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SEPARATION AXIOMS ON κ -OPERATION

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Abstract. In this paper, the concept of spaces on operation κ denoted by κ-T_i ($i = 0, 0', 1/2, 1, 1', 2, 2'$) is introduced and also the properties and relations among the spaces are investigated.

Key words:κ-T_0 space, κ-$T_{0'}$ space, κ-$T_{1/2}$ space, κ-T_1 space, κ-$T_{1'}$ space, κ-T_2 space, κ-$T_{2'}$ space, κ-g.closed.

1. Introduction

In 1979, Kasahara [2] defined the concept of an operation α on a topological space and discussed the concept of an α-closed graph of a function. Following this, Jankovic[1] developed the concept of α-closed sets and further investigated functions with α-closed graphs in 1983. Later in 1991, Ogata[3] defined γ-open sets and studied the related topological properties of the associated topology τ_γ and τ. Being motivated by the above works, we introduced Operation Approches in $\boldsymbol{gs}$-open sets and further this paper deals with spaces. Spaces are useful in reversing the dependence relationship within the sets. Seven new types of spaces are introduced and its properties and relationship among them are analyzed.

2. Preliminaries

Definition 2.1

Let (X, τ) be a topological space. A subset A of a space $(X, \tau)$is called generalized semi closed (gs-closed) set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and Uis semi-open in (X, τ).

Definition 2.2

Let (X, τ) be a topological space.A subset A of a space $(X, \tau)$is called generalized semi open (gs-open) set if X\Ais gs-closed. The collection of all gs-open sets is denoted by $GSO(X,\tau)$.Clearly $\tau \subseteq GSO(X,\tau)$.

Remark 2.3 *Every closed set is* $\boldsymbol{gs}$-closed but the converse not true.

Definition 2.4 *[3]*

Let (X, τ) *be a topological*space. An operation$\gamma: \tau \to P(X)$ is a mapping from τ into the power set of X such that $V \subseteq V^{\gamma}$ for each $V \in \tau$, where V^{γ} denotes the value of γ at V.

Definition 2.5 *[7]*

A subset A of a space (X, τ) will be called a γ-openset of (X, τ) if for each $x \in A$, there exists an open set U such that $x \in U$ and $U^{\gamma} \subset A$. τ_{γ} will denote the set of all γ-open sets. Clearly we have $\tau \supset \tau_{\gamma}$.

Definition 2.6 *[7]*

A subset B of (X, τ) is said to be γ-closedin (X, τ) if X\B is γ-open in(X, τ).

Definition 2.7 *[7]*

A point $x \in X$ is in the γ-closureof a set $A \subseteq X$ if $U^{\gamma} \cap A \neq \phi$ for each open set U of x. The γ closure of a set A is denoted by $Cl_{\gamma}(A)$.

3. *κ* - Open sets

Definition 3.1 *[4]*

Let (X, τ) be a topological space. Am Operation κ is a mapping $\kappa: GSO(X, \tau) \to P(X)$ from the family of generalized semi open sets $GSO(X, \tau)$ to the power set of P(X) such that $V \subseteq V^{\kappa}$ for every $V \in GSO(X, \tau)$ where V^{κ} denotes the value of V under the operation κ.

Definition 3.2 *[4]*

A subset A of a space (X, τ) will be called a κ-open set of (X, τ) if for each $x \in A$, there exists a gs-open neighbourhood U of x and $U^{\kappa} \subseteq A$. $\kappa O(X, \tau)$will denote the set of all κ-open sets.

Remark 3.3 In [8] Theorem 1, Every β-γ-open set of (X, τ) is β-open in (X, τ), i.e.,$\beta O(X)_{\tau} \subseteq \beta O(X)$. But for κ-openset the above result fails. That is if κ is an operation on $GSO(X, \tau)$, then every κ-openset need not be a gs-open set in (X, τ).

Example 3.4

The following example shows that every κ-openset need not be a gs-open set.
Let $X = \{a, b, c\}$, $\tau = \{x, \phi, \{a\}\}$ and $GSO(X, \tau) = \{x, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Let κ be an operation on $GSO(X, \tau)$ such that $\kappa(A) = cl(A)$
Then the κ-open sets are $\{X, \phi, \{b, c\}\}$. Here the κ-open set $\{b, c\}$ is not gs-open.

Remark 3.5 It is also seen from the above example that every gs-open sets need not be κ-open sets. Thus κ-open sets and gs-open sets are independent.

Definition 3.6 *[4]*

A κ-operation $\kappa : GSO(X, \tau) \to P(X)$ is called regular$_\kappa$operation given $x \in X$ and for each pair of gs-open neighbourhoods A and B of x, there exists a gs-open neighbourhood Cof x such that $A^K \cap B^K \supseteq C^K$.

Definition 3.7 *[4]*

A topological space (X, τ) is called κ-regular if for given $x \in X$ and each gs-open neighbourhood U of X, there exists a gs-open neighbourhood V of x such that $V^\kappa \subseteq U$.

Theorem 3.8

Let (X, τ) be a topological space and κ an operation on $GSO(X, \tau)$. Then the following results are equivalent.

(a) $GSO(X, \tau) \subseteq \kappa O(X, \tau)$.

(b) (X, τ) is a κ-regular space.

(c) Given $x \in X$ and every gs-open set B of (X, τ) containing x there exists a κ-open set W of (X, τ) such that $x \in W$ and $W \subseteq B$.

Proof: (a) ⇒ (b)

Let x in X and V, a gs-open neighbourhood of x. By (a), V is κ-open in (X, τ). By Definition 3.2 , there exists a gs-open neighbourhood U ofx such that $U^\kappa \subseteq V$. Hence by Definition 3.8, (X, τ) is κ-regular.

(b) ⇒ (c)

Consider $x \in X$ and a gs-open neighbourhood B of x. By(b), (X, τ) is a κ-regular space. Hence by Definition 3.7, there exists a gs-open neighbourhood W of x such that $W^\kappa \subseteq B$. By Definition 3.1. $W \subseteq W^\kappa$. Hence $x \in W \subseteq W^\kappa \subseteq B$.

Claim: W is κ-open. Let $y \in W$. Implies $y \in X$ and W, be the gs-open neighbourhood of y. Then By (b), there exists a gs-open neighbourhood U of x such that $U^\kappa \subseteq W$. By Definition 3.2, W is κ-open. Hence, there exists a κ-open set W such that $x \in W \subseteq B$, proving (c).

(c) ⇒ (a)

It is left to prove $GSO(X, \tau) \subseteq \kappa O(X, \tau)$. Let A be a gs-open set in (X, τ) and $x \in A$. Then $x \in X$ and By (c), there exists a κ-open set W of (X, τ) such that $x \in W \subseteq A$. (1)

Since W is a κ-open set there exists a gs-open set V such that $x \in V^\kappa \subseteq W$. (2)

(1) and (2) implies $x \in V^{\kappa} \subseteq A$. Implies A is κ-open. Therefore, $GSO(X,\tau) \subseteq \kappa O(X,\tau)$.

Definition 3.9 *[4]*

A subset A of a topological space (X,τ) is called κ-closed whenever $X - A$ is κ-open.

Definition 3.10 *[4]*

Let κ be an operation on $GSO(X,\tau)$. A point $x \in X$ is said to be a κ-closurepoint of the set A if $U^{\kappa} \cap A \neq \phi$ for each gs-open neighbourhood U of x.

$gsCl_{\kappa}(A) = \{x \in X / U^{\kappa} \cap A \neq \phi, \ \forall U, \text{gs- open neighborhood of } x\}$

Definition 3.12 *[4]*

Let κ be an operation on $GSO(X,\tau)$. Then $gs_{\kappa}Cl(A)$is defined as the intersection of all κ-closed sets containingA.

$gs_{\kappa}Cl(A) = \cap \{F \subseteq X/ \ A \subseteq F \text{and} X \backslash F \in \kappa O(X,\tau)\}$.

Definition 3.12 *[4]*

An operation κ on GSO(X,τ) is said to be $open_{\kappa}$operation if for every gs-open neighbourhood U of x ∈ X, there exists a κ-open set V such that x ∈ V and $V \subset U^{\kappa}$.

4. Seperation Axioms

Definition 4.1
A space (X,τ) is called κ-T_0 if for any two distinct points $x, y \in X$, there exists a gs-open set 2 such that either $x \in U$ and $y \notin U^{\kappa}$ or $y \in U$ and $x \notin U^{\kappa}$.

Example 4.2
Let $X = \{a,b,c\}$, $\tau = GSO(X,\tau) = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}$. Thus the space (X,τ) is κ-T_0 space by defining the operation κ as

$$\kappa(A) = \begin{cases} \{a\} & \text{if } A = \{a\} \\ A \cup \{b\} & \text{if } A \neq \{a\} \end{cases}$$

Definition 4.3
A space (X,τ) is called a κ-$T_{0'}$ if for any two distinct points $x, y \in X$, there exists aκ-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

Example 4.4
Let $X = \{a,b,c\}$, $\tau = \{X, \emptyset, \{a\}, \{a,b\}\}$ and $GSO(X,\tau) = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Thus (X,τ) is κ-T_0' space by defining the operation κ as

$$\kappa(A) = \begin{cases} A & \text{if} a \in A \\ gscl(A) & \text{if} a \notin A \end{cases}$$

The κ-open sets obtained are $\{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$.

Definition 4.5
A space (X, τ) is called κ-T_1 if for any two distinct points $x, y \in X$, there exist gs-open sets U and V containing x and y respectively such that $y \notin U^{\kappa}$ and $x \notin V^{\kappa}$.

Example 4.6
Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $GSO(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Thus (X, τ) is κ-T_1 by defining the operation κ as

$$\kappa(A) = \begin{cases} \{a, c\} & \text{if } b \notin A \\ A & \text{Otherwise} \end{cases}.$$

Definition 4.7
A space (X, τ) is called a κ-$T_{1'}$ if for any two distinct points $x, y \in X$, there exist κ-open sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.

Example 4.8
Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $GSO(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Thus (X, τ) is κ-T_1' space by defining the operation κ as

$$\kappa(A) = \begin{cases} \{a, c\} & \text{if } b \notin A \\ A & \text{Otherwise} \end{cases}$$

The κ-open sets obtained are $\{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

Definition 4.9
A space (X, τ) is called κ-T_2 if for any two distinct points $x, y \in X$, there exist gs-open sets U and V such that $x \in U$, $y \in V$ and $U^{\kappa} \cap V^{\kappa} = \emptyset$.

Example 4.10
Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $GSO(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Thus (X, τ) is κ-T_2 space by defining the operation κ as

$$\kappa(A) = \begin{cases} \{a, b\} \text{if } A = \{a\} \\ \{b, c\} & \text{if } A = \{b\} \text{or} \{c\} \\ A & \text{Otherwise} \end{cases}.$$

Definition 4.11
A space (X, τ) is called κ-$T_{2'}$ if for any two distinct points $x, y \in X$, there exist κ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Example 4.12
Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $GSO(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Then (X, τ) is κ-$T_{2'}$ space with $\kappa O(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ by defining κ as

$$\kappa(A) = \begin{cases} A \text{if} b \in A \\ gscl(A) \text{if} b \notin A \end{cases}.$$

Theorem 4.13
A space (X, τ) is a κ-$T_{0'}$ space if and only if for every pair $x, y \in X$ with $x \neq y$, $gs_{\kappa}cl(x) \neq gs_{\kappa}cl(y)$.

Proof: Necessity : Let (X, τ) be a κ-$T_{0'}$ space and say $x \neq y$ belong to X. By definition of κ-$T_{0'}$ we assume that there exists a κ-open set A such that $x \in A$ and $y \notin A$. Therefore $y \in X - A$ and $X - A$ is a κ-closed set. Hence $gs_{\kappa}cl(\{y\}) \subseteq gs_{\kappa}cl(X - A) = X - A$.
Now, $x \in gs_{\kappa}cl(\{x\})$ but $x \notin X - A$. Implies $\notin gs_{\kappa}cl(\{y\})$. Therefore $gs_{\kappa}cl(\{x\}) \neq gs_{\kappa}cl(\{y\})$.
Sufficiency : Consider for any $x \neq y$ we have $gs_{\kappa}cl(\{x\}) \neq gs_{\kappa}cl(\{y\})$. We have to construct a κ-open set U containg x and not containing y. From the assumption there exists a $z \in gs_{\kappa}cl(\{x\})$ and $z \notin gs_{\kappa}cl(\{y\})$. Suppose $x \in gs_{\kappa}cl(\{y\})$ then taking $gs_{\kappa}cl$ on both sides we get $gs_{\kappa}cl(\{x\}) \subseteq$

$gs_\kappa cl(\{y\})$ and hence $z \in gs_\kappa cl(\{y\})$ which is a contradiction. Hence $x \notin gs_\kappa cl(\{y\})$ implies $X - gs_\kappa cl(\{y\}) = U$ which is a κ-open set containing x but not y. Hence (X, τ) is a κ-$T_0{}'$space.

Theorem 4.14
Let κ be a κ-open operation. A space (X, τ) is a κ-T_0 space if and only if for every pair $x, y \in X$ with $x \neq y$, $gscl_\kappa(\{x\}) \neq gscl_\kappa(\{y\})$.
Proof: Necessity: In (X, τ) consider $x \neq y$ and (X, τ) is a D-T_0 space. Then by the definition of κ-T_0 there exists a gs-open set U containing x and U^κ not containing y. Here κ is a κ-open operation, then for every $x \in X$ and for every gs-open set U containing x, there exists a κ-open set S such that $x \in S$ and $S \subseteq U^\kappa$. Hence $y \in X\text{-}U^\kappa \subseteq X - S$ which is a κ-closed set. Therefore $gscl_\kappa(\{y\}) \subseteq gscl_\kappa(X - S) \subseteq X\text{-}S$. Now $x \in gscl_\kappa(\{x\})$ and $x \in S$ implies $x \notin X - S$. Therefore $gscl_\kappa(x) \neq gscl_\kappa(\{y\})$.
Sufficiency : For any $x \neq y \in X$, then we have $gscl_\kappa(\{x\}) \neq gscl_\kappa(\{y\})$. Let $z \in gs_\kappa cl(\{x\})$ and $z \notin gs_\kappa cl(\{y\})$. Suppose $x \in gscl_\kappa(\{y\})$ taking $gscl_\kappa$ on both sides we get $gscl_\kappa(\{x\}) \subseteq gscl_\kappa(\{y\})$. Thus $z \in gs_\kappa cl(\{y\})$ which is a contradiction. Therefore $x \notin gscl_\kappa(\{y\})$, so there exists a gs-open set W such that $x \in W$ and $W^\kappa \cap \{y\} = \emptyset$ that is $y \notin W^\kappa$. Hence (X, τ) is κ-T_0.

Definition 4.15
A subset A of (X, τ) is said to be κ-g. closed if $gscl_\kappa(A) \subseteq U$ whenever $A \subseteq U$ and U is κ-open in (X, τ).

Proposition 4.16 Every κ-closed set is κ-g.closed.
Proof: Let A be a κ-closed set. Then $gscl_\kappa(A) = A$. Therefore $A \subseteq U$ where U is κ-open implies $gscl_\kappa(A) = A \subseteq U$ implies A is κ-g.closed.

Example 4.17
Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $GSO(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.
Then by defining κ as
$$\kappa(A) = \begin{cases} gscl(A) & \text{if } b \notin A \\ A & \text{if } b \in A \end{cases}$$
the κ-open sets are $\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ and κ-g.closed sets are $\{\{a\}, \{b\}, \{b, c\}, \{a, c\}\}$.

Definition 4.18
A space (X, τ) is called a κ-$T_{1/2}$ space if every κ-g.closed set of (X, τ) is κ-closed.
In κ-$T_{1/2}$ space κ-g.closed sets coincide with κ-closed sets.

Example 4.19
Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $GSO(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.
Then (X, τ) is κ-$T_{1/2}$ space with $\kappa O(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ by defining κ as
$$\kappa(A) = \begin{cases} gscl(A) if b \notin A \\ A if b \in A \end{cases}.$$

Definition 4.20
A subset S is said to be gs_κKernal if it is contained in the intersection of all K- open sets. $gs_\kappa \ker(S) = \cap \{V / S \subseteq V, V \in KO(X)\}$ for any subset E of (X, τ).

Theorem 4.21
Let (X, τ) be a topological space and κ be an operation on $GSO(X)$. Then the following statements are equivalent.
(i) A is κ-g.closed in (X, τ).
(ii) $gs_\kappa cl(x) \cap A \neq \emptyset$ for every $x \in gscl_\kappa(A)$.
(iii) $gscl_\kappa(A) \subseteq gs_\kappa ker(A)$ holds, where $gs_\kappa \ker(S) = \cap \{V / S \subseteq V, V \in KO(X)\}$ for any subset E of (X, τ).

Proof:

(i) $\Rightarrow$ (ii) In (X,τ) consider a κ-g.closed set A. If there exists $x \in gscl_\kappa(A)$ with the condition that $gs_\kappa cl(x) \cap A = \emptyset$. Here $gs_\kappa cl(\{x\})$ is κ-closed and $A \subseteq (gs_\kappa cl(\{x\}))^c$. Hence $gscl_\kappa(A) \subseteq \left(gs_\kappa cl(\{x\})\right)^c$ (since A is κ-g.closed). Therefore $x \notin gscl_\kappa(A)$ (since $x \notin (gs_\kappa cl(\{x\}))^c$). This is a contradiction to the assumption. Therefore $gs_\kappa cl(x) \cap A \neq \emptyset$.

(ii) $\Rightarrow$ (iii) Consider a point $x \in gscl_\kappa(A)$. By (ii) Choose $z \in gs_\kappa cl(x) \cap A$. Now U be any κ-open set containing A then $z \in U$ and $z \in gs_\kappa cl(x)$. Implies $U \cap \{x\} \neq \emptyset$. Implies $x \in U$ thus $gs_\kappa ker(A)$. Hence $gscl_\kappa(A) \subseteq gs_\kappa ker(A)$.

(iii) $\Rightarrow$ (i) Consider a κ-open s et U such that $A \subseteq U$. We have to prove $gscl_\kappa(A) \subseteq U$. Let $x \in gscl_\kappa(A)$ By (iii), $x \in gs_\kappa ker(A)$. Hence $x \in U$ because $A \subseteq U$ and U is κ-open.

Theorem 4.22

Let (X,τ) be a topological space and κ an operation on $GSO(X,\tau)$. If a subset A of X is κ-g.closed then $gscl_\kappa(A) - A$ does not contain any non-empty κ-closed set.

Proof: Suppose that there exists a non-empty κ-closed set F such that $F \subseteq gscl_\kappa(A) - A$. Then we have $A \subseteq X - F$ and $X - F$ is κ-open. It follows from the assumption that $gscl_\kappa(A) \subseteq X - F$ and $F \subseteq (gscl_\kappa(A) - A) \cap (X - gscl_\kappa(A))$. Therefore we have $F = \emptyset$.

Remark 4.23 Converse of the above theorem is true if κ is a gs-open operation.

Proof: Let U be a κ-open set and $A \subseteq U$. Since κ is a gs-open operation, $gscl_\kappa(A)$ is κ-closed in (X,τ). Which implies $gscl_\kappa(A) \cap (X - A) = F$ is κ-closed in (X,τ). Now $X - U \subseteq X - A$, $F \subseteq gscl_\kappa(A) - A$. By the converse of Theorem 4.22 above, $F = \emptyset$ which implies $gscl_\kappa(A) \subseteq U$.

Theorem 4.24

For a topological space (X,τ) and an operation κ defined on $GSO(X,\tau)$, for each $x \in X$ either $\{x\}$ is κ- closed or $X - \{x\}$ is κ-g.closed in (X,τ).

Proof: Assume that $\{x\}$ is not κ- closed, then $X - \{x\}$ is not κ-open. Let U be any κ-open set such that $X - \{x\} \subseteq U$. Then $U = X$. Hence $gscl_\kappa(X - \{x\}) \subseteq U$. Therefore, $X - \{x\}$ is a κ-g.closed in (X,τ).

Theorem 4.25

Let (X,τ) be a topological space and κ be an operation on $GSO(X,\tau)$. Then the following properties are equivalent.

(i) A space (X,τ) is κ-$T_{1/2}$.

(ii) For each $x \in X$, $\{x\}$ is κ- closed or κ-open.

Proof:

(i) $\Rightarrow$ (ii) Let $\{x\}$ be not κ- closed in (X,τ). Then $X - \{x\}$ is κ-g.closed. Since (X,τ) is κ-$T_{1/2}$ space, $X - \{x\}$ is κ- closed and so $\{x\}$ is κ-open.

(ii) $\Rightarrow$ (i) Let F be a κ-g.closed set in (X,τ). To prove F is κ- closed it is enough to prove that $gscl_\kappa(F) = F$. Assume that there exists a point x such that $x \in gscl_\kappa(F) - F$. Then by assumption, $\{x\}$ is κ-closed or κ-open.

Case 1: If $\{x\}$ is a κ-closed – for this case, we have a κ-closed set$\{x\}$ such that $\{x\} \subseteq gscl_2(F) - F$. This is a contradiction to Theorem 4.23.

Case 2: If $\{x\}$ is a κ-open – we have $x \in gs_\kappa cl(F)$. Since $\{x\}$ is κ-open, it implies that $\{x\} \cap F \neq \emptyset$. Then $x \in F$ which is a contradiction. Thus we have $gscl_\kappa(F) = F$ and F is κ-closed.

Theorem 4. 26 For a topological space (X,τ), let κ be an operation on $\kappa O(X)$.

(i) Then the following properties are equivalent.

  (1) (X,τ) is κ-T_1.

  (2) For every point $x \in X$, $\{x\}$ is a κ-closed set.

  (3) (X,τ) is κ-T_1'.

(ii) Every κ-T_i' space is □-T_i where $i \in \{0,2\}$.

(iii) Every κ-T_2 space is κ-T_1.

(iv) Every κ-T_1space is κ-$T_{1/2}$.
(v) Every κ-$T_{1/2}$ space is κ-T_0'.
(vi) Every κ-T_i' space is κ-T_{i-1}', where $i \in \{1,2\}$.

Proof:
(i) (1) ⇒ (2) Let $x \in X$ be a point. For each $2 \in X-\{x\}$, there exists a gs-open set U_y such that $y \in U_y$ and $x \notin (U_y)^{\kappa}$. Then $X-\{x\} = \cup\{(U_y)^{\kappa} / y \in X-\{x\}\}$. Each $(U_y)^{\kappa}$ is κ-open and $X-\{x\}$ is a κ-open in (X,τ) implies $\{x\}$ is a κ-closed set.
(2) ⇒ (3) Let x and y be two distinct points of X. By (2), $X-\{x\}$ and $X-\{y\}$ are required κ-open sets such that $y \in X-\{x\}$, $x \notin X-\{x\}$ and $x \in X-\{y\}$, $x \notin X-\{y\}$.
(3) ⇒ (1) It is shown that if $x \in V$, where $V \in \kappa O(X,\tau)$, then there exists a gs-open set U such that $x \in U \subseteq U^{\kappa} \subseteq V$. By using (3), we have that (X,τ) is κ-T_1'.
(ii), (iii), (vi) follows from the definitions.

The proof of (iv) follows from (i) and Theorem 4.25
The proof of (v) follows from Theorem 4.25 and from the definition of κ-T_0'.

Remark 4.27 Following are the examples given to show the counters of Theorem 4.27.
(i) Every κ-T_2 space need not be κ-T_2'.

Let $X = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\}\}$ and $GSO(X,\tau) = \{X,\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\}$. The operation κ is defined as

$$\kappa(A) = \begin{cases} \{a\} & \text{if } A = \{a\} \\ \{b,c\} & \text{if } A = \{b\} \text{or} \{c\} \\ A & \text{Otherwise} \end{cases}$$

The κ-open sets obtained are $\{X,\emptyset,\{a,b\},\{b,c\},\{a,c\}\}$.
Thus the space here is κ-T_2 space but not κ-T_2'.
(ii) Every κ-T_1 space need not be κ-T_2 and Every κ-T_1' space need not be κ-T_2'

Let $X = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$ and $GSO(X,\tau) = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$. The operation κ is defined as

$$\kappa(A) = \begin{cases} \{a,c\} & \text{if } b \notin A \\ A & \text{Otherwise} \end{cases}$$

The κ-open sets obtained are $\{X,\emptyset,\{b\},\{a,b\},\{b,c\},\{a,c\}\}$.
Thus the space here is κ-T_1 space but not κ-T_2' and every κ-T_1' space need not be κ-T_2'
(iii) Following is the example that shows every κ-T_0 space need not be κ-T_0' and Every κ-T_0 space need not be κ-T_1
(iv) Let $X = \{a,b,c\}$, $\tau = GSO(X,\tau) = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,c\}\}$ and. The operation κ is defined as $\kappa(A) = \begin{cases} \{a\} & \text{if } A = \{a\} \\ A \cup \{b\} & \text{if } A \neq \{a\} \end{cases}$

The κ-open sets obtained are $\{X,\emptyset,\{b\},\{a,b\},\{b,c\},\{a,c\}\}$.
Thus the space here is κ-T_0 space but not κ-T_0' and Every κ-T_0 space need not be κ-T_1
(v) Following is the example to show that every κ-T_0' space need not be κ-T_1' and every κT_0- space need not be κ-$T_{1/2}$

Let $X = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\},\{a,b\}\}$ and $GSO(X,\tau) = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,c\}\}$. The operation κ is defined as $\kappa(A) = \begin{cases} A & \text{if} a \in A \\ gscl(A) & \text{if} a \notin A \end{cases}$
The κ-open sets obtained are $\{X,\emptyset,\{a\},\{a,b\},\{a,c\}\}$.
Thus the space here is κ-T_0' space but not κ-T_1' and every κT_0- space need not be κ-$T_{1/2}$
(vi)Every κ-$T_{1/2}$ space need not be κ-T_1
Let $X = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\},\{a,b\}\}$ and $GSO(X,\tau) = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,c\}\}$. The operation κ is defined as

$$\kappa(A) = \begin{cases} \{a, b\} \text{either} A = \{a\} \text{or } \{b\} \\ A \qquad\qquad \text{if} a \notin A \end{cases}$$

The κ-open sets obtained are $\{X, \emptyset, \{a, b\}, \{a, c\}\}$.

Thus the space here is κ-$T_{1/2}$ space but not κ-T_1.

References

[1] Arya, S.P. and Nour, T.M., Characterizations of S-normal spaces, Indian J. Pure. Appl. Math., (1990), 21, 717-719.

[2] Carpintero, C., Rajesh, N. and Rosas, E., Operation approaches on b-open sets and applications, Bol. Sci. Paran. Mat., (2012), 30, 21-33.

[3] Jankovic, D.S., On functions with α-closed graphs, Glasnic Mat., (1983), 18, 161-169.

[4] Jayashree, R. and Sivakamasundari, K., Operation approaches on gs-open sets in topological spaces, Journal of Emerging Technologies and Innovative Research (JETIR), September(2018),Volume 5, Issue 9, 225-229.

[5] Kasahara,S., Operation-compact spaces, Math. Japon., 1979, (24), 97-105.

[6] Ogata, H., Operations on topological spaces and associated topology, Math. Japon., (1991), 36, 175-184.

[7] Sanjay Tahiliani., Operation approach to β-open sets and applications, Math. Commun., (2011), 16, 577-591.