}\Big)]\}$$

For some $q \in \left\{\frac{a_i}{b^2} : n \leq i \leq m-1 \; and b \geq 1\right\}$, then the inequality (3.1.1) becomes

$$\eta[\mathcal{M}(x_n, x_m, x_m, t)] \leq k^n \eta[\mathcal{M}(x_0, x_1, x_1, qt)] + k^{n+1} \eta[\mathcal{M}(x_0, x_1, x_1, qt)] \cdots + k^{m-1} \eta[\mathcal{M}(x_0, x_1, x_1, qt)]$$
$$\leq (k^n + k^{n+1} + \cdots + k^{m-1}) \eta[\mathcal{M}(x_0, x_1, x_1, qt)]$$
$$= k^n (1 + k + \cdots + k^{m-n-1}) \eta[\mathcal{M}(x_0, x_1, x_1, qt)]$$
$$\leq \frac{k^n}{1-k} \eta[\mathcal{M}(x_0, x_1, x_1, qt)],$$

That is, for all $n \in \mathbb{N}, \eta[\mathcal{M}(x_n, x_m, x_m, t)] \leq \frac{k^n}{1-k} \eta[\mathcal{M}(x_0, x_1, x_1, qt)]$.
On taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$0 \leq \lim_{n,m \to \infty} \eta[\,\mathcal{M}(x_n, x_m, x_m, t)] \leq 0.$$

By the continuity of η, we have $\lim_{n,m \to \infty} \eta \mathcal{M}(x_n, x_m, x_m, t) = 1$.
Thus, $\{x_n\}$ is a Cauchy sequence in a closed subset A of a complete generalized b – fuzzy metric space $(X, \mathcal{M}, *, b)$.
Hence there exists some $x^* \in A$ such that $\lim_{n \to \infty} \eta \mathcal{M}(x_n, x^*, x^*, t) = 1,$ for all $t > 0$.
As the sequence $\{x_n\}$ converges to x^*, we obtain that $\mathcal{M}(x^*, Tx_n, Tx_n, t) \to \mathcal{M}(x^*, B, B, t)$.
If, we consider $Tx^* = y$ (say) in B. Since $\{Tx_n\} \subseteq B$ and B is a fuzzy approximately compact with respect to A,$\{Tx_n\}$ has a subsequence which converges to some y in B, therefore $\mathcal{M}(x^*, y, y, t) = \mathcal{M}(A, B, B, t)$ and hence$x^* \in A_0(t)$.
Thus, $\mathcal{M}(x^*, Tx^*, Tx^*, t) = \mathcal{M}(A, B, B, t)$.

For Uniqueness:If there is another point $y^* \neq x^*$ in A.
Then we have $\mathcal{M}(x^*, Tx^*, Tx^*, t) = \mathcal{M}(A, B, B, t)$ and $\mathcal{M}(y^*, Ty^*, Ty^*, t) = \mathcal{M}(A, B, B, t)$

Since, T is generalized b – fuzzy proximal contraction of type – I, so
$\eta[\mathcal{M}(x^*, y^*, y^*, t)] \leq k\eta[\mathcal{M}(x^*, y^*, y^*, t)] < \eta\mathcal{M}(x^*, y^*, y^*, t)]$.
Gives a contradiction. Hence the result.

Theorem 3.2

Let (X, $\mathcal{M}$,* , b) be complete generalized b – fuzzy metric space, let $T : X \to X$be a mapping satisfying

(3.2.1) $\eta[\mathcal{M}(Tx, Ty, Ty, t)] \leq k\eta[\mathcal{M}(x, y, y, t)]$.

(3.2.2) $T$is continuous.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$Converges to x^*.

Proof:
Let $A = B = X$, first we will prove that T is b – fuzzy proximal contraction of type –I.
Let $x, y, u, v \in X$, satisfy the following conditions:

$$\mathcal{M}(u, Tx, Tx, t) = \mathcal{M}(A, B, B, t) \text{and } \mathcal{M}(v, Ty, Ty, t) = \mathcal{M}(A, B, B, t).$$

Since $\mathcal{M}(A, B, B, t) = 1$, so we have $u = Tx$ and $v = Ty$. Since T satisfies condition (3.2.1)

hence $\eta[\mathcal{M}(u, v, v, t)] = \eta[\mathcal{M}(Tx, T, y, Ty, t)] \leq k\, \eta[\mathcal{M}(x, y, y, t)]$

which implies that T is a generalized b – fuzzy proximal contraction of type – I with respect to $\eta \in \Omega$.
If we choose $y = Tx$, then

$$\mathcal{M}(A, B, B, t) = \mathcal{M}(y, Tx, Tx, t) = \mathcal{M}(Tx, Tx, Tx, t).$$

Set B is approximative compact with respect to A, the conditions of Theorem 3.1 are satisfied, so there exists $x^* \in X$ such that $\mathcal{M}(x^*, Tx^*, Tx^*, t) = \mathcal{M}(A, B, B, t)$, which implies that $Tx^* = x^*$.

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